2-D Parameter Estimation Using Arrays With Multidimensional Invariance Structure

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Abstract—A computationally efficient algorithm is presented for narrowband array processing problems where two parameters must be estimated per source (e.g., azimuth and elevation angles of arrival, azimuth angle and wavelength, etc.). The algorithm is applicable to arrays composed of multiple identical subarrays displaced from one another in two dimensions. Important examples include uniform rectangular phased arrays and uniformly sampled linear arrays. Estimates are obtained by performing a least-squares minimization which approximates the optimal Weighted Subspace Fitting (WSF) minimization. While the approximate algorithm yields excellent estimates at low computational cost, significant improvement is realized in many cases by using the estimates to initialize a WSF search. Simulation results illustrating these issues are also presented.

I. INTRODUCTION

WHEN SIGNALS ARE COLLECTED by an array of spatially distributed sensors, a variety of parameters (e.g., amplitude, phase, wavelength, azimuth and elevation angles of arrival, range, etc.) may be estimated per signal source. Although most of the so-called high-resolution algorithms (e.g., MUSIC [1], autoregressive modeling techniques, etc.) have been presented in the context of estimating a single parameter per source, generalizations to the multiple parameter case are relatively straightforward (e.g., for MUSIC, see [2]). The main difficulty associated with these methods is that both computational and storage costs tend to increase rapidly with the dimension of the parameter vector. The increased costs are usually prohibitive even for the two-dimensional case, and the result is that in practice, systems typically employ *non-parametric* techniques (i.e., beamforming, 2D FFTs) to solve what are in reality *parametric* problems. Though these classical techniques are more easily implemented, their performance is known to be poor.

The recently introduced Weighted Subspace Fitting (WSF) algorithm [3, 4, 5] has been shown to provide estimates which asymptotically achieve the Cramér-Rao

bound for parameter estimation problems in narrowband sensor array processing [6, 7]. The main drawback of the WSF approach is that the resulting parameter search must in general be initialized with estimates of reasonable quality. For multiple parameter problems, the initial estimates must also be properly associated; i.e., estimates from different parameter dimensions must be assigned to a particular source. Additionally, since the WSF algorithm ultimately requires a multidimensional parameter search, one would hope that the initial estimates could be obtained as efficiently as possible.

In the single parameter per source case, if the array is composed of two identical but displaced subarrays, the ESPRIT algorithm [8] can be employed to find the initial estimates in a computationally efficient manner. To use ESPRIT for the multiple parameter case, an array with displacement invariances in more than one dimension would be required. However, performing ESPRIT separately for each dimension of the parameter vector would not take advantage of the full invariance structure of such an array. In addition, such an approach provides no mechanism for associating the estimates from different dimensions with a particular source. A similar problem was addressed in [9] in the context of state space models for uniform sampled 2-D time series.

The key to using the geometric ideas of ESPRIT for the multiple parameter problem is recognizing that the ESPRIT algorithm is a special case of WSF for a specific array parameterization [5, 10]. Once this connection is made, it is seen that extending the algorithm to arrays with multidimensional invariances simply amounts to a re-parameterization of the WSF problem. While the ESPRIT/WSF minimization is accomplished very efficiently in the single parameter case, a search technique is required for the multiple invariance parameterization. Consequently, in this paper we develop a *suboptimal*, but computationally efficient algorithm for finding parameter estimates in the multiple invariance case. If the optimal WSF solution is desired, these estimates may be used as initial conditions for the resulting gradient search.

In the next section, a brief description of the narrowband data model is given for arrays with multiple identical subarrays. The WSF algorithm for such arrays is considered in Section 3, and the corresponding suboptimal approach is described in Section 4. Some representative simulation results are given in Section 5. Although

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the discussion will focus on the two-dimensional case, the concepts presented are easily extended to higher dimensional parameter vectors.

