Subspace Alignment Chains and the Degrees of Freedom of the Three-User MIMO Interference Channel

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Abstract

We show that the 3-user $M_T \times M_R$ MIMO interference channel where each transmitter is equipped with M_T antennas and each receiver is equipped with M_R antennas has $d(M,N) \triangleq \min\left(\frac{M}{2-1/\kappa}, \frac{N}{2+1/\kappa}\right)$ degrees of freedom (DoF) normalized by time, frequency, *and space* dimensions, where $M \triangleq \min(M_T, M_R), N \triangleq \max(M_T, M_R), \kappa \triangleq \lceil\frac{M}{N-M}\rceil$. While the DoF outer bound of $d(M,N)$ is established for every M_T, M_R value, the achievability of $d(M,N)$ DoF is established *in general* subject to a normalization with respect to spatial-extensions, i.e., the scaling of the number of antennas at all nodes. Specifically, we show that $qd(M,N)$ DoF are achievable for the 3-user $qM_T \times qM_R$ MIMO interference channel, for some positive integer q which may be seen as a spatial-extension factor. q is the scaling factor needed to make the value $qd(M,N)$ an integer. Given spatial-extensions, the achievability relies only on linear beamforming based interference alignment schemes and requires neither channel extensions nor channel variations in time or frequency. In the absence of spatial extensions, it is shown through examples how essentially the same interference alignment scheme may be applied over time-extensions over either constant or time-varying channels. The central new insight to emerge from this work is the notion of subspace alignment chains as DoF bottlenecks. The subspace alignment chains are instrumental both in identifying the extra dimensions to be provided by a genie to a receiver for the DoF outer bound, as well as in the construction of the optimal interference alignment schemes.

The DoF value $d(M,N)$ is a piecewise linear function of M, N, with either M or N being the bottleneck within each linear segment while the other value contains some redundancy, i.e., it can be reduced without reducing the DoF. The corner points of these piecewise linear segments correspond to two sets, $\mathcal{A} \triangleq \{1/2, 2/3, 3/4, \cdots\}$ and $\mathcal{B} \triangleq \{1/3, 3/5, 5/7, \cdots\}$. The set A contains all those values of M/N and only those values of M/N for which there is redundancy in *both* M and N, i.e., either can be reduced without reducing DoF. The set $\mathcal{B}$ contains all those values of M/N and only those values of M/N for which there is *no redundancy* in either M or N, i.e., neither can be reduced without reducing DoF. Because $\mathcal{A}$ and $\mathcal{B}$ represent settings with maximum and minimum redundancy, essentially they are the basis for the DoF outer bounds and inner bounds, respectively.

Our results settle the question of feasibility of linear interference alignment, introduced previously by Cenk et al., for the 3-user $M_T \times M_R$ MIMO interference channel, completely for all values of M_T, M_R. Specifically, we show that the linear interference alignment problem $(M_T \times M_R, d)^3$ (as defined in previous work by Cenk et al.) is feasible if and only if $d \leq \lfloor d(M,N) \rfloor$. With the exception of the values $M/N \in \mathcal{B}$, and only with that exception, we show that for every M/N value there are proper systems (as defined by Cenk et al.) that are not feasible. Evidently the redundancy contained in all other values of M/N manifests itself as superfluous variables that are not discounted in the definition of proper systems, thus creating a discrepancy between proper and feasible systems.

Our results show that $M/N \in \mathcal{A}$ are the only values for which there is no DoF benefit of joint processing among co-located antennas at the transmitters or receivers. This may also be seen as a consequence of the maximum redundancy in the $M/N \in \mathcal{A}$ settings.

1 Introduction

The number of degrees of freedom (DoF) of a communication network is a metric of great significance as it provides a lens into the most essential aspects of the communication problem. DoF investigations have motivated many fundamental ideas such as interference alignment [4, 7], deterministic channel models [16], rational dimensions [5, 6], information dimensions [20], aligned interference neutralization [1] and manageable interference [2]. The DoF metric is especially valuable as a first order capacity approximation. Since even the smallest gap between the best available DoF inner and outer bounds translates into an unbounded gap in the best available capacity approximations, communication networks where DoF values are not known are some of the least understood problems, and hence, also the most promising research avenues for significant and fundamental discoveries.

DoF characterizations have recently been obtained for a wide variety of wireless networks. Since in this work we are interested primarily in interference channels, it is notable that the DoF of interference channels are known when all nodes are equipped with the same number of antennas, for almost all channel realizations, and regardless of whether the channels are time-varying, frequency-selective or constant. However, if each node may have an arbitrary number of antennas, then a general DoF characterization is not available beyond the 2-user interference channel. One obvious reason is the explosion of the number of parameters in considering arbitrary antenna configurations. However, as we show in this work, the problem involves fundamental challenges even when the number of parameters is restricted by symmetry. In particular, the difficulty of this problem has to do with the new notion of "depth" of overlap between vector subspaces that comes into play on the one hand, and of translating this notion into information theoretic bounds on the other. Specifically, in this work we explore what is perhaps the simplest setting for MIMO interference channels where the DoF remain unknown, and obtain its DoF characterization, while highlighting the challenging nature of the general problem in the process. The assumptions that define our primary focus in this work are:

1. **3-user symmetric MIMO interference channel:** *The number of users is set to $K = 3$, the number of antennas at every transmitter is set to the same value, M_T, and the number of antennas at every receiver is set to the same value, M_R. Thus, the problem space is parameterized by only two variables M_T, M_R.*

2. **Spatially-normalized DoF for almost-all channel realizations:** *The DoF characterizations that we seek are intended in the "almost-surely" sense, with channel coefficients drawn from continuous distributions. Further, the DoF are normalized not only with respect to time and frequency dimensions, but also with respect to the spatial dimension, i.e., we allow channel extensions not only in time and frequency dimensions, but also in the spatial dimension (i.e., through a scaling of antennas).*

As the smallest, and therefore the most elementary interference channel setting where interference alignment becomes relevant, the 3-user interference channel has special significance. The assumption of global channel knowledge, as well as the implicit assumption of comparable signal strengths from all transmitters at all receivers that follows from the definition of the DoF metric (as opposed to Generalized DoF), is most relevant to small clusters of, e.g., no more than 3, mutually interfering users.

The rationale for the remaining assumptions is our interest primarily in generic insights rather than the peculiarities and caveats associated with special structures. While the restriction to almost-all channel realizations is by now a standard assumption for DoF studies, the normalization with respect to spatial dimension is less common. Spatial extension, i.e., proportional scaling of the number of antennas at each node and a corresponding normalization of DoF by the same factor, is appealing in that it allows us to deal with generic channels, thereby revealing generic insights into the geometry of alignments and relative signal space dimensions without having to deal with the added complexity of diagonal or block diagonal channel structures that would result from channel extensions over constant or time-varying channels, or the rational dimension arguments that are often invoked in the absence of sufficient channel diversity. As a further justification for the normalization with respect to spatial dimension, we note that so far, *for every wireless network, with or without multiple antennas, where the DoF characterizations are available for almost-all channel realizations, the DoF characterizations are unaffected by spatial normalization.* Indeed, we conjecture that this should be the case in general, i.e., much like time and frequency dimensions, the DoF of a network (for almost all channel realizations and with global channel knowledge as is assumed here) should also scale with the proportional scaling of the spatial dimension. Notably, the general question of the scaling of DoF with spatial dimension appears as an open problem in [21]. Also notable is the use of spatial normalization to characterize the DoF region for the general MIMO 2-user X channel setting in one of the earliest works on interference alignment [3]. Finally, as explained towards the end of this paper through several examples, the interference alignment solutions developed in this work under spatial extensions may be directly applied to time and/or frequency extensions as well to obtain the same DoF characterizations but without relying on spatial extensions.

2 Background

We are interested in both the information theoretic DoF of MIMO interference channels, as well as the feasibility of linear interference alignment schemes. We start with a summary of relevant work on these topics.

2.1 DoF of MIMO Interference Channels

The two user ($K = 2$) MIMO interference channel, where User 1 has M_1 transmit and N_1 receive antennas, and User 2 has M_2 transmit and N_2 receive antennas, is shown by Jafar and Fakhereddin in [11] to have $\min(M_1 + M_2, N_1 + N_2, \max(M_1, N_2), \max(M_2, N_1))$ DoF. For this result, the achievability is based on linear zero forcing beamforming schemes, and the converse is based on DoF outer bounds for the multiple access channel obtained by providing enough antennas to a receiver so that after decoding and subtracting its own signal, it is able to also decode the interfering signal.

In [7], Cadambe and Jafar introduced an asymptotic interference alignment scheme, referred to as the [CJ08] scheme (see [21] for an intuitive description of the [CJ08] scheme), leading to the result that in the K-user MIMO interference channel, each user can access half-the-cake in terms of DoF[1], for a total DoF value of $KM/2$ DoF, almost surely, when all nodes are equipped with the same number of antennas $M_T = M_R = M$ and when channels are time-varying or frequency-selective. In [23], Motahari et al. introduced the rational dimensions framework based on diophantine approximation theory wherein the [CJ08] scheme is again applied to establish the same DoF result, but with constant channels and without time/frequency extensions. For the 3-user MIMO interference channel setting with $M_T = M_R = M > 1$, Cadambe and Jafar also present a closed form linear

[1] The "cake" refers to the maximum DoF accessible by a user when all interfering users are absent.

beamforming solution in [7] that requires no time-extensions for even M and 2 time-extensions for odd M, to achieve the DoF outer bound value of $M/2$ per user, without the need for channel variations in time/frequency. The DoF outer bound in each of these cases is based on the pairwise outer bounds for any two users, as established previously for the single antenna setting, $M = 1$, by Host-Madsen and Nosratinia in [22] and for the multiple antenna setting, $M > 1$, by Jafar and Fakhereddin in [11].

In [9], Gou and Jafar studied the DoF of the K-user $M_T \times M_R$ MIMO interference channel under the assumption that $\eta = \max(M_T, M_R)/\min(M_T, M_R)$ is an integer and showed that each user has a fraction $\frac{\eta}{\eta+1}$ of the cake[2], for a total DoF value of $K\min(M_T, M_R)\frac{\eta}{\eta+1}$, almost surely, when the number of users $K > \eta$.

Example: The 3-user 1×2 *MIMO interference channel, i.e., the interference channel with 3 users where each transmitter has* 1 *antenna and each receiver has* 2 *antennas, has* $2/3$ *DoF per user, as does the 3-user* 2×1 *MIMO interference channel. The results holds for more than* 3 *users as well, i.e., the* K*-user* 1×2 *and* 2×1 *interference channels have* $2/3$ *DoF per user, for all* $K \geq 3$.

The results of [9], established originally over time-varying/frequency-selective channels, are extended to constant channels without the need for channel extensions, by Ghasemi et al. in [10], by employing the rational dimensions framework. Further, Motahari et al. show that each user in the K-user $M_T \times M_R$ MIMO interference channel, has a fraction $\frac{\eta}{\eta+1}$ of the cake, even when $\eta = \frac{\max(M_T,M_R)}{\min(M_T,M_R)}$ is not an integer, provided the number of users $K \geq \frac{M_T+M_R}{\gcd(M_T,M_R)}$. Interestingly, the achievability of $\frac{\eta}{\eta+1}\min(M_T, M_R)$ DoF per user, or equivalently $\frac{M_T M_R}{M_T+M_R}$ DoF per user, follows from the application of the [CJ08] scheme for every M_T, M_R value, and for any number of users K, and *requires no joint signal processing between the multiple receive (transmit) antennas at any receiver (transmitter).* However, the optimality of these achievable DoF has been shown only when either η is an integer or when $K \geq \frac{M_T+M_R}{\gcd(M_T,M_R)}$ for any η. The outer bounds in each of these cases are based on allowing full cooperation among groups of transmitters/receivers and applying the DoF outer bound for the resulting 2-user MIMO interference channel previously derived by Jafar and Fakhereddin in [11].

Example: Consider the 5-user 2×3 *MIMO interference channel. Allowing full cooperation between users 1, 2, 3 and allowing full cooperation between users 4,5, we have a resulting two user MIMO interference channel where the effective User 1 has 6 transmit and* 9 *receive antennas, and the effective User 2 has* 4 *transmit and* 6 *receive antennas. According to the DoF result for 2-user MIMO interference channel shown by Jafar and Fakhereddin in [11], this channel has 6 DoF. Since cooperation cannot reduce the total DoF, it follows that the original 5 user* 2×3 *interference channel has no more than* $6/5$ *DoF per user. Since* $M_T M_R/(M_T + M_R)$ *DoF are always achievable per user,* $6/5$ *is the optimal value of DoF for this channel. Further, since DoF per user cannot increase with the number of users,* $6/5$ *is the optimal value of DoF per user in the* K *user* 2×3 *MIMO interference channel for all* $K \geq 5$. *The same conclusion applies in the reciprocal* 3×2 *MIMO setting as well.*

Note that the DoF value of the 2×3 or the 3×2 MIMO interference channel is not known if the number of users, K, is 3 or 4. As a special case of our results in this work, we will show that $6/5$ is the optimal DoF value per user in the 2×3 or 3×2 MIMO interference channel for all $K > 2$, thereby resolving the DoF value for all K in the 2×3 and 3×2 settings. Since outer bounds based on full cooperation are not enough, the challenge in this case will be to identify the genie signals that will lead us to this conclusion.

[2] Here, the "cake" corresponds to $\min(M_T, M_R)$ DoF.

2.2 Feasibility of Linear Interference Alignment without Time/Frequency/Space Extensions

While the DoF of MIMO interference channels are of fundamental interest, the achievable schemes are often built upon theoretical constructs such as the rational dimensions framework or Renyi information dimension, whose physical significance and robustness is not yet clear. On the other hand, linear beamforming schemes are well understood based on the abundance of MIMO literature. As a consequence, there is much interest in the DoF achievable through linear beamforming schemes, i.e., through linear interference alignment schemes. A central question in this research avenue is the feasibility (almost surely) of linear interference alignment based on only spatial beamforming, i.e., without the need for channel extensions or variations in time/frequency/space. The feasibility problem is introduced by Gomadam et al. in [24], where iterative algorithms were proposed to test the feasibility of desired alignments. Recognizing the feasibility problem as equivalent to the solvability of a system of polynomial equations, Cenk et al. in [12] draw upon classical results in algebraic geometry about the solvability of *generic* polynomial equations, to classify an alignment problem as *proper* if and only if the number of independent variables in every set of equations is at least as large as the number of equations in that set. While the polynomial equations involved in the feasibility of interference alignment are not strictly generic, Cenk et al. appeal to the intuition that proper systems are likely to be feasible and improper systems to be infeasible. For a K user $M_T \times M_R$ MIMO interference channel where each user desires d DoF, denoted as the $(M_T \times M_R, d)^K$ setting, Cenk et al. identified the system as proper if and only if

$$d \leq \frac{M_T + M_R}{K+1} \tag{1}$$

The conjectured correspondence between proper/improper systems and feasible/infeasible systems is settled completely in one direction, and partially in the other direction, in recent works by Bresler et al. in [13] and Razaviyayn et al. in [14], who show that:

1. Improper systems are infeasible [13, 14].
2. If M_T, M_R are divisible by d then proper systems are feasible [14].
3. For square channels, $M_T = M_R$, proper systems are feasible [13].

While the properness of a system seems to work fairly well as an indicator of the feasibility of linear interference alignment in most cases studied so far, it is also remarkable that based on existing results it is possible to find examples of proper systems that are not feasible. For example, consider the 3-user interference channel where $(M, N) = (4, 8)$ and each user desires $d = 3$ DoF. According to (1) this is a *proper* system, and according to [9] it is *infeasible* because the information theoretic DoF outer bound value for this channel is only 8/3 per user. The DoF outer bound is easily found by allowing two of the users to cooperate fully, so that the resulting 2-user MIMO interference channel with $(M_1, N_1, M_2, N_2) = (8, 16, 4, 8)$ has a total DoF value of 8 according to [11]. Since cooperation does not hurt, and linear schemes (or any other scheme for that matter) cannot beat an information-theoretic outer bound, it is clear that the $(4 \times 8, d)^3$ linear interference alignment problem is infeasible for $d \geq 8/3$, and in particular for $d = 3$.

The observation that some proper systems are not feasible is also made by Cenk et al. in [12] who suggest including known information theoretic DoF outer bounds to further expand the set of infeasible systems. Interestingly, so far, *all known DoF outer bounds for K user MIMO interference* *channels come directly from the DoF result for the 2-user MIMO interference channel* [11], applied after allowing various subsets of users to cooperate, while eliminating other users to create a 2-user interference channel (as also illustrated by the preceding example). As we show in this work, these DoF outer bounds do not suffice, even for the symmetric 3-user MIMO interference channel for all M_T, M_R values. Thus, the feasibility of proper systems remains an open problem in general, even if restricted to the 3 user setting.

In this work, for the 3-user $M_T \times M_R$ MIMO interference channel, we settle the issue of feasibility of linear interference alignment. Somewhat surprisingly within this setting, especially considering systems near the threshold of proper/improper distinction, we show that *most proper systems are infeasible.*

3 System Model and Metrics

We consider a fully connected three-user MIMO interference channel where there are M_T and M_R antennas at each transmitter and each receiver, respectively. As shown in Fig. 1, each transmitter sends one independent message to its own desired receiver. Denote by $\mathbf{H}_{ji}$ the $M_R \times M_T$ channel matrix from transmitter i to receiver j where $i, j \in \{1, 2, 3\}$. We assume that the channel coefficients are independently drawn from continuous distributions. While our results are valid regardless of whether the channel coefficients are constant or varying in time/frequency, we assume that the channel coefficients are constant in this paper. Global channel knowledge is assumed to be available at all the nodes.

At time index $t \in \mathbb{Z}^+$, Transmitter i sends a complex-valued $M_T \times 1$ signal vector $\bar{X}_i(t)$, which satisfies an average power constraint $\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\|\bar{X}_i(t)\|^2] \leq \rho$ for T channel uses. At the receiver side, User j receives an $M_R \times 1$ signal vector $\bar{Y}_j(t)$ at time index t, which is given by:

$$\bar{Y}_j(t) = \sum_{i=1}^{3} \mathbf{H}_{ji}\bar{X}_i(t) + \bar{Z}_j(t)$$

where $\bar{Z}_j(t)$ an $M_R \times 1$ column vector representing the i.i.d. circularly symmetric complex additive white Gaussian noise (AWGN) at Receiver j, each entry of which is an i.i.d. Gaussian random variable with zero-mean and unit-variance.

Let $R_k(\rho)$ denote the achievable rate of User k where ρ is also referred to as the Signal-to-Noise Ratio (SNR). The capacity region $\mathcal{C}(\rho)$ of this network is the set of achievable rate tuples $R(\rho) = (R_1(\rho), R_2(\rho), R_3(\rho))$, such that each user can simultaneously decode its desired message with arbitrarily small error probability. The maximum sum rate of this channel is defined as $R_\Sigma(\rho) = \max_{R(\rho)\in\mathcal{C}(\rho)}\sum_{k=1}^{3} R_k(\rho)$, and $R(\rho) = R_\Sigma(\rho)/3$ denotes the maximum rate normalized per user. The sum DoF are defined as $d_\Sigma(M_T, M_R) = \lim_{\rho\to\infty} R_\Sigma(\rho)/\log(\rho)$, and $\text{DoF}(M_T, M_R) = d_\Sigma(M_T, M_R)/3$ stands for the normalized DoF per user. Throughout this paper, we also write $\text{DoF}(M_T, M_R)$ as $d(M_T, M_R)$ for simplicity. Moreover, we use $d_k(M_T, M_R)$ to denote the number of DoF associated with User k, and the user index k is interpreted modulo 3 so that, e.g., User 0 is the same as User 3. Furthermore, we define the DoF normalized by the spatial dimension, $\overline{\text{DoF}}_\Sigma(M_T, M_R)$ as

$$\overline{\text{DoF}}_\Sigma(M_T, M_R) \quad = \quad \max_{q\in\mathbb{Z}^+} \frac{d_\Sigma(qM_T, qM_R)}{q}$$

and $\overline{\text{DoF}}(M_T, M_R) = \overline{\text{DoF}}_\Sigma(M_T, M_R)/3$ is similarly defined. The dependence on M_T, M_R may be dropped for compact notation when no ambiguity would be caused, i.e., instead of $\overline{\text{DoF}}(M_T, M_R)$ we may write just $\overline{\text{DoF}}$.

Notation: For the matrix $\mathbf{A}$, $\mathbf{A}(i,:)$ and $\mathbf{A}(:,j)$ denote its i^{th} row and j^{th} column vector, respectively; $\mathbf{A}(i, m:n)$ denotes a $1\times(n-m+1)$ row vector obtained from the i^{th} row and the m^{th} to n^{th} columns of the matrix $\mathbf{A}$; $\mathbf{A}(m:n,:)$ denotes a matrix obtained from the m^{th} to n^{th} rows of the matrix $\mathbf{A}$; $\mathbf{A}^T$, $\mathbf{A}^H$ and $\det(\mathbf{A})$ denote transpose, conjugate transpose and the determinant and of the matrix $\mathbf{A}$, respectively. We use the notation $o(x)$ to represent any function $f(x)$ such that $\lim_{x\to\infty} f(x)/x = 0$. Further, we define $M = \min(M_T, M_R), N = \max(M_T, M_R)$.

4 Main Results

In order to present the DoF results for the 3-user $M_T \times M_R$ MIMO interference channel in a compact form, let us define the quantity $\text{DoF}^\star$ as follows.

Definition 1

$$\text{DoF}^\star \quad \triangleq \quad \min\left(\frac{M}{2-1/\kappa_M}, \frac{N}{2+1/\kappa_N}\right) \tag{2}$$

where $M = \min(M_T, M_R), N = \max(M_T, M_R), \kappa_M = \lfloor \frac{N}{N-M} \rfloor$ *and* $\kappa_N = \lceil \frac{M}{N-M} \rceil$.

The quantities κ_N and κ_M denote the length of the *subspace alignment chain* in the network with $M_T < M_R$ and $M_T > M_R$, respectively. Subspace alignment chains are a central notion for this work and will be described in the next section. As a sanity check, note that as N and M become approximately equal, i.e., both κ_N and κ_M become large, $\text{DoF}^\star$ converges to the "half the cake" value, $M/2$.

Remark: Note that an equivalent expression, $\text{DoF}^\star = \min\left(\frac{M}{2-1/\kappa}, \frac{N}{2+1/\kappa}\right)$, with $\kappa = \lceil \frac{M}{N-M} \rceil$ is used in the abstract of this paper to avoid multiple (κ_M, κ_N) parameters. However, as we explain the result in terms of alignment chains, the distinct values of κ_M, κ_N will be more insightful.

To arrive at the main result of this paper, we proceed through two intermediate results, presented here as lemmas.

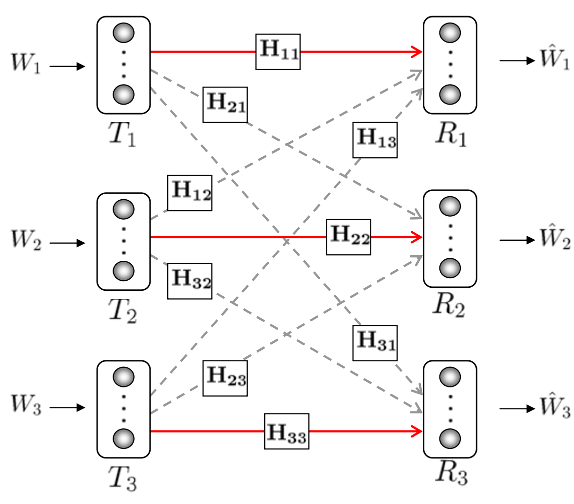


Figure 1: Three-User MIMO Interference Channel

Lemma 1 (Outer Bound) *For the 3-user $M_T \times M_R$ MIMO interference channel, the DoF value per user is bounded above as:*

$$\text{DoF} \leq \text{DoF}^\star \tag{3}$$

The proof of Lemma 1 is presented in Section 6 and Section 7.

Remark: Note that the outer bound holds for *arbitrary* values of M_T, M_R *without* any spatial normalization. However, also note that the outer bound does scale with spatial dimension, e.g., if we double the number of antennas at each node the value $\text{DoF}^\star$ would be doubled as well. In other words, the outer bound holds both with and without spatial normalization.

Remark: Unlike all previously used DoF outer bounds for MIMO interference channels, this outer bound does not follow from allowing cooperation among groups of users. Instead, what is needed is a careful choice of genie signals that provide a receiver access to parts of the signal space originating at interfering transmitters. Precisely which parts of a signal space can be provided as side information to produce the tight DoF outer bounds, is perhaps the most challenging technical aspect that we deal with in this paper. As such, the outer bounds represent the most significant contribution and a majority of this paper is devoted to their derivation.

Lemma 2 (Inner Bound) *For the 3-user $M_T \times M_R$ MIMO interference channel, the DoF per user value $\lfloor \text{DoF}^\star \rfloor$ is achievable with linear beamforming over constant channels without the need for symbol extensions in time, frequency or space.*

The proof of Lemma 2 is presented in Section 8.

Remark: Note that the achievability result stated in Lemma 2 is limited to integer values of DoF. Typically, the issue of achievability for non-integer values of DoF is resolved by using symbol extensions over time or frequency dimensions, often with the need for time-varying/frequency-selective channels to create sufficient diversity. Time/frequency extensions can be used here as well, as will be discussed in Section 8.3. However, since our primary interest is in generic channels rather than the structured (block-diagonal) channels that result from time/frequency extensions, we will focus on spatial extensions instead.

Lemma 1 and Lemma 2 lead us immediately to our main results, stated as theorems.

Theorem 1 (Spatially-Normalized DoF) *For the 3-user $M_T \times M_R$ MIMO interference channel, the spatially-normalized degrees of freedom value per user is given by:*

$$\overline{\text{DoF}} \;=\; \text{DoF}^\star \tag{4}$$

Proof: The proof of Theorem 1 follows directly from Lemma 1 and Lemma 2. Clearly, whenever $\text{DoF}^\star$ is an integer, we have an exact DoF characterization, $\text{DoF} = \text{DoF}^\star$, without the need for any spatial extensions. For the cases where $\text{DoF}^\star$ is not an integer, let us express it in its rational form p/q. Then, scaling the number of antennas by q, we have a 3-user $qM_T \times qM_R$ MIMO interference channel, for which the value $\text{DoF}^\star = p$ is both achievable and an outer bound, i.e., it is optimal. Since the normalized DoF outer bound is not affected by spatial scaling, no other spatial extension can improve the spatially-normalized DoF, and we have the result of Theorem 1. ∎

To understand the result of Theorem 1, the following equivalent, but more explicit representation of $\overline{\text{DoF}}$ will be useful:

$$\overline{\text{DoF}} = \begin{cases} M, & 0 \leq \frac{M}{N} \leq 1/3 \\ N/3, & 1/3 \leq \frac{M}{N} \leq 1/2 \\ 2M/3, & 1/2 \leq \frac{M}{N} \leq 3/5 \\ 2N/5, & 3/5 \leq \frac{M}{N} \leq 2/3 \\ 3M/5, & 2/3 \leq \frac{M}{N} \leq 5/7 \\ 3N/7, & 5/7 \leq \frac{M}{N} \leq 3/4 \\ 4M/7, & 3/4 \leq \frac{M}{N} \leq 7/9 \\ 4N/9, & 7/9 \leq \frac{M}{N} \leq 4/5 \\ \vdots & \vdots \leq \frac{M}{N} \leq \vdots \end{cases} \tag{5}$$

or in a compact form:

$$\overline{\text{DoF}} = \begin{cases} \frac{p}{2p-1}M, & \frac{p-1}{p} \leq \frac{M}{N} \leq \frac{2p-1}{2p+1} \\ \frac{p}{2p+1}N, & \frac{2p-1}{2p+1} \leq \frac{M}{N} \leq \frac{p}{p+1} \end{cases} \quad p \in \mathbb{Z}^+. \tag{6}$$

While Theorem 1 allows spatial extensions to avoid dealing with structured channels resulting from symbol extensions in time/frequency, our results are not limited to settings with spatial extensions. As we illustrate through numerous examples, the achievable schemes translate directly from spatial extensions to channel extensions over time/frequency dimensions (asymmetric complex signaling may be required in some cases [27, 15]) instead. A sample of our DoF results, *without spatial extensions* (and with only constant channels in every case except the SISO setting), appears in Fig. 2.

Lastly, we consider the most restricted setting where only linear alignment schemes are allowed, and no channel extensions are allowed in time/frequency/space. The next theorem settles the issue of feasibility of linear interference alignment for the 3-user $M_T \times M_R$ MIMO interference channel.

Theorem 2 (Feasibility of Linear Interference Alignment) *For the 3-user $M_T \times M_R$ MIMO interference channel, the DoF demand per user, d, is feasible with linear interference alignment if and only if $d \leq \text{DoF}^\star$.*

Remark: Note that the feasibility of linear interference alignment is intended here in the same sense as studied previously by Cenk et al. in [12], Bresler et al. in [13] and Razaviyayn et al. in [14], i.e., without symbol extensions in time/frequency/space.

Proof: The proof of Theorem 2 also follows directly from Lemma 1 and Lemma 2. Since the feasibility of linear interference alignment only concerns integer values of DoF per user, where the inner and outer bounds are tight, and the inner bound is achievable with linear interference alignment without the need for symbol extensions in time, frequency or space, the result of Theorem 2 follows immediately, and will be presented in Section 8.4 in detail. ■

5 Understanding the Result

In this section we provide an intuitive understanding of the main results based on linear dimension counting arguments, specifically by introducing the notion of subspace alignment chains, and highlight the key observations that follow from the main results.

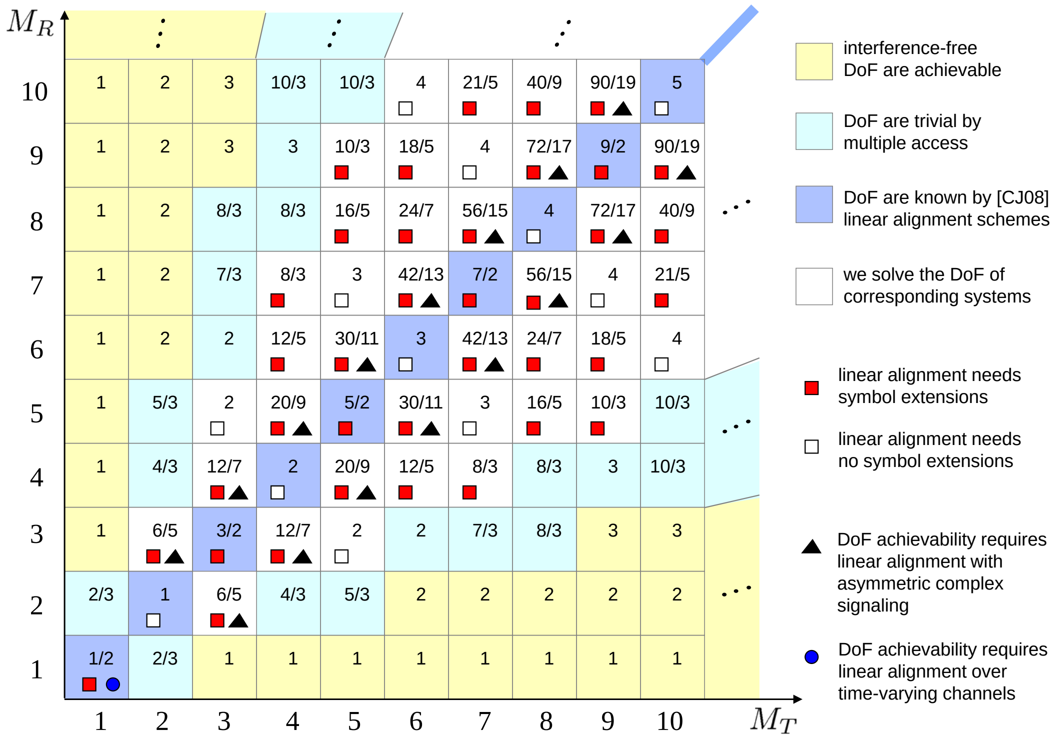


Figure 2: DoF per User (without spatial-extensions) of the Three-User $M_T \times M_R$ MIMO Interference Channel for $M_T, M_R \leq 10$. Except for the SISO case, only linear schemes and constant channels are involved. The SISO case requires either non-linear schemes or channel variations.

5.1 The Simple Cases: $M/N \leq 1/2$ and $M = N$

If $M = N$, then the DoF are already available, corresponding to the half-the-cake value reported in [7].

If $0 < M/N \leq 1/2$, i.e., $2M \leq N$, then interference alignment is not required. Consider first the setting $0 < M/N \leq 1/3$, i.e., $3M \leq N$. Here, if $3M_T \leq M_R$ then each receiver has enough antennas to zero force all interference at no cost to desired signals, and if $3M_R \leq M_T$ then each transmitter has enough antennas to zero force all unintended receivers. Therefore, in this case every user achieves all of the cake, i.e., his interference-free DoF, M. Next, consider the case $1/3 \leq M/N \leq 1/2$, i.e., $2M \leq N \leq 3M$. Allowing any two users to cooperate, and using the DoF result for the resulting 2 user MIMO interference channel, we find the total DoF outer bound value of N. This value is easily seen to be achievable with only zero forcing at the receivers ($M_R > M_T$) or at the transmitters ($M_T > M_R$), e.g., with users 1 and 2 achieving M DoF each and the third user achieving $N - 2M$ DoF. Note that the DoF assigned to each user can always be made equal by equal time-sharing between all permutations of the users for a given DoF allocation.

For the remaining cases, i.e., $1/2 < M/N < 1$, it turns out that we need interference alignment. The challenge lies not only in constructing interference alignment schemes for this setting, but also, and to a greater extent, in finding the required DoF outer bounds. A new notion that emerges in this setting and that plays a central role in limiting the DoF values, is the notion of subspace alignment chains, which is introduced next.

5.2 Subspace Alignment Chains

The main DoF outer bound, presented in Lemma 1, consists of two outer bounds.

$$\text{DoF} \leq \frac{N}{2 + 1/\kappa_N} \tag{7}$$

$$\text{DoF} \leq \frac{M}{2 - 1/\kappa_M} \tag{8}$$

We will refer to these bounds as the N-bound and the M-bound, respectively. The parameter κ_N and κ_M denote the length of the *subspace alignment chain* in the channel with $M_T < M_R$ and $M_T > M_R$ respectively, a notion to be introduced in this section. Note that the first outer bound value is limited by N and is an increasing function of κ_N while the second outer bound value is limited by M and is a decreasing function of κ_M. The significance of this will become clear in the following description.