II. DATA MODEL

We will assume an M -element array of sensors, d narrowband far-field emitters, and we will define $\mathbf{a}(\theta) \in \mathbb{C}^M$ to be the array response for a narrowband emitter at DOA θ . The output $\mathbf{z} \in \mathbb{C}^M$ of the array at time t is given by

$$\mathbf{z}(t) = \mathbf{G}\mathbf{s}(t) + \mathbf{n}(t),$$

where $\mathbf{s}(t) \in \mathbb{C}^d$ is the amplitude and phase of the signals at time t , $\mathbf{n}(t)$ is additive noise, and where

$$\mathbf{G} \stackrel{\text{def}}{=} [\mathbf{a}(\theta_1) \cdots \mathbf{a}(\theta_d)].$$

Assuming the noise is spatially white² and uncorrelated with the signals, we have

$$\mathbf{R}_{zz} = \mathcal{E}\{\mathbf{z}(t)\mathbf{z}^*(t)\} = \mathbf{G}\mathbf{R}_{ss}\mathbf{G}^* + \sigma^2\mathbf{I},$$

where $\mathcal{E}$ denotes expectation, $\mathbf{R}_{ss}$ is the covariance matrix of the emitter signals, and σ^2 is the noise variance at each sensor. The covariance $\mathbf{R}_{ss}$ is assumed to be full rank d (no unity correlated signals)³ and the columns of $\mathbf{G}$ are assumed to be linearly independent. The eigendecomposition of $\mathbf{R}_{zz}$ has the following form:

$$\mathbf{R}_{zz} = \sum_{i=1}^M \lambda_i \mathbf{e}_i \mathbf{e}_i^* = \mathbf{E}_s \mathbf{\Lambda}_s \mathbf{E}_s^* + \sigma^2 \mathbf{E}_n \mathbf{E}_n^* \quad (1)$$

where $\mathbf{E}_s = [\mathbf{e}_1 \cdots \mathbf{e}_d]$, $\mathbf{E}_n = [\mathbf{e}_{d+1} \cdots \mathbf{e}_M]$, and $\lambda_1 \geq \cdots \geq \lambda_d > \lambda_{d+1} = \cdots = \lambda_M = \sigma^2$. The span of the d eigenvectors $\mathbf{E}_s$ defines the signal subspace, and the orthogonal complement spanned by $\mathbf{E}_n$ defines the noise subspace. This terminology is a consequence of the fact that $\text{span}(\mathbf{E}_s) = \text{span}(\mathbf{G}) \perp \text{span}(\mathbf{E}_n)$. This implies that a $d \times d$ matrix $\mathbf{T}$ exists satisfying $\mathbf{E}_s = \mathbf{G}\mathbf{T}$.

To examine the special structure that results for arrays with multiple invariances, assume the array is composed of p identical subarrays of m sensors, each displaced in time or space from the reference subarray. The total number of sensors M need not be equal to mp since overlapping subarrays are allowed ($M < mp$ in such cases). The structure inherent in the signal subspace may be described by the following equation:

$$\mathbf{J}\mathbf{E}_s \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{E}_0 \\ \mathbf{E}_1 \\ \mathbf{E}_2 \\ \vdots \\ \mathbf{E}_{p-1} \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ \mathbf{A}\Phi_1 \\ \mathbf{A}\Phi_2 \\ \vdots \\ \mathbf{A}\Phi_{p-1} \end{bmatrix} \mathbf{T}, \quad (2)$$

²As usual, the assumption of spatially white noise is not necessary; the extension to an arbitrary noise covariance $\sigma^2 \Sigma$ is straightforward, provided Σ is known.

³Although the parameters of fully correlated signals may be estimated using the multiple invariance parameterization of the WSF algorithm presented later, for simplicity we will restrict our discussion in this paper to cases where $\mathbf{R}_{ss}$ is full rank. The coherent case is considered in [4, 10].

where $\mathbf{J}$ is an $mp \times M$ selection matrix, and Φ_i is a diagonal matrix of phase delays which describes the displacement of the i^{th} subarray from the reference.