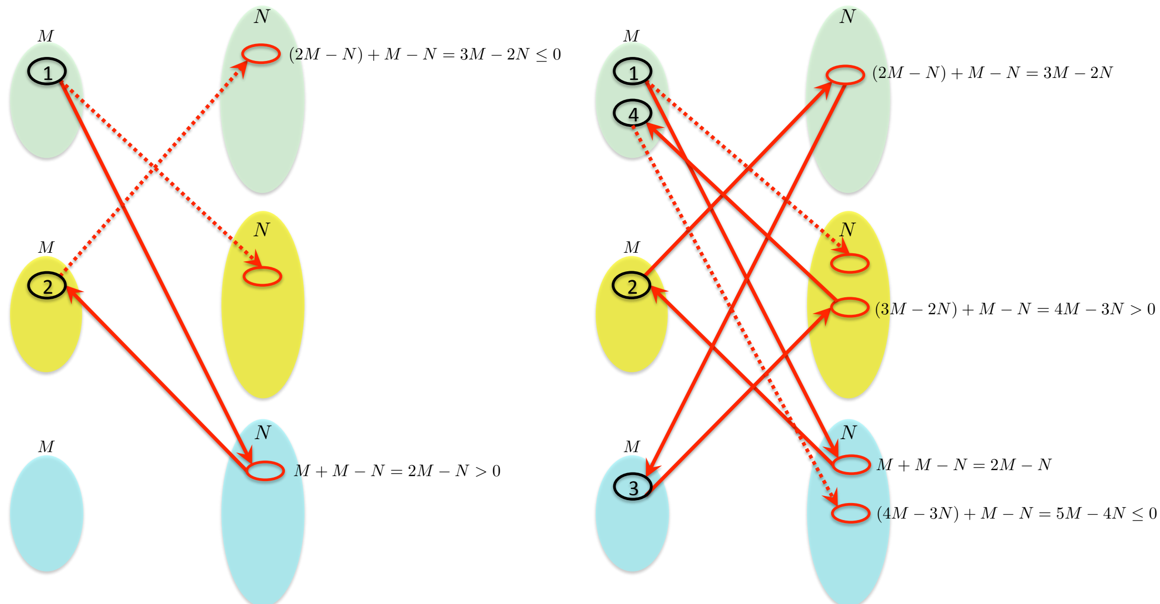
5.2.1 The N-bound: DoF $\leq \frac{\kappa_N}{2\kappa_N+1}N$

Consider the first outer bound, DoF $\leq \frac{\kappa_N}{2\kappa_N+1}N$. Since linear dimension counting arguments are identical for reciprocal networks [24], without loss of generality let us focus on the setting $M_T < M_R$, so that $M = M_T, N = M_R$. Now, since each receiver has at least as many antennas as any transmitter, zero-forcing of signals by the transmitters is not possible. Since interference cannot be eliminated, the next best thing is to align interference. Ideally, since each transmitter causes interference at two receivers, it should align with another interference vector at each of those undesired receivers. For example, let us consider a vector $\mathbf{V}_{1(1)}$ sent by Transmitter 1 that causes interference at Receiver 2. This vector should align with a vector $\mathbf{V}_{3(1)}$ sent by Transmitter 3, which is also undesired at Receiver 2. Now, the vector $\mathbf{V}_{3(1)}$ also causes interference at Receiver 1, so it should align there with a vector $\mathbf{V}_{2(1)}$ sent by Transmitter 2. The vector $\mathbf{V}_{2(1)}$ in turn also causes interference to Receiver 3, so it should align with a vector $\mathbf{V}_{1(2)}$ sent from Transmitter 1. Continuing like this, we create a chain of desired alignments:

$$\mathbf{V}_{1(1)} \xleftrightarrow{\text{Rx 2}} \mathbf{V}_{3(1)} \xleftrightarrow{\text{Rx 1}} \mathbf{V}_{2(1)} \xleftrightarrow{\text{Rx 3}} \mathbf{V}_{1(2)} \xleftrightarrow{\text{Rx 2}} \mathbf{V}_{3(2)} \xleftrightarrow{\text{Rx 1}} \mathbf{V}_{2(2)} \xleftrightarrow{\text{Rx 3}} \cdots$$

The main question is whether this ideal scenario is possible, i.e., can we extend this subspace alignment chain indefinitely? Consider, for example the setting $M = N$ that is previously solved by Cadambe and Jafar in [7]. Cadambe and Jafar create this infinite chain of alignments using the asymptotic alignment scheme for $M = N = 1$ and implicitly create an infinite alignment chain in the non-asymptotic solution for, e.g., $M = N = 2$, as the chain closes upon itself to form a loop, i.e., $\mathbf{V}_{1(1)} = \mathbf{V}_{1(2)}$. The chain closes upon itself mainly because in this setting (as well as all cases where $M = N > 1$) the optimal signal vectors are eigenvectors of the cumulative channel encountered in traversing the alignment chain starting from any transmitter and continuing until we return to the same transmitter. Thus, as shown by Cadambe and Jafar [7], the ideal solution of perfect alignment, achieved by an infinite (or closed-loop) alignment chain, is possible when $M = N$.

When $M \neq N$, it turns out that the subspace alignment chain can neither be continued indefinitely, nor be made to close upon itself. The length of the subspace alignment chain is therefore limited to a finite value κ_N that is a function of M and N. The limited length of the subspace alignment chain creates the bottleneck on the extent to which interference can be aligned, and is ultimately the main factor limiting the DoF value.



(a) $1/2 < M/N \le 2/3$, Chain length $\kappa_N = 2$ (b) $3/4 < M/N \le 4/5$, Chain length $\kappa_N = 4$

Figure 3: Subspace alignment chains leading to the N-bound, DoF $\le \frac{\kappa_N}{2\kappa_N+1}N$ per user. Dashed arrows indicate unaligned interference. Numbered signal spaces shown in black ovals indicate the position of the signal space in the alignment chain. The directions of the solid arrows indicate the progression of the alignment chain.

Consider the setting $1/2 < M/N \le 2/3$, as shown in Fig. 3(a), from a linear dimension counting perspective. Users 1, 2, and 3 are shown in green, yellow and blue in the figure. Let us start our subspace alignment chain with a signal space at Transmitter 1, shown in the figure as a black oval with the number 1. Consider the alignment that can occur at Receiver 3 with a corresponding signal space (marked with the number 2 to indicate the second signal in the alignment chain) originating at Transmitter 2. Note that interference can be aligned at each receiver only within that subspace which is accessible from both interfering transmitters. Since each transmitter can access only a M dimensional subspace of the N dimensional signal space available to Receiver 3, and because generic subspaces do not overlap any more than they have to, it follows that the size of the signal space accessible from both Transmitters 1 and 2, where interference alignment can take place, is no more than $M + M - N = 2M - N$ dimensional. Since $1/2 < M/N$, we note that $2M - N$ is a positive number, i.e., such a space exists. However, the accounting for aligned dimensions does not stop here. Let us continue the alignment chain to see if further alignment is possible. The $2M - N$ dimensional signal space 2 originating at Transmitter 2, can align with a corresponding signal space from Transmitter 3 at Receiver 1, in no more than $(2M - N) + M - N = 3M - 2N$ dimensions. However, since $M/N \le 2/3$, we note that $3M-2N$ is not a positive number, i.e., such a space does not exist. Thus, the alignment chain must stop here, and the maximum length of the alignment chain is $\kappa_N = 2$, representing the number of transmitted signal spaces that participate in the alignment chain. Now, to find corresponding DoF outer bound, let us account for the signal space dimensions. Fig. 3(a) may be helpful here. Assuming each transmitted signal space is d_0 dimensional, the total number of dimensions transmitted is $d_0 + d_0 = 2d_0$, and the number of dimensions occupied by interference at all three receivers is $d_0 + d_0 + d_0 = 3d_0$. Since the desired signal spaces must remain distinct from interference, the sum of desired signal dimensions at all receivers is $2d_0$. Thus, the total number of receive dimensions needed to accommodate both the desired signals and the interference is at least $5d_0$. Since the total number of receive dimensions available is $3N$ we must have $5d_0 \leq 3N$, i.e., $d_0 \leq 3N/5$. Since the total number of transmitted dimensions *per user* is $2d_0/3$, we have the corresponding outer bound value, DoF $\leq 2N/5$ per user, which corresponds to $\frac{\kappa_N}{2\kappa_N+1}N$, as expected.

As the next example, consider the setting $3/4 < M/N \leq 4/5$, shown in Fig. 3(b). Again, we start our alignment chain with the subspace numbered 1 at Transmitter 1. As seen from Receiver 3, this space can align with a corresponding signal space (numbered 2) originating at Transmitter 2, in no more than $2M - N > 0$ dimensions. Continuing the chain, the next alignment must occur at Receiver 1, where the $2M - N$ dimensional space (numbered 2) originating at Transmitter 2, can align with a corresponding signal space (numbered 3) originating at Transmitter 3, in no more than $(2M-N)+M-N = 3M-2N$ dimensions, which is also a positive number since $M/N > 3/4 > 2/3$, i.e., such a space exists. At this point the alignment chain has length 3. Continuing it further we note that the next alignment must occur at Receiver 2, where the $3M - 2N$ dimensional space (numbered 3) originating at Transmitter 3 can align with a corresponding signal space (numbered 4) originating at Transmitter 1, in no more than $(3M - 2N) + M - N = 4M - 3N$ dimensions, which is also a positive number since $M/N > 3/4$, i.e., such a space exists. Now the length of the alignment chain is 4. In order to continue the alignment chain further, we next consider Receiver 3, where the $4M - 3N$ dimensional space (numbered 4) originating at Transmitter 1 can align with a corresponding space originating at Transmitter 2 in no more than $(4M-3N)+M-N = 5M-4N$ dimensions, which is not a positive number since $M/N \leq 4/5$, i.e., such a space does not exist. Thus, the alignment chain ends here, and the maximum length of the alignment chain is $\kappa_N = 4$ for this example. Now, lets compute the implied DoF outer bound. Assuming each transmitted signal space is d_0 dimensional, the total number of dimensions transmitted is $4d_0$, and the number of dimensions occupied by interference at all three receivers is $5d_0$. Including the desired signal dimensions which must remain distinct from interference, the total number of receive dimensions needed is at least $4d_0 + 5d_0 = 9d_0$. Since the total number of receive dimensions available is $3N$ we must have $9d_0 \leq 3N$, i.e., $d_0 \leq N/3$. Since the total number of transmitted dimensions *per user* is $4d_0/3$, we have the corresponding outer bound, DoF $\leq 4N/9$ per user, which again corresponds to $\frac{\kappa_N}{2\kappa_N+1}N$, as expected.

These examples can be generalized in a straightforward manner, to verify that for the setting $\frac{p-1}{p} < \frac{M}{N} \leq \frac{p}{p+1}$, the length of the subspace alignment chain $\kappa_N = p$, which can also be expressed as $\kappa_N = \lceil \frac{M}{N-M} \rceil$. The corresponding outer bound value is calculated as follows. A total of $\kappa_N d_0$ dimensions are transmitted, creating interference spaces of total dimension $(\kappa_N + 1)d_0$. The desired signal and interference together need a total of $\kappa_N d_0 + (\kappa_N + 1)d_0 = (2\kappa_N + 1)d_0$ dimensions. Since only $3N$ dimensions are available, we must have $d_0 \leq \frac{3N}{2\kappa_N+1}$. Now, since the number of dimensions transmitted per user is $\kappa_N d_0/3$, we have the N-bound, DoF $\leq \frac{\kappa_N}{2\kappa_N+1}N$.

We make an interesting comparison between the subspace alignment chains introduced above and the work of Bresler and Tse in [8]. Bresler and Tse considered linear interference alignment problem for a three-user SISO Gaussian interference channel with time-varying/frequency-selective channel coefficients when the maximum diversity order that the channel can provide, e.g., the number of sub-carriers, is limited to L. They show that the maximum DoF achievable per user through linear interference alignment is a strictly increasing function of L that monotonically converges to the information theoretic outer bound value of $1/2$ as L approaches infinity. Finite length chains of aligned vectors appear in the derivation of Bresler and Tse's result as well. However, in spite of this superficial similarity, the alignment chains used by Bresler and Tse are fundamentally different from the subspace alignment chains in this work. First, the alignment chains of Bresler and Tse identify the limitations of DoF achievable through *linear* interference alignment schemes, but it is known that these limitations are surpassed by rational alignment schemes [6] that can achieve the information theoretic outer bound value of $1/2$ without the need for any channel variations. On the other hand, the subspace alignment chains presented in this work are much more fundamental in that they identify *information theoretic* outer bounds, i.e., these bounds cannot be beaten by linear alignment, rational alignment or any other scheme to be invented in the future. Second, the alignment chains in Bresler and Tse's work reach their maximum length when it becomes impossible to separate the desired signal from interference. Alignment of interference, per se, is not a challenge in their setting. On the other hand, the subspace alignment chains discussed above reach their maximum length when it becomes impossible to align interference any further. Keeping the desired signal separate from interference is not the main concern in our setting.

5.2.2 The M-bound: DoF $\leq \frac{\kappa_M}{2\kappa_M - 1}M$

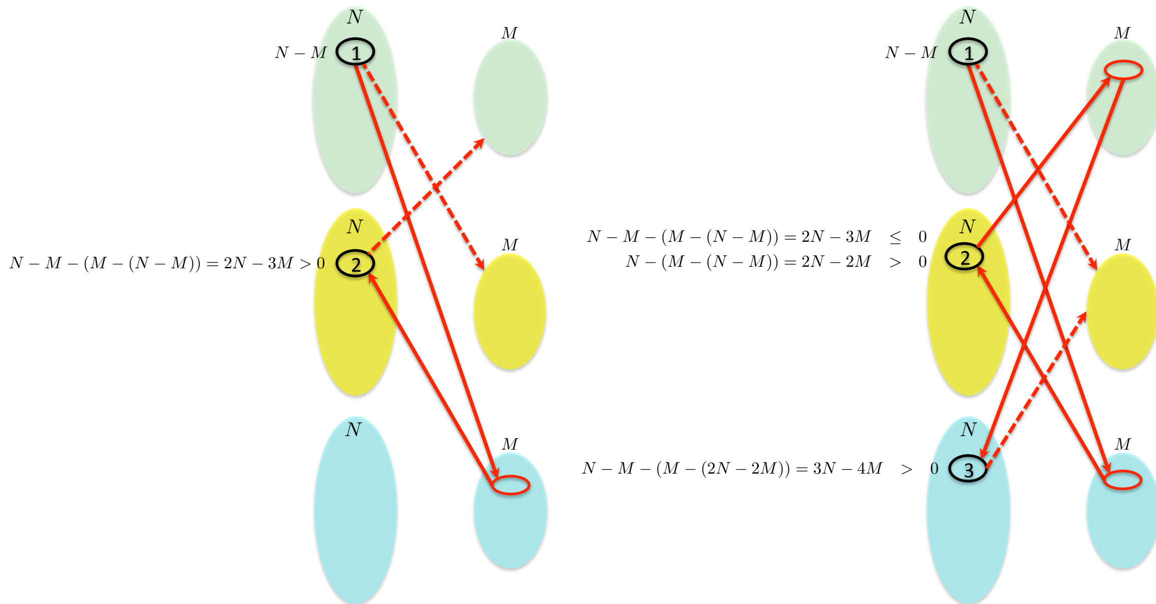
Here we explain the second outer bound, DoF $\leq \frac{\kappa_M}{2\kappa_M-1}M$, from linear dimension counting arguments. Since linear dimension counting arguments are identical for reciprocal networks [24], without loss of generality let us focus on the setting $M_T > M_R$, so that $M = M_R, N = M_T$. Now, since each transmitter has more antennas than any receiver, zero-forcing of signals by the transmitters is possible. Ideally, we would like to zero force all interference. However, since $M_T < 2M_R$, it is not possible for any transmitter to simultaneously zero-force its transmitted signal to both unintended receivers. The next best thing is to zero-force interference to the extent possible, and then align the remaining interference that cannot be zero-forced. This observation gives rise to a slightly different type of subspace alignment chains. The two ends of the alignment chain correspond to transmitted signals that are zero forced at one unintended receiver and cause interference at the other unintended receiver. These two non-zero-forceable interference terms are connected through a subspace alignment chain to alleviate the impact of the non-zero-forceable interference. Since zero-forcing is preferable to interference alignment, smaller interference alignment chains are preferable. Therefore, in this case, we will be limited by how quickly the semi-zero-forced signals can be connected through a subspace alignment chain. Therefore, this perspective gives rise to an outer bound $\frac{\kappa_M}{2\kappa_M-1}M$ which is a *decreasing* function of the length of the subspace alignment chain.

Consider the setting $1/2 \leq M/N < 2/3$, illustrated in Fig. 4(a). We start the alignment chain at Transmitter 1, where the number of transmitted dimensions that can be zero-forced at Receiver 2 is no more than $N - M$. Since $N - M \leq M$, these dimensions cannot be simultaneously zero-forced at Receiver 3. This creates the non-zero-forceable interference at Receiver 3 which initiates the subspace alignment chain. Recall that we would like the alignment chain to be as short as possible. This means that ideally, we would like the next signal in the chain (marked as number 2 in the figure), which must originate at Transmitter 2, to simultaneously accomplish the following objectives:

1. Signal space 2 should align with pre-existing interference at Receiver 3.

2. Signal space 2 should be zero-forced at Receiver 1.

Note that if it is possible to accomplish these objectives, signal 2 would not increase the interference space at any undesired receiver, which is why this is the preferred goal in choosing signal space 2.



(a) $1/2 \leq M/N < 2/3$, Chain length $\kappa_M = 2$

(b) $2/3 \leq M/N < 3/4$, Chain length $\kappa_M = 3$

Figure 4: Subspace alignment chains leading to the M-bound, DoF $\leq \frac{\kappa_M}{2\kappa_M-1}M$ per user. Dashed arrows indicate zero-forced interference. Numbered signal spaces shown in black ovals indicate the position of the signal space in the alignment chain. The directions of the solid arrows indicate the progression of the alignment chain.

Let us see if this is possible. At Receiver 3, interference already spans $N-M$ dimensions, leaving an interference free space of $M-(N-M)$ dimensions. For signal space 2 to align with pre-existing interference at Receiver 3, Transmitter 2 must zero-force these $M-(N-M)$ interference-free dimensions for Receiver 3. In addition, if signal space 2 is to be zero-forced at Receiver 1, then another M dimensions must be zero forced. With generic signal spaces, the total number of dimensions to be zero forced is $M+(M-(N-M))$. Since Transmitter 2 has only N antennas, it leaves $N-[M+(M-(N-M))] = 2N-3M$ dimensions within which both objectives can be accomplished. Since $M/N < 2/3$, we note that $2N-3M > 0$, i.e., such a space exists and it is possible to terminate the alignment chain. The resulting alignment chain length is $\kappa_M = 2$. Now, let us compute the implied DoF bound. With d_0 dimensions assigned to each transmitted signal space, the total number of transmitted dimensions is $2d_0$ and the total number of interference dimensions is d_0. Adding up the desired signal dimensions and the interference dimensions (because the two must not overlap) we need a total of $3d_0$ dimensions at the three receivers. The total number of dimensions at the three receivers is $3M$, which gives us the outer bound $3d_0 \leq 3M$, or $d_0 \leq M$. Since the number of transmitted dimensions *per user* is $2d_0/3$ we have the outer bound, DoF$\leq 2M/3$ per user, which coincides with $\frac{\kappa_M}{2\kappa_M-1}M$, as expected.

Next, let us consider the setting $2/3 \leq M/N < 3/4$, illustrated in Fig. 4(b). Once again, we start the alignment chain at Transmitter 1 and continue up to signal space 2. However, note that this time, because $2/3 \leq M/N$, there does not exist a signal space 2 which can be simultaneously aligned with interference at Receiver 3 and be zero-forced at Receiver 1. Thus, the alignment chain cannot be terminated at length 2. The next best thing is to only achieve interference alignment with signal space 2 and extend the subspace alignment chain. Recall that in order to align interference at

Receiver 3, Transmitter 2 must zero force the $M-(N-M)$ dimensional null-space of interference at Receiver 3, leaving exactly $N-[M-(N-M)] = 2N-2M$ dimensions at Transmitter 2 that satisfy the interference alignment requirement. Since $2N-2M \leq M$, note that the aligned dimensions cannot be zero forced at Receiver 1. Thus, the corresponding interference space at Receiver 1 is $2N-2M$ dimensional. Now, we extend the subspace alignment chain to signal space 3, which originates at Transmitter 3 (see Fig. 4(b)), and again we check if the alignment chain can be terminated. For the alignment chain to end with signal space 3, it must simultaneously satisfy the following two objectives:

1. Signal space 3 should align with pre-existing interference at Receiver 1.

2. Signal space 3 should be zero-forced at Receiver 2.

Let us see if this is possible. At Receiver 1, the space accessible by interference produced so far occupies no more than $2N-2M$ dimensions. Equivalently, an interference-free space of at least $M-(2N-2M)$ dimensions must be maintained at Receiver 1. For signal space 3 to align with pre-existing interference at Receiver 1, Transmitter 3 must avoid the interference-free dimensions available to Receiver 1, i.e., it needs to zero-force these $M-(2N-2M)$ interference-free dimensions. In addition, if signal space 3 is to be zero-forced at Receiver 2, then another M dimensions must be zero forced. With generic signal spaces, the total number of dimensions to be zero forced is $M+(M-(2N-2M))$. Since Transmitter 3 has only N antennas, it leaves $N-[M+(M-(2N-2M))] = 3N-4M$ dimensions within which both objectives can be accomplished. Since $M/N < 3/4$, $3N-4M > 0$, i.e., such a space exists and it is possible to terminate the alignment chain. The resulting alignment chain length is $\kappa_M = 3$. Now, let us compute the implied DoF bound. With d_0 dimensions assigned to each transmitted signal space, the total number of transmitted dimensions is $3d_0$ and the total number of interference dimensions is $2d_0$. Adding up the desired signal dimensions and the interference dimensions (because the two must not overlap) we need a total of $5d_0$ dimensions at the three receivers. The total number of dimensions at the three receivers is $3M$, which gives us the outer bound $5d_0 \leq 3M$, or $d_0 \leq 3M/5$. Since the number of transmitted dimensions *per user* is d_0 we have the outer bound, DoF$\leq 3M/5$ per user, which coincides with $\frac{\kappa_M}{2\kappa_M-1}M$, as expected.

Continuing along the same lines, these examples can also be generalized in a straightforward manner to verify that for the setting $\frac{p-1}{p} \leq \frac{M}{N} < \frac{p}{p+1}$, the length of the subspace alignment chain $\kappa_M = p$, which can also be expressed as $\kappa_M = \lfloor \frac{N}{N-M} \rfloor$. The corresponding outer bound value is calculated as follows. A total of $\kappa_M d_0$ dimensions are transmitted, creating interference spaces of total dimension $(\kappa_M - 1)d_0$. The desired signal and interference together need a total of $\kappa_M d_0 + (\kappa_M - 1)d_0 = (2\kappa_M - 1)d_0$ dimensions at the receivers. Since only $3M$ dimensions are available at the receivers, we must have $d_0 \leq \frac{3M}{2\kappa_M - 1}$. Now, since the number of dimensions transmitted per user is $\kappa_M d_0/3$, we have the outer bound, DoF $\leq \frac{\kappa_M}{2\kappa_M-1}M$.

We conclude this discussion with two plots of the DoF characterization presented in Theorem 1, shown in Fig. 5 and Fig. 6. Both figures are plotted as a function of the ratio $\gamma \stackrel{\triangle}{=} M/N$. Clearly, as M, N become less disparate, γ increases, and the length of the subspace alignment chain, both κ_N and κ_M increase as well, approaching infinity as $\gamma \to 1$. While the two figures are equivalent, the normalization with respect to N in Fig. 5 and the normalization with respect to M in Fig. 6 highlight the role of interference-alignment and zero-forcing, respectively. As γ increases from 0 to 1, subspace alignment chains become longer, a desirable outcome for interference alignment, which is reflected in Fig. 5. On the other hand, as γ increases from 0 to 1, alignment chains become longer, an undesirable outcome for zero forcing, which is reflected in Fig. 6. The interplay between the two bounds is evident in the piecewise analytical nature of the DoF function, with either M or N being the bottleneck within each analytical segment. A number of interesting observations can be made from the DoF result. These observations are the topic of the remainder of this section.

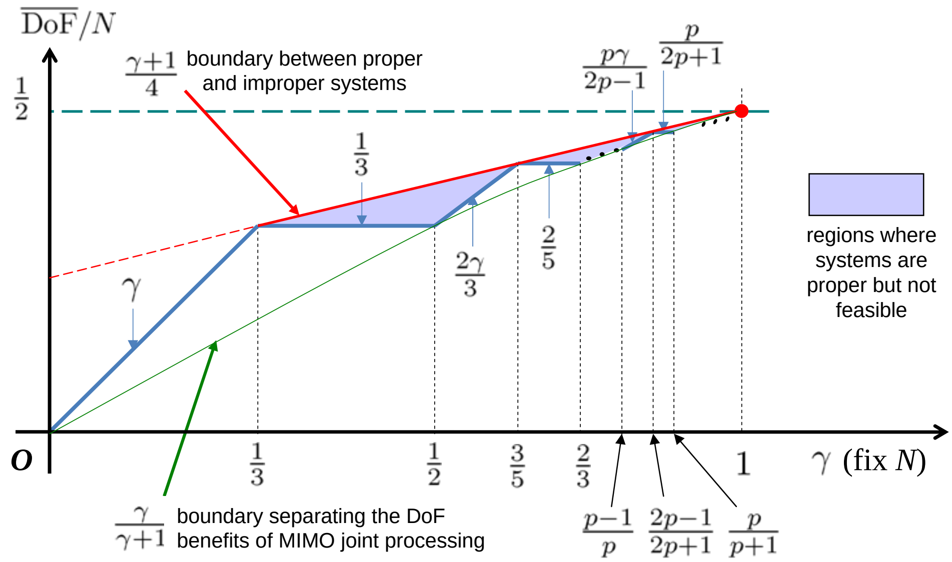


Figure 5: $\overline{\text{DoF}}/N$ as a function of $\gamma = \frac{M}{N}$

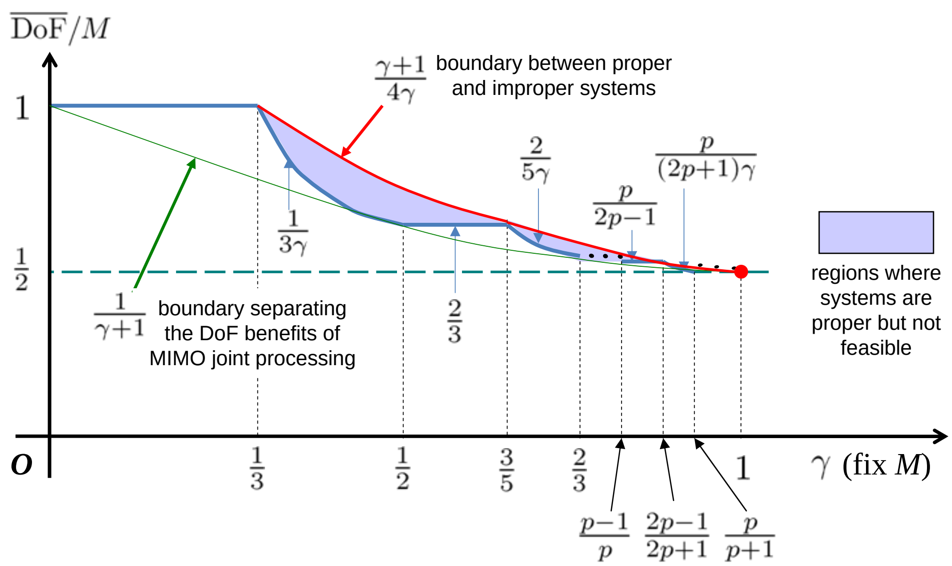


Figure 6: $\overline{\text{DoF}}/M$ as a function of $\gamma = \frac{M}{N}$

5.3 Other Key Observations

For these observations, we will refer to the $\overline{\text{DoF}}$ characterization in Theorem 1 and its depiction as the piecewise linear curve in Fig. 5.

5.3.1 Redundant Dimensions

A particularly interesting observation from (6) and Fig. 5 is that within each piecewise linear interval, the $\overline{\text{DoF}}$ value depends only on either M or N. This makes the other parameter somewhat redundant. In other words, within each piecewise linear section of the $\overline{\text{DoF}}$ curve, either the number of dimensions at the transmitter or receiver can be increased/decreased without changing the $\overline{\text{DoF}}$ value. For example, consider the 3-user 7×10 MIMO interference channel. This channel lies in the range $2/3 \leq M/N \leq 5/7$, where the $\overline{\text{DoF}}$ value $3M/5$ depends only on M. Therefore, there is some redundancy in N. Since the redundant dimensions are fractional, they are much more explicitly seen in a larger space. So let us consider the 3-user 35×50 MIMO interference channel, which is simply a spatially scaled version of the original 7×10 setting. Now, we know that the 35×50 setting has exactly 21 DoF (no spatial scaling required, since this is an integer value). However, note that the 35×49 setting also has only 21 DoF. Incidentally, the 31×49 setting achieves the 21 DoF with only linear beamforming based interference alignment, i.e., without the need for symbol extensions in time/frequency/space. Therefore, clearly, the 50^{th} receive antenna is redundant from a DoF perspective.

The previous example illustrates the situation for M/N values that fall strictly inside the piecewise linear intervals. Now let us consider M/N values that fall at the boundary points of the piecewise linear intervals. For example, consider the 3-user 10×15 MIMO interference channel, which corresponds to the M/N value $2/3$, i.e., a corner point, and has DoF value 6. However, note that the 3 user 9×15 MIMO interference channel also has 6 DoF. Thus, the 10^{th} transmit antenna is redundant from a DoF perspective. Alternatively, consider the 3-user 10×14 MIMO interference channel, which also has 6 DoF. Thus, evidently, the 15^{th} receive antenna is redundant in the 10×15 setting. Thus, one can lose either a transmit antenna or a receive antenna (but not both) without losing DoF in the 3-user 10×15 MIMO interference channel. This is because the 10×15 setting, which corresponds to $M/N = 2/3$, sits at the boundary of two piece-wise linear segments where the redundancies in M and the redundancies in N meet. Therefore it contains both redundancies. The same observation is true for corner points $M/N = 2/3, 3/4, 4/5, \cdots$.

Now consider the other set of corner values $M/N = 1/3, 3/5, \cdots$. As it turns out, these are the only values where neither the transmitter, nor the receiver has any redundant dimensions. We summarize the observations below:

1. For $M/N \in (0, 1/3), (1/2, 3/5), (2/3, 5/7), \cdots$, the value of M is the bottleneck, but the value of N includes redundant dimensions that can be sacrificed without losing DoF.

2. For $M/N \in (1/3, 1/2), (3/5, 2/3), (5/7, 3/4), \cdots$, the value of N is the bottleneck, but the value of M includes redundant dimensions that can be sacrificed without losing DoF.

3. For $M/N \in \mathcal{A} \stackrel{\triangle}{=} \{1/2, 2/3, 3/4, 4/5, \cdots\}$, and only for these values, both M and N include redundant dimensions, either of which can be sacrificed without losing DoF.

4. For $M/N \in \mathcal{B} \stackrel{\triangle}{=} \{1/3, 3/5, 5/7, 7/9, \cdots\}$, and only for these values, neither M nor N contains any redundant dimensions, i.e., reducing either will lead to loss of DoF.

Thus, the sets $\mathcal{A}$ and $\mathcal{B}$ represent maximally and minimally redundant settings. The redundancy, and the lack thereof, has interesting implications. For instance, the set $\mathcal{A}$ corresponds to maximal redundancy and is also the precise set of M/N values for which no joint processing is needed among the co-located antennas at any transmitter or receiver. On the other hand the set $\mathcal{B}$ corresponds to no-redundancy, and is also the precise set of M/N values for which all proper systems are feasible from a linear interference alignment perspective. Next we elaborate upon these observations.

5.3.2 The DoF Benefit of MIMO Processing

In the K-user MIMO interference channel setting with $M = N$, i.e., equal number of antennas at every node, Cadambe and Jafar have shown [7] that there is no DoF benefit of joint processing among multiple antennas, because the network has $KM/2$ DoF even if each user is split into M users, each with a single transmit and single receive antenna and with only independent messages originating at each transmitter. For the K-user $M_T \times M_R$ MIMO interference channel setting, Ghasemi et al. have shown in [10] that $\frac{M_T M_R}{M_T+M_R} = \frac{MN}{M+N}$ DoF are achievable even if transmitter is split into M_T single-antenna transmitters, each receiver is split into M_R single antenna receivers, and there are no common messages. Note that while in general this achievable DoF value is not optimal, Ghasemi et al. have shown that $\frac{M_T M_R}{M_T+M_R}$ is the optimal DoF value per user for the K-user $M_T \times M_R$ MIMO interference channel when the number of users $K \geq \frac{M_T+M_R}{\gcd(M_T,M_R)}$. Here we make related observations for our setting.

1. The 3-user $M_T \times M_R$ MIMO interference channel has $\text{DoF}(M_T, M_R) = \frac{M_T M_R}{M_T+M_R}$ whenever $M/N \in \mathcal{A}$, i.e., $\frac{M}{N} = \frac{p}{p+1}$. Note that this is a statement about regular DoF, i.e., without requiring spatial extensions. This is because the outer bound from Lemma 1 matches the achievability result of Ghasemi et al. in [10] for these settings.

2. The $K > 2$ user $M_T \times M_R$ MIMO interference channel has $\text{DoF}(M_T, M_R) = \frac{M_T M_R}{M_T+M_R}$ whenever $\frac{M}{N} = \frac{p}{p+1}$, for some $p \in \mathbb{Z}^+$. Note that this statement is for *any number of users* (greater than 2), and not just for 3 users. Thus, we have a DoF characterization for any number of users whenever $\frac{M}{N} = \frac{p}{p+1}$. The outer bound holds for any number of users because the DoF per user cannot increase with the number of users. The achievability is already established for any number of users. Since the two match, we know the DoF for any number of users. The result significantly strengthens the previous DoF characterization by Ghasemi et al. in [10]. As an example, consider the $(M, N) = (2, 3)$ setting. Ghasemi et al. show that the this setting has $6/5$ DoF per user if the number of users is 5 or higher. However, our result shows that this setting has $6/5$ DoF if the number of users is 3 or higher. The improvement is even more stark for larger values of M, N. For example, Ghasemi et al. show that the setting $(M, N) = (9, 10)$ has exactly $90/19$ DoF per user if the number of users is 19 or higher, whereas our result shows that this setting has exactly $90/19$ DoF if the number of users is 3 or higher.

3. There are no DoF benefits of MIMO processing whenever $\frac{M}{N} = \frac{p}{p+1}$, for $p \in \mathbb{Z}^+$. This can be seen as follows. For our result illustrated in Fig. 5, the piecewise linear $\overline{\text{DoF}}/N$ curve is bounded below by the smooth curve shown with a light solid line plotting the value $\frac{\gamma}{\gamma+1}$. Since $\gamma = M/N$, note that $N\frac{\gamma}{\gamma+1} = \frac{MN}{M+N}$ corresponds to $\frac{MN}{M+N}$ DoF, i.e., the achievable DoF without any joint processing across multiple antennas, or the DoF achievable with independent coding/decoding at each transmit/receive antenna. Since the $\overline{\text{DoF}}/N$ curve touches the $\frac{\gamma}{\gamma+1}$ curve whenever $\gamma = \frac{p}{p+1}$, there is no DoF benefit of MIMO processing in these settings.