The parameterization of (2) is very general and can be made to apply to a wide variety of problems. As an example, consider the 5×5 rectangular array depicted in Figure 1. For this case, the sensor outputs at time

Figure 1: 5×5 Rectangular Array

t are stacked in a column vector $\mathbf{x}(t)$, and the selection matrix $\mathbf{J} = [\mathbf{J}_0^T \mathbf{J}_1^T \mathbf{J}_2^T \mathbf{J}_3^T]^T$ is used to define the four overlapping subarrays. The phase delays for subarray 3 with respect to the reference subarray are simply the sum of the delays for subarrays 1 and 2. Thus $\Phi_3 = \Phi_1 \Phi_2$, and the model for the data becomes:

$$\mathbf{J}\mathbf{x}(t) = \begin{bmatrix} \mathbf{A} \\ \mathbf{A}\Phi_1 \\ \mathbf{A}\Phi_2 \\ \mathbf{A}\Phi_1\Phi_2 \end{bmatrix} \mathbf{s}(t) + \mathbf{n}(t).$$

If the azimuth angle θ and the elevation angle γ are referenced to a vertical line in the plane of the page and a line normal to the page, respectively, the elements of Φ_1 and Φ_2 can be expressed as

$$\begin{aligned} \Phi_1 &= \text{diag}\{\exp(-j2\pi|\Delta_1|\sin\theta_i/\lambda)\} \\ \Phi_2 &= \text{diag}\{\exp(-j2\pi|\Delta_2|\cos\theta_i\sin\gamma_i/\lambda)\} \end{aligned}$$

for $i = 1, \dots, d$. Figure 1 could just as easily represent samples of a 2-D time series composed of a sum of cisoids, in which case

$$\begin{aligned} \Phi_1 &= \text{diag}\{\exp(\alpha_{1,i} - j\omega_{1,i})\} \\ \Phi_2 &= \text{diag}\{\exp(\alpha_{2,i} - j\omega_{2,i})\} \end{aligned}, \quad (3)$$

where $\alpha_{k,i}$ and $\omega_{k,i}$ are respectively the frequency and decay factor for the i^{th} signal in the k^{th} dimension.

In practice, the sample covariance $\hat{\mathbf{R}}_{zz}$ defined by

$$\hat{\mathbf{R}}_{zz} = \frac{1}{N} \sum_{k=1}^N \mathbf{z}(t_k) \mathbf{z}^*(t_k)$$

is used to estimate $\mathbf{R}_{zz}$, and the corresponding estimate $\hat{\mathbf{E}}_s$ will not exactly satisfy the relationship of (2). To estimate the signal parameters, the weighted subspace fitting method performs a least-squares fit of the model of (2) to the matrix $\mathbf{J}\hat{\mathbf{E}}_s$, as described in the next section.

III. WEIGHTED SUBSPACE FITTING FRAMEWORK

All algorithms within the recently introduced class of *subspace fitting* algorithms [3, 4, 5] solve a minimization problem of the following common form:

$$\min_{\boldsymbol{\eta}, \mathbf{T}} \|\mathbf{M} - \tilde{\mathbf{G}}(\boldsymbol{\eta})\mathbf{T}\|_F^2, \quad (4)$$

where $\mathbf{M}$ is a matrix derived from the data, $\tilde{\mathbf{G}}(\boldsymbol{\eta})$ describes the parameterization of the array response, and where $\mathbf{T}$ attempts to map a particular array response onto the data. Algorithms which belong to this class include classical beamforming, deterministic maximum likelihood, MUSIC, and ESPRIT.

Referring back to the parameterization of (2), we see that the appropriate WSF minimization for the multiple parameter problem is as follows:

Given p subspace estimates $\hat{\mathbf{E}}_0, \hat{\mathbf{E}}_1, \dots, \hat{\mathbf{E}}_{p-1} \in \mathbb{C}^{m \times d}$, find a matrix $\mathbf{A} \in \mathbb{C}^{m \times d}$, a matrix $\mathbf{T} \in \mathbb{C}^{d \times d}$, and (unitary) diagonal matrices $\boldsymbol{\Phi}_1, \dots, \boldsymbol{\Phi}_{p-1} \in \mathbb{C}^{d \times d}$ to minimize

$$V = \left\| \begin{bmatrix} \hat{\mathbf{E}}_0 \\ \hat{\mathbf{E}}_1 \\ \hat{\mathbf{E}}_2 \\ \vdots \\ \hat{\mathbf{E}}_{p-1} \end{bmatrix} \mathbf{W}^{1/2} - \begin{bmatrix} \mathbf{A} \\ \mathbf{A}\boldsymbol{\Phi}_1 \\ \mathbf{A}\boldsymbol{\Phi}_2 \\ \vdots \\ \mathbf{A}\boldsymbol{\Phi}_{p-1} \end{bmatrix} \mathbf{T} \right\|_F^2. \quad (5)$$