4. Conversely, MIMO processing has DoF benefits whenever $\frac{M}{N} \neq \frac{p}{p+1}$, for some $p \in \mathbb{Z}^+$. This is because for all these (M, N) values, the $\overline{\text{DoF}}$ curve is strictly above the $\frac{MN}{M+N}$ value.

5. MIMO processing can enable linear achievable schemes when otherwise asymptotic alignment would be needed. We illustrate this with an example. Consider the 3-user 10×15 interference channel (where $M/N = 2/3$), which we now know, has 6 DoF per user. We also know from the work of Ghasemi et al. that this DoF value can be achieved without any MIMO processing, but it requires the [CJ08] asymptotic interference alignment scheme [25] either in the rational dimensions framework or over time-varying channels. On the other hand, as we will show in this work, the 6 DoF per user can be achieved purely through linear interference alignment based on beamforming without requiring any symbol extensions in space, time or frequency, by exploiting the MIMO benefit of joint processing among antennas located on the same node.

5.4 Infeasibility of Proper Systems without Symbol Extensions

Our DoF results settle the issue of feasibility/infeasibility of linear interference alignment for the 3-user $M_T \times M_R$ MIMO interference channel. As stated in Section 2 and Theorem 2, we emphasize here again that the feasibility/infeasibility of linear interference alignment we study in this paper is for the constant channels without symbol extensions in time/frequency/space. Prior work on this topic is summarized in Section 2. As stated earlier, it is known that improper systems are infeasible, and also under certain conditions proper systems are feasible. The feasibility of proper systems is believed to be much more widely true. Somewhat surprisingly, our results show that for the 3-user MIMO interference channel, the relationship between proper systems and feasibility of linear interference alignment is very weak, in the sense that *most* strictly proper systems are infeasible.

Specializing the characterization of proper systems by Cenk et al. [12] to the 3-user $M_T \times M_R$ MIMO interference channel, the system is proper if and only if the desired DoF per user

$$d \leq \frac{M_T + M_R}{4} \tag{9}$$

We will say that a system is strictly proper if we have equality in (9). In Fig. 5, the $\overline{\text{DoF}}/N$ curve is bounded above by a solid red line that plots the value $\frac{\gamma+1}{4}$. Note that this line only starts from $\gamma = 1/3$ because for $\gamma \in (0, 1/3)$ (denoted as the dashed red line) the DoF per user is bounded above by the single user bound. This solid red line differentiates proper systems from improper systems. All systems above the curve are improper while those below the curve are proper. However, the DoF outer bound lies strictly below the curve except when $\gamma = \frac{M}{N} = \frac{2p-1}{2p+1}$ for $p \in \mathbb{Z}^+$. Therefore, *many proper systems and most strictly proper systems are infeasible.* Remarkably, these systems are not only infeasible in terms of the restricted goal of achieving the desired DoF values through interference alignment with linear beamforming and without relying on channel extensions in time/frequency, but also they are infeasible in terms of the much more relaxed goal of achieving the desired DoF through any possible achievable scheme, linear or non-linear, using interference-alignment or otherwise, using time-varying or constant channels, and using scalar or vector coding. This is because the solid blue curve is the information-theoretic DoF outer bound.

Let us make this observation explicit with a few examples. Consider the $(M, N) = (8, 12)$ setting (where $\gamma = M/N = 2/3$), with desired DoF value $d = 5$ per user. This is a strictly proper system because $d = \frac{M+N}{4}$. But from Lemma 1 we know that the information theoretic DoF per user for this channel is bounded above by $24/5$. Therefore $d = 5$ is clearly infeasible. Similarly, for any value of $\gamma \notin \mathcal{B}$, i.e., that is not of the form $\frac{2p-1}{2p+1}$, one can create an infeasible proper system. For example, suppose $\gamma = 0.61$. Equivalently, $\frac{M}{N} = \frac{61}{100}$. Choosing $(M, N) = (244, 400)$, we arrive at the desired

DoF value $d = \frac{644}{4} = 161$ per user which would make the system strictly proper. However, we know from Lemma 1 that the information theoretic DoF (per user) outer bound for $(M, N) = (244, 400)$ is $800/5 = 160$. Thus, once again the strictly proper system is infeasible. As yet another example consider the choice $\gamma = 0.74$, i.e., $\frac{M}{N} = \frac{74}{100}$. Choose the setting $(M, N) = (148, 200)$ for which the information-theoretic DoF per user is bounded above by $\frac{600}{7} = 85.7....$ Therefore, if the desired DoF value per user is 87, the system is strictly proper and infeasible, if the desired DoF value per user is 86, the system is proper and infeasible. Proceeding similarly, we arrive at the following conclusions.

1. For every value of γ except $\gamma \in \mathcal{B}$, we can find proper systems that are infeasible, not only in terms of linear interference alignment, but also information theoretically infeasible.

2. For the values $\gamma \in \mathcal{B}$, proper systems are always feasible, i.e., the DoF demand $d = (M + N)/4$ per user, which is also the information theoretic outer bound, is actually achievable with only linear interference alignment, without the need for symbol extensions in time/frequency/space.

These observations, especially the infeasibility of proper systems for $M/N \neq 1/3, 3/5, 5/7, \cdots$, can be understood in terms of the redundant dimensions explained in the previous section. Recall that Cenk et al. [12] make the distinction between proper/improper systems based on the number of variables involved in the system of polynomial equations. From the observations on redundant dimensions, we know that except for $M/N = 1/3, 3/5, 5/7, \cdots$, every other setting contains redundant dimensions either in M or N. Evidently these redundant dimensions contribute superfluous variables which inflate the variable count, thereby qualifying a system as proper even when it is not feasible. We suspect that this observation may have significant implications in algebraic geometry where the solvability of systems of polynomial equations remains an unsolved problem.

5.5 Degrees of Freedom without Spatial Normalization

From our results it is clear that for all (M, N) settings where DoF* takes an integer value, we have a precise characterization of DoF (without the need for spatial normalization). Precise DoF characterizations are also available for (M, N) settings where $\gamma = \frac{M}{N} = \frac{p}{p+1}$ for some $p \in \mathbb{Z}^+$, because in all these settings the outer bound of coincides with the achievability result of Ghasemi et al. [10], and the DoF value is $\frac{MN}{M+N}$. For the remaining cases, we propose a linear beamforming construction for achieving the DoF outer bound without relying on spatial extensions. Because the DoF outer bound is a fractional value, symbol extensions over time/frequency are needed to make the DoF value a whole number over the extended channel. Interestingly, the construction is always non-asymptotic, as in, the number of symbol extensions needed is only enough to make the DoF value an integer. We show analytically that while in many cases symbol extensions in time over constant channels are sufficient to achieve the information theoretic DoF outer bound, there are also cases where time-variations/frequency-selectivity of the channel is needed to achieve the DoF outer bound with linear interference alignment schemes. The feasibility of interference alignment can be settled in every case through a numerical test. We carry out this test to establish the DoF values for all (M, N) values upto $M, N \leq 10$. In general, we end with the conjecture that in all cases, the DoF outer bound value is tight, even with constant channels, and may be achieved with non-linear interference alignment schemes, e.g., exploiting the rational dimensions framework or the Renyi information dimensions framework.

6 DoF Outer Bounds: Preliminaries

Recall from the discussion on redundant dimensions, that the settings corresponding to $M/N = 1/2, 2/3, 3/4, 4/5, \cdots$, are the ones that contain the most redundant dimensions, i.e., for these settings, and only for these settings, it is possible to reduce *either* M or N without losing DoF. It follows then, that in order to prove the strongest DoF outer bounds, it is these settings that must be considered. Indeed, as we will see in the information-theoretic derivations of the DoF outer bounds, essentially the outer bounds must be shown for cases corresponding to $M/N = 1/2, 2/3, 3/4, \cdots$, and then with only a little additional effort, these outer bounds can be extended to all M, N values.

A key step for the information-theoretic DoF outer bound proof is to first perform a change of basis operation, corresponding to invertible linear transformations at both the transmitters and receivers. While invertible linear transformations at the transmitters and/or receivers do not affect the DoF, the change of basis identifies the subspace alignments which in turn helps identify the side information to be provided by a genie for the DoF outer bound. Next, we will present the invertible linear transformations for the case of $(M_T, M_R) = (p, p+1)$. Since the change of basis operations are linear transformations, they are also directly applicable to the reciprocal channel according to the dual nature of reciprocal channels [24].

6.1 Change of Basis for $(M_T, M_R) = (2, 3)$

We begin with the simplest case, i.e., the 2×3 MIMO interference channel. The linear transformations are carried out by multiplying an invertible square matrix at each transmitter and each receiver. The procedure of designing these transformation matrices is illustrated as follows. For brevity, we use $\mathbf{T}_k$ to denote the 2×2 invertible square matrix at Transmitter k, and $\mathbf{R}_k$ to denote the 3×3 transformation matrix at Receiver k.

Step 1: Consider the 3×3 square matrix $\mathbf{R}_k$ which has three rows. First, we determine its first row and its last row, corresponding to the first and the third antennas at Receiver k. The linear transformation is designed in such a manner that the first antenna of Receiver k does not hear Transmitter $k+1$ and the third antenna of Receiver k does not hear Transmitter $k-1$. This is illustrated in Fig.7(a). This operation is guaranteed by the fact that $\mathbf{H}_{k(k+1)}$ and $\mathbf{H}_{k(k-1)}$ are both 3×2 matrices, and the left null space of each of them has one dimension. Therefore, the first row

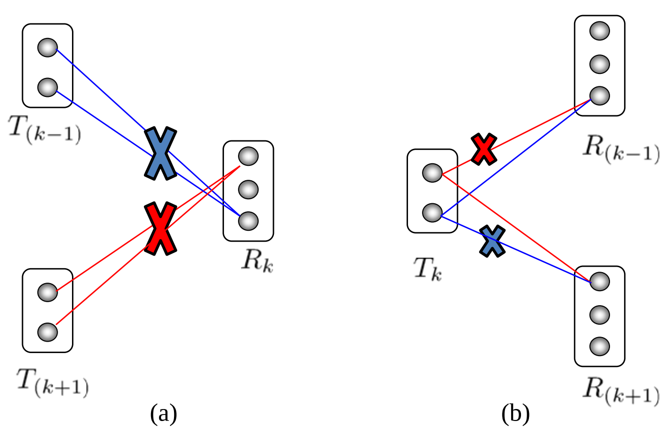


Figure 7: Invertible Linear Transformations for the 2×3 Case, (a) Step 1, (b) Step 2

of $\mathbf{R}_k$ lies in the left null space of $\mathbf{H}_{k(k+1)}$, and the last row lies in the left null space of $\mathbf{H}_{k(k-1)}$. That is,

$$\begin{aligned}\mathbf{R}_k(1,:)\mathbf{H}_{k(k+1)} &= \mathbf{0} \quad (10)\\ \mathbf{R}_k(3,:)\mathbf{H}_{k(k-1)} &= \mathbf{0}. \quad (11)\end{aligned}$$

If we choose the first entry of $\mathbf{R}_k(1,:)$ and the last entry of $\mathbf{R}_k(3,:)$ to be one, then their remaining entries can be solved through following equations.

$$\begin{aligned}\mathbf{R}_k(1,2:3) &= -\left(\mathbf{H}_{k(k+1)}(2:3,:)\right)^{-1}\mathbf{H}_{k(k+1)}(1,:) \quad (12)\\ \mathbf{R}_k(3,1:2) &= -\left(\mathbf{H}_{k(k-1)}(1:2,:)\right)^{-1}\mathbf{H}_{k(k-1)}(3,:). \quad (13)\end{aligned}$$

Step 2: After the first and the third rows of $\mathbf{R}_k$ are specified, we switch to the transmitter side and determine the two columns of $\mathbf{T}_k$, which correspond to the two antennas at Transmitter k. Our goal is to ensure that the first antenna of Transmitter k is not heard by the last antenna of Receiver $k-1$ but is heard by first antenna of Receiver $k+1$, while the last antenna of Transmitter k is not heard by first antenna of Receiver $k+1$ but is heard by the last antenna of Receiver $k-1$. This is illustrated in Fig. 7(b). As a result, the first column of $\mathbf{T}_k$ is chosen in the null space of the channel associated with the last antenna of Receiver $k-1$ (because the channels are generic, this is *not* in the null space of the first antenna of Receiver $k+1$). Similarly, the last column of $\mathbf{T}_k$ is chosen in the null space of the channel associated with the first antenna of Receiver $k+1$. Mathematically,

$$\begin{aligned}\mathbf{R}_{k-1}(3,:)\mathbf{H}_{(k-1)k}\mathbf{T}_k(:,1) &= 0 \quad (14)\\ \mathbf{R}_{k+1}(1,:)\mathbf{H}_{(k+1)k}\mathbf{T}_k(:,2) &= 0. \quad (15)\end{aligned}$$

Since $\mathbf{R}_{k-1}(3,:)\mathbf{H}_{(k-1)k}$ and $\mathbf{R}_{k+1}(1,:)\mathbf{H}_{(k+1)k}$ are 1×2 vectors, the null space of each of them has one dimension. Because the channel coefficients are generic, they are linearly independent almost surely. Therefore, if we choose the first entry of $\mathbf{T}_k(:,1)$ and last entry of $\mathbf{T}_k(:,2)$ to be one, then their remaining entries can be solved from following equations:

$$\begin{aligned}\mathbf{T}_k(2,1) &= -\mathbf{R}_{k-1}(:,1)\mathbf{H}_{(k-1)k}/\mathbf{R}_{k-1}(:,2)\mathbf{H}_{(k-1)k} \quad (16)\\ \mathbf{T}_k(1,2) &= -\mathbf{R}_{k+1}(:,2)\mathbf{H}_{(k+1)k}/\mathbf{R}_{k+1}(:,1)\mathbf{H}_{(k+1)k}. \quad (17)\end{aligned}$$

After this operation, the matrix $\mathbf{T}_k$ is uniquely determined.

Step 3: Now let us switch back to the receiver side again, to determine the remaining row (the second row) of the $\mathbf{R}_k$ at Receiver k. In order for the transformation to be invertible, we need to choose this row such that it is linearly independent with the other rows. For simplicity, we set the first and last entries to be zero and the second to be one[3]. Therefore, the linear transformation matrices at both transmitter and receiver sides have been determined. Note that the invertibility of these transformations is guaranteed by the generic nature of channel coefficients.

The resulting network connectivity after the change of basis operations is shown in Fig. 8.

Since the channel coefficients of the original network are generic, the coefficients of the connected links in Fig. 8 are non-zero almost surely. For brevity, we use "a, c" at the transmitters and "a, b, c" at the receivers to denote the corresponding antennas.

[3]Note that the manner we choose the second row of $\mathbf{R}_k$ is not unique. In Section 8 we will consider another way to design this row for a simpler pattern of channel connectivity of the resulting network.

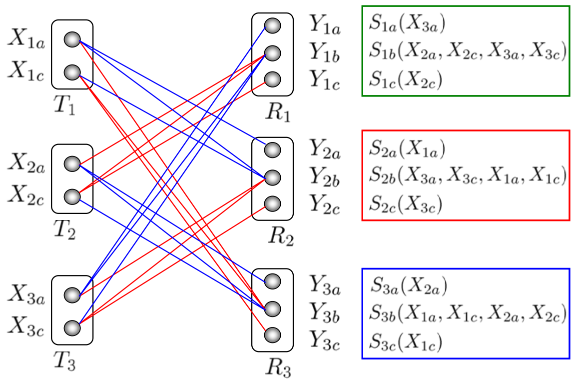


Figure 8: Three-User 2×3 MIMO Interference Channel after the Change of Basis Operation

Next, we show that these linear transformation matrices have full rank almost surely and are therefore invertible. To do that, we need to prove the determinant of each linear transformation matrix is non-zero almost surely. Notice that from (12), (13), (16) and (17), except for some entries which are chosen as specific values, all the remaining entries of the transformation matrices are in the form of the ratio of two polynomials of the channel coefficients. Therefore, the determinant of the linear transformation matrix can be also expressed as a ratio of two polynomials of channel coefficients. Further, multiplication with the denominator changes it to a polynomial. To prove that the determinant is non-zero almost surely, it suffices to show that the polynomial is not the zero polynomial. This can be easily verified through constructing specific channel coefficients for this resulting channel. Such a specific construction is as follows.

$$\mathbf{H}_{(k-1)k} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{H}_{(k+1)k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{18}$$

It is easily seen that the linear transformation for such channel is the identity matrix whose determinant is non-zero. Therefore, the polynomial is not zero polynomial and thus it is non-zero almost surely across all realizations of the channels.

In the new network obtained after the change of basis, each receiver, after decoding its own message and then reconstructing its signal vector sent from its own transmitter, can subtract it from its received signal vector, leaving only the interference from the other undesired transmitters. In Fig. 8, we use $S_{(\cdot)}$ to denote the noisy interference terms at the receiver antenna $(\cdot)$, thus obtaining the interference signal vector $\bar{S}_k = [S_{ka}\ S_{kb}\ S_{kc}]^T$ where subscripts are associated with corresponding receive antennas. The transmit symbols contained within each remaining interference term are indicated explicitly, reflecting the connectivity of the network after the change of basis operation.