The word *unitary* appears in parentheses since it may not apply to time series problems (as in (3)). With reference to equation (4), the parameter vector $\boldsymbol{\eta}$ for this minimization problem is

$$\boldsymbol{\eta} = [\theta_1, \dots, \theta_d, \gamma_1, \dots, \gamma_d, \text{vec}(\tilde{\mathbf{I}}\tilde{\mathbf{A}})^T, \text{vec}(\tilde{\mathbf{I}}\tilde{\mathbf{A}})^T], \quad (6)$$

where θ_i and γ_i are, say, the azimuth and elevation angle of the i^{th} signal, $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{A}}$ are respectively the real and imaginary parts of $\mathbf{A}$, $\text{vec}(\cdot)$ is the column stacking operator, and where $\tilde{\mathbf{I}} = [0_{(m-1) \times 1} \ 1]$ picks out all but the first row of the matrix to its right. Algorithms for implementing the minimization of (5) can be found in [4, 10].

Note that the square root of a positive definite weighting matrix $\mathbf{W}$ has been included in the cost function

V. Ottersten and Viberg [3, 4, 5] have made a comprehensive study of the asymptotic properties of estimates obtained from this type of minimization problem for a general weighting $\mathbf{W}$. A key result of this analysis is that the optimal choice of $\mathbf{W}$ is

$$\mathbf{W}_{\text{opt}} = (\hat{\mathbf{A}}_s - \hat{\sigma}^2 \mathbf{I})^2 \hat{\mathbf{A}}_s^{-1}, \quad (7)$$

where $\hat{\sigma}^2$ is an estimate of the noise variance and where $\hat{\mathbf{A}}_s$ are the “signal” eigenvalues of the sample covariance $\hat{\mathbf{R}}$ as in equation (1). Optimal here means that asymptotically,

$$\mathbf{C}(\mathbf{W}_{\text{opt}}) \leq \mathbf{C}(\mathbf{W}) \quad (8)$$

for all matrices $\mathbf{W}$, where

$$\mathbf{C}(\mathbf{W}) = N \mathcal{E}\{(\hat{\boldsymbol{\eta}}(\mathbf{W}) - \boldsymbol{\eta}_0)^*(\hat{\boldsymbol{\eta}}(\mathbf{W}) - \boldsymbol{\eta}_0)\},$$

is the normalized covariance of the estimation error and $\boldsymbol{\eta}_0$ is the true parameter vector. In fact, in [6, 7] it is shown that $\mathbf{C}(\mathbf{W}_{\text{opt}})$ is asymptotically identical to the Cramér-Rao bound (CRB) for non-overlapping subarray structures⁴. For overlapping structures, $\mathbf{C}(\mathbf{W}_{\text{opt}}) \neq \text{CRB}$, and equation (8) has not been established analytically. However, in all simulations with overlapping subarrays performed to date, it has been observed that $\mathbf{C}(\mathbf{I}) > \mathbf{C}(\mathbf{W}_{\text{opt}})$.

Before discussing an approximate solution to (5), it is necessary to reformulate the expression for the cost function V . If the definitions $\tilde{\boldsymbol{\Psi}}_i = \mathbf{T}^{-1}\boldsymbol{\Phi}_i\mathbf{T}$, $\tilde{\boldsymbol{\Psi}} = [\mathbf{I} \mid \tilde{\boldsymbol{\Psi}}_1 \mid \dots \mid \tilde{\boldsymbol{\Psi}}_{p-1}]$, and $\tilde{\mathbf{E}}_W = [\mathbf{E}_0 \mathbf{W}^{1/2} \mid \dots \mid \mathbf{E}_{p-1} \mathbf{W}^{1/2}]$ are made, equation (5) may be rewritten as

$$V = \|\tilde{\mathbf{E}}_W \mathbf{P}_{\tilde{\boldsymbol{\Psi}}}^\perp\|_F^2, \quad (9)$$

where

$$\mathbf{P}_{\tilde{\boldsymbol{\Psi}}}^\perp = \mathbf{I} - \tilde{\boldsymbol{\Psi}}^*(\tilde{\boldsymbol{\Psi}}\tilde{\boldsymbol{\Psi}}^*)^{-1}\tilde{\boldsymbol{\Psi}}.$$