6.2 Change of Basis for $(M_T, M_R) = (3, 4)$

Consider the linear transformation for the 3×4 setting as another example. We label antennas from the top to the bottom of Transmitter k as ka_1, kb_1, kc_1, respectively. The corresponding columns in the transformation matrix are labeled in the same manner. Also, we label the antennas from the top to the bottom of Receiver k as ka_1, ka_0, kc_0, kc_1, respectively, and the corresponding rows of the transformation matrix are labeled in the same manner as well.

Step 1: We start with the linear transformations at the receiver. First, we ensure that antenna ka_1 does not hear Transmitter $k+1$ and antenna kc_1 does not hear Transmitter $k-1$. This operation is guaranteed because we can choose the columns ka_1 and kc_1 in the left null space of the channel matrix $\mathbf{H}_{k(k+1)}$ and $\mathbf{H}_{k(k-1)}$, respectively. Since channel matrices are 4×3, these two columns are uniquely determined as the corresponding one dimensional left null spaces.

Step 2: After determining rows ka_1 and kc_1 of $\mathbf{R}_k$, we can design columns ka_1 and kc_1 of $\mathbf{T}_k$ at Transmitter k. The goal is to ensure antenna ka_1 of Transmitter k is *not* heard by antenna $(k-1)c_1$ of Receiver $k-1$. Similarly, antenna kc_1 of Transmitter k is *not* heard by antenna $(k+1)a_1$ of Receiver $k+1$. Note that in the three-dimensional signal space at Transmitter k, there is a two-dimensional subspace orthogonal to antenna $(k-1)c_1$ and another two-dimensional subspace orthogonal to antenna $(k+1)a_1$. These two two-dimensional subspaces have one-dimensional intersection within the three-dimensional space seen from the transmitter. Therefore, we choose columns ka_1 and kc_1 as the directions within their corresponding two-dimensional subspaces but *not* in the one-dimensional intersection subspace. Moreover, we will restrict remaining columns at Transmitter k to lie in the one-dimensional intersection. In other words, the remaining antennas of Transmitter k can see neither the antenna $(k+1)a_1$ nor $(k-1)c_1$. In this case, since only one antenna kb_1 remains, the column kb_1 is fixed as the one-dimensional intersection of those two 2-dimensional subspaces.

Step 3: Now we can switch to receivers again to determine rows ka_0 and kc_0 of $\mathbf{R}_k$. First, in order for rows ka_0 and kc_0 to be linearly independent with rows ka_1 and kc_1 and thereby for the transformations to be invertible, the subspace spanned by rows ka_0 and kc_0 should have only null intersection with that spanned by rows ka_1 and kc_1. For simplicity, we set the first and last columns of these two rows to zero. Then our goal is to ensure that ka_0 at the receiver does not hear $(k+1)b_1$, and kc_1 at the receiver does not hear $(k-1)b_1$. This can be easily achieved by choosing the rows of ka_1 and kc_1 to be orthogonal to the channel vectors associated with the antenna $(k+1)b_1$ and $(k-1)b_1$, respectively.

Similar to the 2×3 setting, the invertibility of channel transformation matrices is guaranteed almost surely owing to the generic property of channel coefficients. The resulting channel connectivity after the change of basis operations, as well as the interference as the function of transmit signals are shown in Fig. 9.

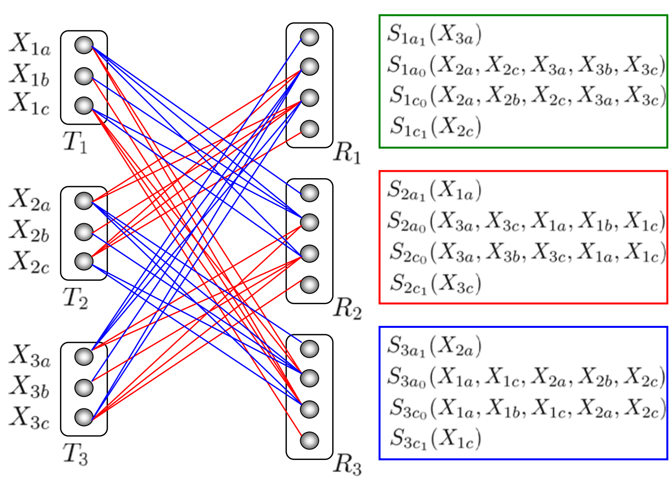


Figure 9: Three-User 3×4 MIMO Interference Channel after the Change of Basis Operation

6.3 Change of Basis for $(M_T, M_R) = (4, 5)$

The third example is the 4×5 MIMO interference channel. As we will see shortly, this case has a recursive relation with the 2×3 setting. As before, we label the antennas of Transmitter k from the top to the bottom as ka_2, ka_1, kc_1, kc_2 and the antennas at Receiver k from top to bottom as $ka_2, ka_1, kb_1, kc_1, kc_2$. The corresponding columns and rows at transmitters and receivers are labeled in the same manner.

Step 1: As before, we start from the first and last antennas at the receiver side. In the transformed basis, the first antenna at Receiver k, ka_2, is placed into the null space of Transmitter $k+1$ such that it does not hear Transmitter $k+1$. The last antenna at Receiver k, kc_2, is in the null space of Transmitter $k-1$ such that it does not hear Transmitter $k-1$.

Step 2: After the first and last rows of the transformation matrix at each receiver are designed, we now turn to the first and last columns of the transformation matrix at each transmitter. The goal is to ensure that antennas ka_2 and kc_2 at Transmitter k cannot see receiver antennas $(k-1)c_2$ and $(k+1)a_2$. Since there are 4 antennas at each transmitter, there are two 3-dimensional subspaces, orthogonal to $(k-1)c_2$ and $(k+1)a_2$, respectively, which have a 2-dimensional intersecting subspace. To achieve the goal, we choose the columns ka_2 and kc_2 as linearly independent directions in these two spaces, respectively. Moreover, we will restrict the space spanned by remaining two columns (ka_1 and kc_1) to be the 2-dimensional intersection subspace. As a result, antennas ka_1 and kc_1 will not be heard by either $(k+1)a_2$ or $(k-1)c_2$.

After Step 1 and Step 2, we have specified the first and last columns and rows of the linear transformation matrices at each transmitter and receiver, respectively. To determine the remaining three rows of transformation matrices at each receiver, we first need to ensure that they are linearly independent with the first and last rows. One way to achieve this is to set the first and last columns of these three rows to zero. Therefore, although the receiver has five antennas, the remaining three antennas are restricted to access only three dimensional subspaces within this five-dimensional signal space. Now ignoring the first and last antennas, the receiver is equivalent to a receiver with 3 antennas. On the other hand, since the remaining two antennas at each transmitter are restricted to access only a two-dimensional subspace in the four-dimensional transmitted signal space, it is equivalent to a transmitter with 2 antennas. Therefore, we obtain a core of 2×3 interference channel. Notice that the change of basis operations so far do not depend on the channel coefficients of the inner 2×3 MIMO interference channel. Since the original inner 2×3 channel coefficients are generic, the resulting inner 2×3 channel after first two steps are still generic. Thus, the linear transformations designed for the 2×3 case previously can be applied here to determine the remaining columns and rows at each transmitter and receiver.

Again, it is easy to show that the linear transformation matrices at each user have full rank almost surely due to the generic property of the channel coefficients. The channel connectivity after the change of basis operations and the interference as the function of transmit signals plus noise are shown in Fig. 10.

6.4 Change of Basis for $(M_T, M_R) = (p, p+1)$

After showing three specific examples, we present the change of basis for the general $p \times (p+1)$ setting in this subsection. As in the 4×5 setting, the linear transformations for the general setting can be determined in a recursive manner. Specifically, for the $p \times (p+1)$ setting, we first design the first and last columns and rows for the linear transformation matrices at each transmitter and receiver, respectively. Once designed, there remain $(p-2)$ and $(p-1)$ columns and rows

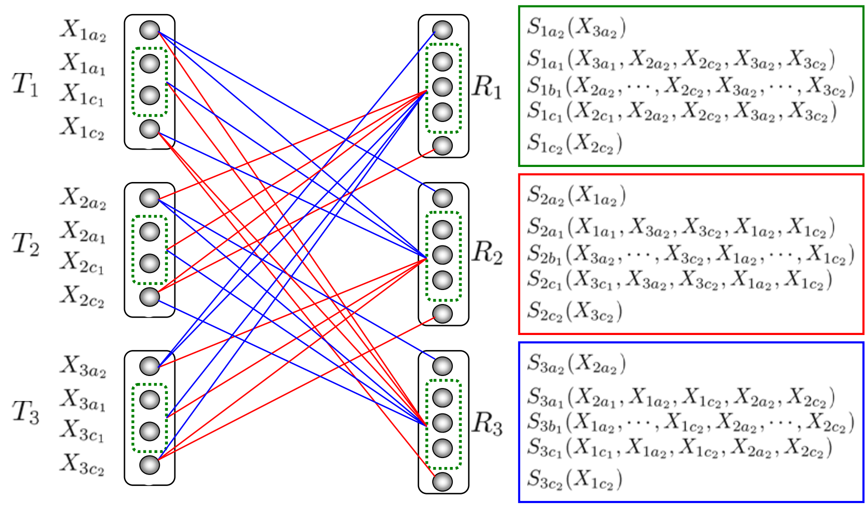


Figure 10: Three-User 4×5 MIMO Interference Channel after the Change of Basis Operation

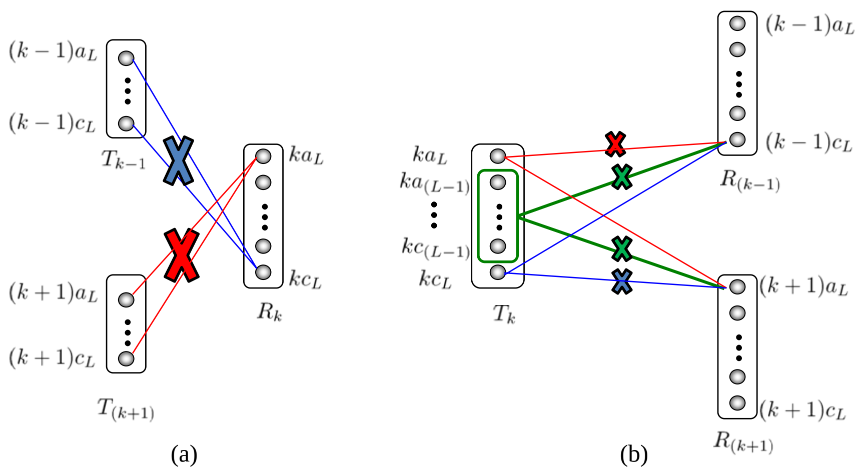


Figure 11: Invertible Linear Transformations for the $2L \times (2L+1)$ setting, (a) Step 1, (b) Step 2

in the transformation matrices to be designed at each transmitter and each receiver, respectively. Essentially, this is the $(p-2) \times (p-1)$ setting if we ignore the two antennas in the outer shell, and hereby the design for the $(p-2) \times (p-1)$ case can be applied. As a result, we only need to specify the design of the first and the last columns and rows of transformation matrices at the transmitter and receiver sides, respectively, for the general $p \times (p+1)$ case. Since the $p \times (p+1)$ case can be reduced to the $(p-2) \times (p-1)$ case, we further need to consider two subcases. One is the $(2L, 2L+1)$ setting which can be recursively boiled down to the 2×3 case discussed earlier, while the other is $(2L+1, 2L+2)$ and can be recursively boiled down to the 3×4 case discussed earlier, $\forall L \in \mathbb{Z}^+$. The algorithm for these two groups are essentially identical. In the following, we first consider the $(2L, 2L+1)$ setting.

6.4.1 Change of Basis for $(M, N) = (2L, 2L+1)$

We label the antennas from the top to the bottom, as ka_L, ka_{L-1}, $\cdots$, ka_1, kc_1, $\cdots$, kc_L at Transmitter k, and as $ka_L, ka_{L-1}, \cdots, ka_1, kb_1, kc_1, \cdots, kc_L$ at Receiver k. Each antenna corresponds to one column at each transmitter and one row at each receiver, respectively. Therefore, the column labeling is from the left to the right of the transformation matrices at each transmitter, and the row labeling is from the top to the bottom of that at each receiver. We will always first design the first and last rows of the transformation matrix at each receiver, and then based on that design determine the first and last columns of the transformation matrix at each transmitter.

Step 1: Consider Receiver k. The goal is to ensure that the first antenna ka_L and the last antenna kc_L do not hear transmitters $k-1$ and $k+1$, respectively, as illustrated in Fig. 11(a). This can be done by choosing row ka_L in the left null space of $\mathbf{H}_{k(k-1)}$ and row kc_L in the left null space of $\mathbf{H}_{k(k+1)}$. Since these two matrices are $(2L+1) \times 2L$, the left null space has one dimension and thus two rows are uniquely determined. After that, the remaining $2L-1$ rows are restricted to span the $(2L-1)$-dimensional subspace that does not overlap with the subspace spanned by the first and last rows such that they are linearly independent. One way to achieve this is to set the first and last columns, i.e., columns ka_L and kc_L, of all the remaining rows to be zero.

Step 2: Once the rows ka_L and kc_L of $\mathbf{R}_k$ are determined, we switch to determine the first and last columns of the transformation matrix $\mathbf{T}_k$. As illustrated in Fig. 11(b), the goal is to ensure that antenna ka_L of Transmitter k cannot see antenna $(k-1)c_L$ of Receiver $k-1$, while the antenna kc_L cannot see the antenna $(k+1)a_L$ of Receiver $k+1$. Since there are $2L$ antennas at each transmitter, there are two $(2L-1)$-dimensional subspaces orthogonal to $(k-1)c_L$ and $(k+1)a_L$, respectively. However, these two subspaces have an intersection of $2L-2$ dimensions at the transmitter in general, implying that there is only one dimension in each subspace that does not overlap with the other. Therefore, we choose rows ka_L and kc_L in that one dimension in each subspace. Moreover, we restrict the space spanned by remaining $2L-2$ columns to be the $(2L-2)$-dimensional intersection subspace. As a result, antennas $ka_{L-1}, \cdots, ka_1, kc_1, \cdots, kc_{L-1}$ will not see either $(k+1)a_L$ or $(k-1)c_L$, as shown in Fig. 11(b).

After Step 1 and Step 2, we essentially still need to design the $(2L-2) \times (2L-1)$ setting. We can repeat these two steps until we reach the 2×3 core for which the linear transformations have been illustrated before. Also, it is not difficult to see these linear transformations are invertible, i.e., $\mathbf{T}_k$ and $\mathbf{R}_k$ have full rank, almost surely.

6.4.2 Change of Basis for $(M, N) = (2L+1, 2L+2)$

The linear transformation for $(2L+1) \times (2L+2)$ is essentially the same as the case $2L \times (2L+1)$. By designing the first and last rows of the transformation matrix at each receiver first and then designing the first and last columns at each transmitter in the same manner as the $2L \times (2L+1)$ case, we end up with the $(2L-1) \times 2L$ case. Repeating such a procedure ultimately brings us to the 3×4 setting for which the linear transformations have been illustrated. Moreover, it is not difficult to see these linear transformations are invertible as well.

Remark: Note that this transformation process looks like layer-by-layer onion peeling, as shown in Fig. 12. Specifically, for the $p \times (p+1)$ setting, by peeling out the outer shell at each user, i.e., the first and the last antennas, the remaining part is a $(p-2) \times (p-1)$ interference channel. Repeating this procedure, we eventually obtain the core of the 2×3 interference network if p is even as shown in Fig. 12(a), or the core of the 3×4 interference network if p is odd as in Fig. 12(b). The onion peeling intuition is important for the change of basis, and it also help us identify

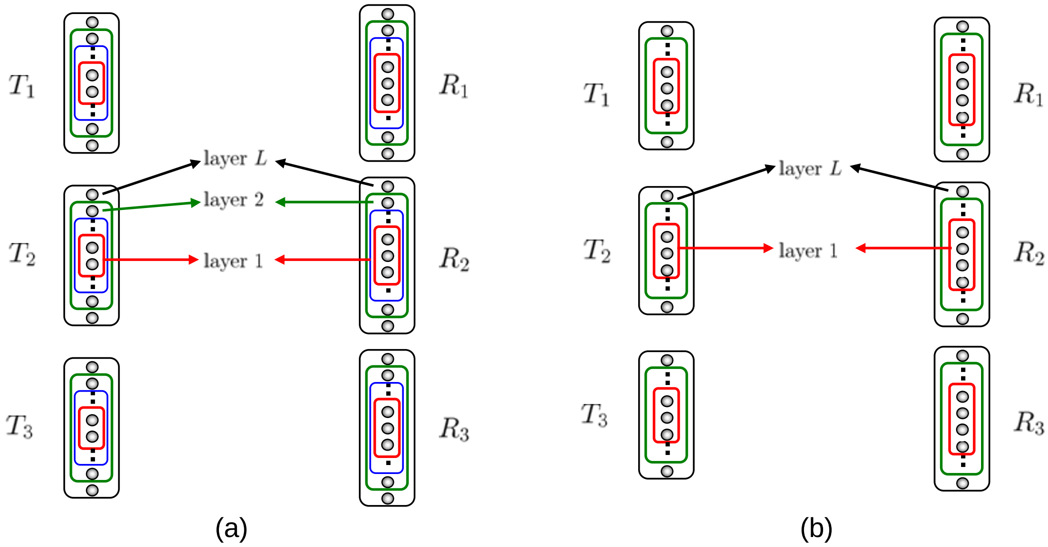


Figure 12: Intuition of Onion Peeling, (a) M is even, (b) M is odd

the genie signals provided to each receiver for deriving the information theoretic DoF outer bounds in Section 7.