For the rectangular array example just presented, $\tilde{\boldsymbol{\Psi}} = [\mathbf{I} \mid \tilde{\boldsymbol{\Psi}}_1 \mid \tilde{\boldsymbol{\Psi}}_2 \mid \tilde{\boldsymbol{\Psi}}_1\tilde{\boldsymbol{\Psi}}_2]$. This is not, of course, the only way to divide a uniform rectangular array of identical sensors into subarrays. If a 3×3 subarray size had been chosen instead, $\tilde{\boldsymbol{\Psi}}$ would have had the following form:

$$\tilde{\boldsymbol{\Psi}} = [\mathbf{I} \mid \tilde{\boldsymbol{\Psi}}_1 \mid \tilde{\boldsymbol{\Psi}}_1^2 \mid \tilde{\boldsymbol{\Psi}}_2 \mid \tilde{\boldsymbol{\Psi}}_2\tilde{\boldsymbol{\Psi}}_1 \mid \tilde{\boldsymbol{\Psi}}_2\tilde{\boldsymbol{\Psi}}_1^2 \mid \tilde{\boldsymbol{\Psi}}_2^2 \mid \tilde{\boldsymbol{\Psi}}_2^2\tilde{\boldsymbol{\Psi}}_1 \mid \tilde{\boldsymbol{\Psi}}_2^2\tilde{\boldsymbol{\Psi}}_1^2].$$

As smaller and smaller subarray sizes are chosen, better use is made of the information about the physical structure of the array, and consequently improved performance is expected. In the limit, $m = 1$, $p = M$, and the physical structure of the array is completely exploited. However, as p increases, there is a corresponding computational increase for the suboptimal method proposed in the next section.

⁴We note here that the distribution of the estimates obtained from the single-invariance ESPRIT algorithm has been conjectured to be asymptotically independent of $\mathbf{W}$ [11].

IV. A SUBOPTIMAL APPROACH

The difficulty in obtaining a direct solution to (9) is the requirement that the elements $\Psi_1, \dots, \Psi_{p-1}$ of $\tilde{\Psi}$ obey certain constraints. The first type of constraint that arises is due to the specific structure of $\tilde{\Psi}$ which must reflect the phase relationships between each subarray and the reference. From the rectangular array example of the previous section, a *structural* constraint of this type is $\tilde{\Psi}_3 = \tilde{\Psi}_1 \tilde{\Psi}_2$.

The second type of constraint occurs because $\Psi_i = \mathbf{T}^{-1} \Phi_i \mathbf{T}$ implies that all of the sub-blocks of $\tilde{\Psi}$ must have a common set of eigenvectors. Since no two sources can share an identical set of parameters, this constraint translates into the requirement that all matrices in $\{\Psi_i\}_{i=1}^{p-1}$ commute. The parameters of interest are obtained from the matrices $\{\Phi_i\}_{i=1}^{p-1}$, which are the eigenvalues of the matrices $\{\Psi_i\}_{i=1}^{p-1}$. Since the eigenvalue ordering for each Ψ_i is arbitrary, enforcement of the *commutativity* constraint provides for the association of multiple parameters with a single source. That is, the k^{th} element $\Phi_{i,kk}$ of the diagonal matrix Φ_i is associated with the element $\Phi_{l,kk}$ of the diagonal matrix $\Phi_l = \mathbf{T} \Psi_l \mathbf{T}^{-1}$, $l = 1, \dots, p-1$. A third type of constraint arises in direction of arrival estimation problems where the matrices $\{\Phi_i\}_{i=1}^{p-1}$ have diagonal elements on the unit circle.

The minimization of (5) can thus be written as the constrained problem

$$\min_{\tilde{\Psi}} \|\bar{\mathbf{E}}_W P_{\tilde{\Psi}}^\perp\|_F^2 \quad (10)$$

subject to

$$\mathbf{H}_k(\Psi_1, \dots, \Psi_{p-1}) = 0, \quad k = 1, \dots, q.$$

To find an approximate solution, the following minimization problem which approximates that given in (10) will be solved:

1. Find $\tilde{\tilde{\Psi}} = [\mathbf{I} | \tilde{\tilde{\Psi}}_1 | \dots | \tilde{\tilde{\Psi}}_{p-1}]$ by solving the unconstrained problem

$$\tilde{\tilde{\Psi}} = \arg \min_{\tilde{\tilde{\Psi}}} \|\bar{\mathbf{E}}_W P_{\tilde{\tilde{\Psi}}}^\perp\|_F^2.$$

2. Find $\hat{\Psi} = [\mathbf{I} | \hat{\Psi}_1 | \dots | \hat{\Psi}_{p-1}]$ "close" to $\tilde{\tilde{\Psi}}$ such that

$$\mathbf{H}_k(\hat{\Psi}_1, \dots, \hat{\Psi}_{p-1}) = 0, \quad k = 1, \dots, q.$$

Details on implementing this two-step procedure can be found in [12, 13]. The computational advantage of the algorithm is that the solution at each step is obtained directly, via two low order eigendecompositions (or SVDs) and QR decompositions. When the subarrays are configured so that p is small, each of these computations is considerably less costly than the initial eigendecomposition required to obtain $\bar{\mathbf{E}}_s$.

V. SIMULATION RESULTS

In this section, some Monte Carlo simulation results are presented to illustrate the performance of the techniques described in the preceding sections. The algorithms are compared with MUSIC, and in all cases are shown to have significantly superior performance in terms of resolution and RMS error of the angle estimates.

In each of the following cases, a 5x5 uniform rectangular array with half-wavelength interelement spacing was simulated. Overlapping subarrays of size 4x4 were used for the approximate subspace fitting technique, exactly as depicted in Figure 1. Estimates obtained from the approximate minimization were used to initialize a fully calibrated WSF search; i.e., the WSF minimization exploited the full structure of the array. In each case, two emitters were present at the (azimuth, elevation) coordinates $(-10^\circ, 10^\circ)$ and $(-5^\circ, 7^\circ)$. The weighting used for both subspace fitting algorithms was $\mathbf{W} = \mathbf{W}_{\text{opt}}$. The emitter signals were generated as constant amplitude planewaves with random phase, uniformly distributed on $[0, 2\pi]$, and the sensor array elements were assumed to have unity gain in the direction of the impinging signals. In all cases it was assumed that the detection problem was correctly solved; i.e., a signal subspace of dimension 2 was always used.

Case 1: Uncorrelated Sources

For this case the sources were completely uncorrelated, and simulations were performed for the following values of N , the number of snapshots: $N = 25, 50, 100$, and 250. A total of 250 Monte Carlo trials was conducted for each value of N . The RMS error of the estimates is plotted in Figures 2 and 3 versus N . The percentages located next to the curves for MUSIC indicate the algorithm's failure rate. A failure was said to occur if MUSIC was unable to locate two peaks in the square region bounded by $[-20^\circ, 5^\circ]$ in the azimuth dimension and $[-5^\circ, 20^\circ]$ in the elevation dimension. The fully calibrated WSF parameterization provides estimates with RMS error roughly 30-40% lower than that obtained by the suboptimal algorithm. There is a significant difference in performance between these methods and MUSIC.

Case 2: Correlated Sources

This case is identical to the previous one, except that the two sources were correlated. Simulations were conducted for three values of the correlation coefficient ρ : $\rho = 0.3, 0.6$, and 0.9, and N was fixed at 100 for all cases. The RMS error of the estimates is plotted in Figures 4 and 5. The result for $\rho = 0$ was obtained from Case 1. Again, the subspace fitting methods are far more effective than MUSIC in estimating the emitter DOAs. The advantage of the WSF algorithm is especially evident for highly correlated signals.

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Figure 2: RMS Error of Azimuth Estimates for Case 1

Figure 4: RMS Error of Azimuth Estimates for Case 2

Figure 3: RMS Error of Elevation Estimates for Case 1

Figure 5: RMS Error of Elevation Estimates for Case 2

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