7 Information Theoretic DoF Outer Bound

In this section, we will derive the information theoretic DoF outer bound claimed in Lemma 1. We organize this section as follows. We begin with the case when $M/N \leq 3/5$ where cooperation outer bound is sufficient. For the remaining $M/N > 3/5$ case, we first consider the $M_T < M_R$ setting, i.e., networks that have more antennas at the receiver nodes than at the transmitters. We only show some specific examples to convey the central idea to derive the information theoretic DoF outer bounds, and defer the general proof to Appendix A. Since the duality property has not been shown for information theoretic statements, we also consider the $M_T > M_R$ setting. Since the essential idea still follows the onion peeling intuition, we defer the detailed proof to Appendix B. We start with the $M/N \leq 3/5$ setting as follows.

DoF Outer Bound for $M/N \leq 3/5$: Consider Fig. 2 where the squares are shaded with light yellow, green and grey colors:

(A) $M/N \in (0, 1/3] \Rightarrow d \leq M$

(B) $M/N \in (1/3, 1/2] \Rightarrow d \leq N/3$

(C) $M/N \in (1/2, 3/5] \Rightarrow d \leq 2M/3$

While we have shown in Section 5.1 that the single user or cooperation DoF outer bounds are sufficient for case (A) and (B), we still need to consider the case (C). Fortunately, it turns out that for this case, the cooperation DoF bound is still sufficient. Specifically, the resulting two user interference channel by allowing any two of three users to cooperate where User 1 has M_T and M_R antennas and User 2 has $2M_T$ and $2M_R$ antennas has been proved to have $\min(3M_T, 3M_R, \max(M_T, 2M_R), \max(2M_T, M_R))$ DoF [11], which is equal to $2\min(M_T, M_R) =$

$2M$. Since cooperation among antennas cannot reduce the channel capacity, $2M/3$ is the DoF per user bound for the original channel in case (C).

DoF Outer Bound for $M/N > 3/5$: While applying cooperation outer bounds for $M/N \leq 3/5$ is sufficient to establish the sum DoF results (the achievability will be shown in Section 8), the DoF remain open for $M/N > 3/5$ cases. The nature of the DoF outer bound derivation is recursive. To set up the recursion we will first solve the core cases and then construct a recursive argument for the general proof built upon the reduction to these core cases. The complexity of the problem is such that some partitioning into disjoint groups, each of which must be studied separately, is necessary. First, because no claim to duality can be made a-priori for information theoretic DoF results, we need to deal with the cases $M_T > M_R$ and $M_R > M_T$ separately. Indeed the outer bound proof for the two settings require different approaches. Second, within each of these cases, e.g., with $M_R > M_T$ there are also two groups of channels, $(2L, 2L+1)$ and $(2L+1, 2L+2)$, each of which must be considered separately through its own recursive reduction. As such we will need to establish the core results for each of these groups, which will make the overall proof of Lemma 1 quite lengthy, rather unavoidably so.

In the following, we will first show the information theoretic DoF outer bounds for three specific examples of $(M, N) = (p, p+1)$ where $p = 2, 3, 4$. Then based on the observation on the proofs for these examples, we generalize the information theoretic DoF outer bound proof to $(M, N) = (pq, (p+1)q)$ where $p \in \mathbb{Z}^+, p > 4$ and $q \in \mathbb{Z}^+$ in Appendix A. Since results for $q > 1$ follow from $q = 1$ due to a simple spatial expansion, we only need to consider $q = 1$, i.e., $(M, N) = (p, p+1)$.

Before proceeding to the specific examples, we first introduce the following two lemmas that will be frequently used in the converse argument in the remainder of this section.

Lemma 3 *Consider an arbitrary M-dimensional random vector $\bar{X}$ whose distribution may depend on ρ, and independent M-dimensional Gaussian vectors $\bar{Z} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$ and $\bar{Z}' \sim \mathcal{CN}(\mathbf{0}, \mathbf{K})$, that are both independent of $\bar{X}$. $\mathbf{K}$ is a non-singular covariance matrix held fixed as $\rho \to \infty$. We have*

$$h(\bar{X} + \bar{Z}) = h(\bar{X} + \bar{Z}') + o(\log \rho). \tag{19}$$

Proof: Since $\bar{Z}'$ is independent of both $\bar{X}$ and $\bar{Z}$, we have

$$h(\bar{X} + \bar{Z}) = h(\bar{X} + \bar{Z} + \bar{Z}' | \bar{Z}') \leq h(\bar{X} + \bar{Z} + \bar{Z}'). \tag{20}$$

On the other hand, since $\bar{X} + \bar{Z} + \bar{Z}'$ is a degraded version of $\bar{X} + \bar{Z}$, we have

$$\begin{aligned} 0 &\leq I(\bar{X}; \bar{X} + \bar{Z}) - I(\bar{X}; \bar{X} + \bar{Z} + \bar{Z}') && (21) \\ &= h(\bar{X} + \bar{Z}) - h(\bar{X} + \bar{Z} | \bar{X}) - h(\bar{X} + \bar{Z} + \bar{Z}') + h(\bar{X} + \bar{Z} + \bar{Z}' | \bar{X}) && (22) \\ &= h(\bar{X} + \bar{Z}) - h(\bar{X} + \bar{Z} + \bar{Z}') - h(\bar{Z}) + h(\bar{Z} + \bar{Z}') && (23) \\ &= h(\bar{X} + \bar{Z}) - h(\bar{X} + \bar{Z} + \bar{Z}') + \log(\det(\mathbf{K} + \mathbf{I})). && (24) \end{aligned}$$

Combining (20) and (24) produces

$$h(\bar{X} + \bar{Z}) \leq h(\bar{X} + \bar{Z} + \bar{Z}') \leq h(\bar{X} + \bar{Z}) + \log(\det(\mathbf{K} + \mathbf{I})) \tag{25}$$

where $\log(\det(\mathbf{K} + \mathbf{I}))$ is a constant which does not depend on ρ. Thus, we have

$$h(\bar{X} + \bar{Z}) = h(\bar{X} + \bar{Z} + \bar{Z}') + o(\log \rho). \tag{26}$$

Following the same approach, we can also obtain

$$h(\bar{X} + \bar{Z}') = h(\bar{X} + \bar{Z} + \bar{Z}') + o(\log \rho). \tag{27}$$

Thus, we have

$$h(\bar{X} + \bar{Z}) = h(\bar{X} + \bar{Z}') + o(\log \rho). \tag{28}$$

■

Remark: As shown in Lemma 3, non-singular noise terms whose power does not depend on ρ, only contribute an $o(\log \rho)$ term to the differential entropy, which does not affect the DoF. Based on this observation, in the remainder of this paper, we will often use the phrase "subject to noise distortion" to indicate the widely used (see e.g., [11, 3]) DoF outer bound argument whereby reducing noise at a node by an amount that is SNR independent (and therefore inconsequential for DoF) allows it to decode a message.

Lemma 4 *If a genie provides a subset of noisy transmitted signals, denoted as $\mathcal{G}$, to Receiver k, such that it can decode all three messages from the observation $(\bar{Y}_k^n, \mathcal{G})$, then we can always outer bound the mutual information term $I(W_1, W_2, W_3; \bar{Y}_k^n, \mathcal{G})$ as follows:*

$$\begin{aligned}
I(W_1, W_2, W_3; \bar{Y}_k^n, \mathcal{G}) &= I(W_1, W_2, W_3; \bar{Y}_k^n) + I(W_1, W_2, W_3; \mathcal{G}|\bar{Y}_k^n) && (29)\\
&\leq M_R\ n \log \rho + I(W_1, W_2, W_3; \mathcal{G}|\bar{Y}_k^n) + n\ o(\log \rho) && (30)\\
&\leq M_R\ n \log \rho + h(\mathcal{G}|\bar{Y}_k^n) - h(\mathcal{G}|W_1, W_2, W_3, \bar{Y}_k^n) + n\ o(\log \rho) && (31)\\
&= M_R\ n \log \rho + h(\mathcal{G}|\bar{Y}_k^n) + n\ o(\log \rho) + o(n). && (32)
\end{aligned}$$

Proof: In the derivations above, (29) follows from the mutual information chain rule. (30) is obtained because Receiver k has only M_R antennas. (31) follows from the entropy chain rule, and (32) is obtained from Lemma 3 since given all the three messages we can reconstruct the genie signals $\mathcal{G}$ subject to noise distortion. ■

7.1 Case: $(M, N) = (2, 3)$ $\Rightarrow$ DoF $\leq \frac{6}{5}$

After the change of basis operation introduced in Section 6, we obtain the network with connectivity in Fig. 8, shown again in Fig. 13 for the reader's convenience.

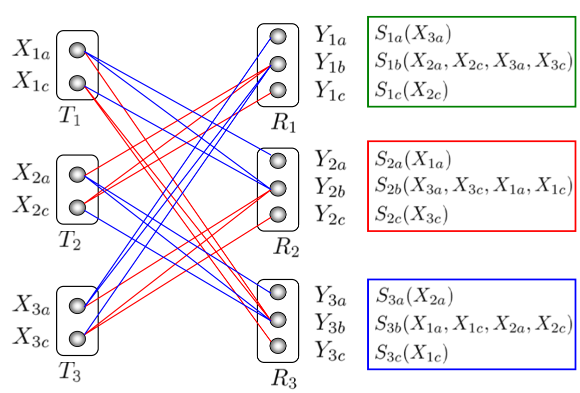


Figure 13: Three-User 2×3 MIMO Interference Channel

First, a genie provides the signal $\mathcal{G}_1 = \{X_{2a}^n + \tilde{Z}_{2a}^n\}$ to Receiver 1, where $\tilde{Z}_{2a}^n \sim \mathcal{CN}(0,1)$ is an artificial i.i.d. Gaussian noise. Let us consider Receiver 1. Since we are dealing with a converse argument, it follows by assumption that the receiver is able to decode and subtract out its desired signal. It therefore has $S_{1c}^n(X_{2c}^n) = X_{2c}^n + Z_{2c}^n$ at the antenna "1c". Thus, with the provided genie signal, Receiver 1 can decode W_2 from the observation $(X_{2a}^n + \tilde{Z}_{2a}^n, S_{1c}^n(X_{2c}^n))$ subject to the noise distortion. After decoding W_2 we can reconstruct the transmitted signals (X_{2a}^n, X_{2c}^n) and subtract them from S_{1b}^n to obtain $S_{1b}'^n$ which is a linear combination of transmitted signals (X_{3a}^n, X_{3c}^n) plus the noise. Now by the two linearly independent observations of $(S_{1a}^n(X_{3a}^n), S_{1b}'^n(X_{3a}^n, X_{3c}^n))$, Receiver 1 can resolve (X_{3a}^n, X_{3c}^n) and thus can decode W_3 as well, subject to noise distortion. Since the genie information $\mathcal{G}_1$ provided to Receiver 1 allows it to decode all three messages subject to noise distortion, we have:

$$
\begin{aligned}
n(R_1 + R_2 + R_3) &\leq I(W_1, W_2, W_3; \bar{Y}_1^n, \mathcal{G}_1) + n\ o(\log\rho) + o(n) && (33)\\
&\leq Nn\log\rho + h(X_{2a}^n + \tilde{Z}_{2a}^n | \bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (34)\\
&\leq Nn\log\rho + h(X_{2a}^n + \tilde{Z}_{2a}^n | X_{2c}^n + Z_{2c}^n) + n\ o(\log\rho) + o(n) && (35)\\
&\leq Nn\log\rho + nR_2 - h(X_{2c}^n + Z_{2c}^n) + n\ o(\log\rho) + o(n) && (36)
\end{aligned}
$$

where (33) follows from Fano's inequality. (34) follows from Lemma 4. (35) follows from the fact that dropping the conditioning cannot decrease the differential entropy. Thus, we only keep S_{1c}^n as the conditioning term, which is $X_{2c}^n + Z_{2c}^n$. (36) is obtained because from the observations of $(X_{2a}^n + \tilde{Z}_{2a}^n, X_{2c}^n + Z_{2c}^n)$ we can decode W_2 subject to noise distortion. By advancing the user indices, we therefore obtain the following three inequalities:

$$
\begin{aligned}
nR_\Sigma &\leq Nn\log\rho + nR_2 - h(X_{2c}^n + Z_{2c}^n) + n\ o(\log\rho) + o(n) && (37a)\\
nR_\Sigma &\leq Nn\log\rho + nR_3 - h(X_{3c}^n + Z_{3c}^n) + n\ o(\log\rho) + o(n) && (37b)\\
nR_\Sigma &\leq Nn\log\rho + nR_1 - h(X_{1c}^n + Z_{1c}^n) + n\ o(\log\rho) + o(n). && (37c)
\end{aligned}
$$

Since we always use "advance the user indices" in a circularly symmetric way, from now on we will use compact notations

$$
\begin{aligned}
R &= R_\Sigma/3\\
h(X_{(\cdot)}) &= [h(X_{1(\cdot)}) + h(X_{2(\cdot)}) + h(X_{3(\cdot)})]/3
\end{aligned}
$$

where quantities without user index represent the average of all rotated indices. Thus (37) can be rewritten as

$$
3nR \leq Nn\log\rho + nR - h(X_c^n + Z_c^n) + n\ o(\log\rho) + o(n). \tag{38}
$$

Next, let a genie provide the signal $\mathcal{G}_2 = \{X_{3c}^n + \tilde{Z}_{3c}^n\}$ to Receiver 1 where $\tilde{Z}_{3c}^n \sim \mathcal{CN}(0,1)$ is an artificial i.i.d. Gaussian noise. Similarly, by providing $\mathcal{G}_2$ to Receiver 1, it can first decode W_3 subject to noise distortion from the observation $(S_{1a}^n(X_{3a}^n), X_{3c}^n + \tilde{Z}_{3c}^n)$. After decoding W_3, Receiver 1 can reconstruct transmitted signals (X_{3a}^n, X_{3c}^n) and then subtract them from S_{1b}^n to obtain a noisy linear combination of (X_{2a}^n, X_{2c}^n), from which together with $S_{1c}^n(X_{2c}^n)$ Receiver 1 can decode W_2 as well, subject to noise distortion. Since Receiver 1 again can decode all three messages, we have:

$$
\begin{aligned}
nR_\Sigma &\leq I(W_1, W_2, W_3; \bar{Y}_1^n, X_{3c}^n + \tilde{Z}_{3c}^n) + n\ o(\log\rho) + o(n) && (39)\\
&\leq Nn\log\rho + h(X_{3c}^n + \tilde{Z}_{3c}^n | \bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (40)\\
&\leq Nn\log\rho + h(X_{3c}^n + \tilde{Z}_{3c}^n) + n\ o(\log\rho) + o(n). && (41)
\end{aligned}
$$

The derivation above is similar to that for the previous bound. By advancing the user indices, we obtain the second inequality as follows.

$$3nR \leq Nn \log \rho + h(X_c^n + \tilde{Z}_c^n) + n\ o(\log \rho) + o(n). \tag{42}$$

Adding up the two inequalities of (38) and (42), each obtained by averaging over user indices, subject to noise distortion we have:

$$6nR \leq 2Nn \log \rho + nR + n\ o(\log \rho) + o(n). \tag{43}$$

where $h(X_c^n + \tilde{Z}_c^n) - h(X_c^n + Z_c^n) = n\ o(\log \rho) + o(n)$. By arranging terms of (43) we have:

$$5nR \leq 2Nn \log \rho + n\ o(\log \rho) + o(n). \tag{44}$$

By dividing $n \log \rho$ on both sides of (44), and letting $n \to \infty$ followed by $\rho \to \infty$, we obtain:

$$d \leq \frac{2N}{5} = \frac{6}{5}. \tag{45}$$

Note that the genie signals provided to receivers contain Gaussian noise. Since we are interested in the DoF characterization and due to the "subject to noise distortion" statement, we will not explicitly mention the noise terms in the genie signal and received signals in the remainder of this exposition for notational simplicity.

Remark: In order to obtain the inequalities (38) and (42), a genie provides the signal set $\mathcal{G}_1$ and $\mathcal{G}_2$ to Receiver 1, respectively (also to other receivers by advancing user indices). In other words, with the genie signal provided to Receiver 1, it can always decode all the three messages. Moreover, in the remaining of this paper, unless otherwise specified, it is not difficult to see that by providing the genie signal set that will be specified later to Receiver 1, it can always decode all three messages subject to the noise distortion.

7.2 Case: $(M, N) = (3, 4) \ \Rightarrow$ DoF $\leq \frac{12}{7}$

The second example is the $(M, N) = (3, 4)$ setting. After taking the linear transformation introduced in Section 6, we obtain the network with connectivity shown in Fig. 14, and the resulting equivalent channel coefficients are non-zero.

First, a genie provides the signal set $\mathcal{G}_1 = \{X_{2a}^n, X_{3c}^n\}$ to Receiver 1. As in the $(M, N) = (2, 3)$ case, it is easy to see by providing this set of genie signals to Receiver 1, it can decode all three messages subject to the noise distortion. Consider the sum rate of three messages:

$$\begin{aligned}
nR_\Sigma &\leq I(W_1, W_2, W_3; \bar{Y}_1^n, X_{2a}^n, X_{3c}^n) + n\ o(\log \rho) + o(n) && (46)\\
&\leq Nn \log \rho + h(X_{2a}^n, X_{3c}^n | \bar{Y}_1^n) + n\ o(\log \rho) + o(n) && (47)\\
&\leq Nn \log \rho + h(X_{2a}^n) + h(X_{3c}^n) + n\ o(\log \rho) + o(n) && (48)
\end{aligned}$$

where (46) follows from Fano's inequality, and (47) is because of Lemma 4. (48) follows from the chain rule and dropping the condition terms cannot decrease the differential entropy. By averaging over the user indices, we obtain the first inequality as follows.

$$3nR \leq Nn \log \rho + h(X_a^n) + h(X_c^n) + n\ o(\log \rho) + o(n). \tag{49}$$

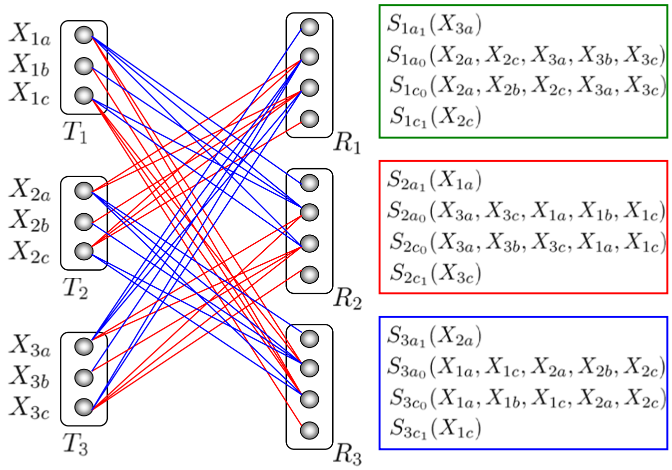


Figure 14: Three-User 3×4 MIMO Interference Channel

Next, a genie provides the signal set $\mathcal{G}_2 = \{X_{2a}^n, X_{2b}^n\}$ to Receiver 1, then the sum rate of three messages is bounded above by:

$$\begin{aligned} nR_\Sigma &\leq I(W_1, W_2, W_3; \bar{Y}_1^n, X_{2a}^n, X_{2b}^n) + n\ o(\log\rho) + o(n) && (50)\\ &\leq Nn\log\rho + h(X_{2a}^n, X_{2b}^n|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (51)\\ &= Nn\log\rho + h(X_{2a}^n|\bar{Y}_1^n) + h(X_{2b}^n|X_{2a}^n, \bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (52)\\ &\leq Nn\log\rho + h(X_{2a}^n|X_{2c}^n) + h(X_{2b}^n|X_{2a}^n, X_{2c}^n) + n\ o(\log\rho) + o(n) && (53)\\ &\leq Nn\log\rho + nR_2 - h(X_{2c}^n) + n\ o(\log\rho) + o(n). && (54)\end{aligned}$$

The derivation above is similar to that for the previous bound. Note that the last inequality also follows because subject to the noise distortion, we can decode the message W_2 from the observation of $(X_{2a}^n, X_{2b}^n, X_{2c}^n)$. By averaging over the user indices, we obtain the second inequality as follows.

$$3nR \leq Nn\log\rho + nR - h(X_c^n) + n\ o(\log\rho) + o(n). \tag{55}$$

Similarly, if a genie provides the signal set $\mathcal{G}_3 = \{X_{3b}^n, X_{3c}^n\}$ to Receiver 1, then we have:

$$\begin{aligned} nR_\Sigma &\leq I(W_1, W_2, W_3; \bar{Y}_1^n, X_{3b}^n, X_{3c}^n) + n\ o(\log\rho) + o(n) && (56)\\ &\leq Nn\log\rho + h(X_{3b}^n, X_{3c}^n|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (57)\\ &= Nn\log\rho + h(X_{3c}^n|\bar{Y}_1^n) + h(X_{3b}^n|X_{3c}^n, \bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (58)\\ &\leq Nn\log\rho + h(X_{3c}^n|X_{3a}^n) + h(X_{3b}^n|X_{3c}^n, X_{3a}^n) + n\ o(\log\rho) + o(n) && (59)\\ &\leq Nn\log\rho + nR_3 - h(X_{3a}^n) + n\ o(\log\rho) + o(n). && (60)\end{aligned}$$

where the last inequality follows because subject to noise distortion we can decode W_3 from the observation of $(X_{3a}^n, X_{3b}^n, X_{3c}^n)$. By averaging over the user indices, we obtain the third inequality:

$$3nR \leq Nn\log\rho + nR - h(X_a^n) + n\ o(\log\rho) + o(n). \tag{61}$$

Adding up the three inequalities (49), (55) and (61), we have:

$$9nR \leq 3Nn\log\rho + 2nR + n\ o(\log\rho) + o(n). \tag{62}$$

By arranging terms of (62) we have:

$$7nR \leq 3Nn \log \rho + n\ o(\log \rho) + o(n). \tag{63}$$

By dividing $n \log \rho$ on both sides of (63), and letting $n \to \infty$ followed by $\rho \to \infty$, we obtain:

$$d \leq \frac{3N}{7} = \frac{12}{7}. \tag{64}$$

7.3 Case: $(M, N) = (4, 5) \ \Rightarrow$ DoF $\leq \frac{20}{9}$

We show the third example is this subsection. Let us consider the $(M, N) = (4, 5)$ setting. Similar to the two examples we have shown, we again take linear transformations introduced in Section 6, thus obtaining the network with resulting connectivity shown in Fig. 15, and the equivalent channel coefficients are non-zero.

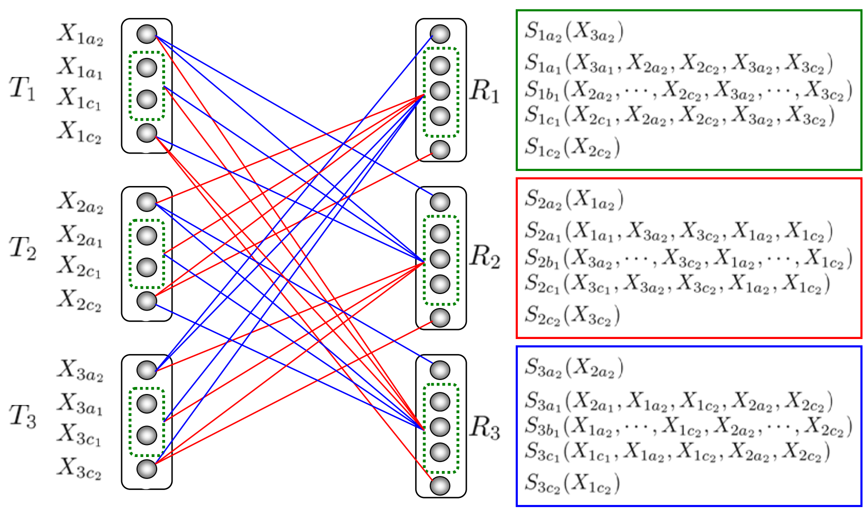


Figure 15: Three-User 4×5 MIMO Interference Channel

Just as we have shown in Section 6 that the linear transformations of the 4×5 setting follow in a recursive manner (onion peeling insights) from the 2×3 setting, in this subsection the reader will note that the genie signals in the 4×5 network also have a recursive relationship with those for the $(M, N) = (2, 3)$ setting. We will emphasize this observation in the next subsection.

First, a genie provides the signal set $\mathcal{G}_1 = \{X^n_{2a_2}, X^n_{3c_2}, X^n_{2a_1}\}$ to Receiver 1. Let us consider the sum rate of the three messages:

$$\begin{aligned}
nR_\Sigma &\leq I(W_1, W_2, W_3; \bar{Y}^n_1, X^n_{2a_2}, X^n_{3c_2}, X^n_{2a_1}) + n\ o(\log \rho) + o(n) && (65)\\
&\leq Nn \log \rho + h(X^n_{2a_2}, X^n_{3c_2}, X^n_{2a_1} | \bar{Y}^n_1) + n\ o(\log \rho) + o(n) && (66)\\
&\leq Nn \log \rho + h(X^n_{2a_2} | \bar{Y}^n_1) + h(X^n_{3c_2} | \bar{Y}^n_1) + h(X^n_{2a_1} | \bar{Y}^n_1, X^n_{2a_2}, X^n_{3c_2}) + n\ o(\log \rho) + o(n) && (67)\\
&\leq Nn \log \rho + h(X^n_{2a_2} | X^n_{2c_2}) + h(X^n_{3c_2}) + h(X^n_{2a_1} | X^n_{2a_2}, X^n_{2c_2}, X^n_{2c_1}) + n\ o(\log \rho) + o(n) && (68)\\
&= Nn \log \rho + h(X^n_{3c_2}) + nR_2 - h(X^n_{2c_2}) - h(X^n_{2c_1} | X^n_{2a_2}, X^n_{2c_2}) + n\ o(\log \rho) + o(n) && (69)
\end{aligned}$$

where (65) follows from Fano's inequality, and (66) is due to Lemma 4. (68) is obtained because dropping conditioning terms does not decrease the differential entropy. By averaging over the user indices, we therefore obtain the first inequality as follows.

$$\begin{aligned} 3nR &\leq Nn\log\rho + h(X_{c_2}^n) + nR - h(X_{c_2}^n) - h(X_{c_1}^n|X_{a_2}^n, X_{c_2}^n) + n\ o(\log\rho) + o(n) \\ &\leq Nn\log\rho + nR - h(X_{c_1}^n|X_{a_2}^n, X_{c_2}^n) + n\ o(\log\rho) + o(n). \end{aligned} \tag{70}$$

Second, a genie provides the signal set $\mathcal{G}_2 = \{X_{2a_2}^n, X_{3c_2}^n, X_{3c_1}^n\}$ to Receiver 1. With the similar analysis we have the sum rate bound:

$$\begin{aligned} nR_\Sigma &\leq I(W_1, W_2, W_3; \bar{Y}_1^n, X_{2a_2}^n, X_{3c_2}^n, X_{3c_1}^n) + n\ o(\log\rho) + o(n) && (71) \\ &\leq Nn\log\rho + h(X_{2a_2}^n, X_{3c_2}^n, X_{3c_1}^n|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (72) \\ &\leq Nn\log\rho + h(X_{2a_2}^n) + h(X_{3c_2}^n) + h(X_{3c_1}^n|\bar{Y}_1^n, X_{2a_2}^n, X_{3c_2}^n) + n\ o(\log\rho) + o(n) && (73) \\ &\leq Nn\log\rho + h(X_{2a_2}^n) + h(X_{3c_2}^n) + h(X_{3c_1}^n|X_{3a_2}^n, X_{3c_2}^n) + n\ o(\log\rho) + o(n) && (74) \end{aligned}$$

where the last inequality is obtained because knowing $\bar{Y}_1^n$ we can decode W_1, and thus we obtain $S_{1a_2}^n$ which is a noisy version of $X_{3a_2}^n$. Then we use the fact that dropping the condition terms cannot decrease the differential entropy. By averaging over the user indices, we have the second inequality:

$$3nR \leq Nn\log\rho + h(X_{a_2}^n) + h(X_{c_2}^n) + h(X_{c_1}^n|X_{a_2}^n, X_{c_2}^n) + n\ o(\log\rho) + o(n). \tag{75}$$

Third, a genie provides signals $\mathcal{G}_3 = \{X_{2a_2}^n, X_{2a_1}^n, X_{2c_1}^n\}$ to Receiver 1. Thus, we have:

$$\begin{aligned} nR_\Sigma &\leq I(W_1, W_2, W_3; \bar{Y}_1^n, X_{2a_2}^n, X_{2a_1}^n, X_{2c_1}^n) + n\ o(\log\rho) + o(n) && (76) \\ &\leq Nn\log\rho + h(X_{2a_2}^n, X_{2a_1}^n, X_{2c_1}^n|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (77) \\ &\leq Nn\log\rho + h(X_{2a_2}^n, X_{2a_1}^n, X_{2c_1}^n|X_{2c_2}^n) + n\ o(\log\rho) + o(n) && (78) \\ &= Nn\log\rho + nR_2 - h(X_{2c_2}^n) + n\ o(\log\rho) + o(n) && (79) \end{aligned}$$

and by averaging over the user indices, we have the third inequality:

$$3nR \leq Nn\log\rho + nR - h(X_{c_2}^n) + n\ o(\log\rho) + o(n). \tag{80}$$

Finally, a genie provides signals $\mathcal{G}_4 = \{X_{3a_1}^n, X_{3c_1}^n, X_{3c_2}^n\}$ to Receiver 1, and we have:

$$\begin{aligned} nR_\Sigma &\leq I(W_1, W_2, W_3; \bar{Y}_1^n, X_{3a_1}^n, X_{3c_1}^n, X_{3c_2}^n) + n\ o(\log\rho) + o(n) && (81) \\ &\leq Nn\log\rho + h(X_{3a_1}^n, X_{3c_1}^n, X_{3c_2}^n|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (82) \\ &\leq Nn\log\rho + h(X_{3a_1}^n, X_{3c_1}^n, X_{3c_2}^n|X_{3a_2}^n) + n\ o(\log\rho) + o(n) && (83) \\ &= Nn\log\rho + nR_3 - h(X_{3a_2}^n) + n\ o(\log\rho) + o(n) && (84) \end{aligned}$$

and we obtain the last inequality after averaging over the user indices:

$$3nR \leq Nn\log\rho + nR - h(X_{a_2}^n) + n\ o(\log\rho) + o(n). \tag{85}$$

Adding up all the sum rate inequalities in (70), (75), (80) and (85) we have:

$$12nR \leq 4Nn\log\rho + 3nR + n\ o(\log\rho) + o(n). \tag{86}$$

By arranging terms of (86) we have:

$$9nR_\Sigma \leq 4Nn\log\rho + n\ o(\log\rho) + o(n). \tag{87}$$

By dividing $n\log\rho$ on both sides of (87), and letting $n \to \infty$ followed by $\rho \to \infty$, we obtain:

$$d \leq \frac{4N}{9} = \frac{20}{9}. \tag{88}$$

7.4 Onion Peeling: The Intuition of the DoF Outer Bound Proof

While we have only shown the information theoretic DoF outer bounds for $(M, N) = (p, p+1)$, $p = 2, 3, 4$ settings, it is not difficult to see that they share some similarities in both genie signals and sum rate bounding techniques. In this subsection, we will specify these similarities and build the intuition behind the outer bound derivations so as to solve general $(M, N) = (p, p+1)$ cases where $p \in \mathbb{Z}$. There are two questions that we will answer:

1. How should we design the genie signal sets with minimum cardinality such that each receiver with each genie signal set can decode all three messages?

2. In order to finalize the converse arguments, how many sum rate bounds do we need totally?

Answer to Question 1: In the three examples that we have shown, we always provide genie signals to Receiver 1 so as to upper bound the sum rate of three users, then we obtain the other two sum rate inequalities by advancing user indices and eventually have one inequality by averaging over three users for compact notations. For an $M \times N$ interference channel, since each receiver is able to decode its own message, it can subtract the signal carrying its desired message and obtain N linearly independent combinations of the interference signals from the $2M$ transmit antennas (dimensions) of the two interferers. Thus, in order for the receiver to decode the other two undesired messages, a genie needs to provide a signal set containing at least $\max(0, 2M - N)$ dimensional linearly independent signals to that receiver such that it can recover the signal vectors sent from the two interferers directly or by an invertible linear transformation subject to noise distortion. Therefore, as in $(M, N) = (2, 3)$, $(3, 4), (4, 5)$ cases, each genie signal set has one, two, three dimensions, respectively.

Now that we know that each genie signal set has $\max(0, 2M-N)$ dimensions, we still need to specify which $\max(0, 2M-N)$ dimensions a genie provides. Let us first recall 2×3 and 4×5 cases. We only consider genie signals to Receiver 1 because those for Receiver 2 and 3 are obtained by simply advancing user indices. For the 2×3 setting, a genie provides X_{2a}^n and X_{3c}^n, producing two outer bounds. For the 4×5 setting, if we peel out the two antennas in the outer shell, i.e., by providing the corresponding transmitted signals $(X_{2a_2}^n, X_{3c_2}^n)$ in the information theoretic statements, the remaining part is an embedded $(M, N) = (2, 3)$ setting. For the 2×3 setting, the two genie signals are $X_{2a_1}^n$ and $X_{3c_1}^n$, respectively, and thus for the 4×5 setting, at least we should have two sets of genie signals, which include $(X_{2a_2}^n, X_{3c_2}^n)$ and one of $X_{2a_1}^n$ and $X_{3c_1}^n$, respectively. After doing this, the remaining task is to bound the sum rate of the signals sent from the peeled out layer. For example, in the $(M, N) = (4, 5)$ case, we have four sets of genie signals, two more than that in the $(M, N) = (2, 3)$ case. In fact, for the general $(M, N) = (p, p + 1)$ case, as in the proofs that we show for general cases, the genie provides a total of p sets of signals to Receiver 1. In the $p - 2$ sets among them, the signals include that designed for the $(p - 2, p - 1)$ setting plus the two signals sent from the first antenna of Transmitter 2 and the last antenna of Transmitter 3.

Answer to Question 2: The number of sum rate inequalities depends on how many sets of signals that genie provides to Receiver 1 (also for Receiver 2 and 3 by advancing user indices). In the cases of $(M, N) = (2, 3)$ and $(3, 4)$, the number of genie signal sets is 2 and 3, respectively, i.e., identical to the value of M[4]. Essentially, one set of genie signals produces one sum rate inequality, and thus we have a total of M sum rate inequalities. From the three examples, it can be seen that

[4]The number of genie signal sets provided to each receiver is associated with the length of subspace alignment chain κ_N. In the special $(M, N) = (p, p + 1)$ case where $p \in \mathbb{Z}^+$, this number is equal to M.

in each inequality, on the right-hand-side, there is an $Nn\log\rho$ term plus extra terms. These extra terms, if added through the total of M inequalities, produce the term $(M-1)nR$ subject to the noise distortion, i.e.,

$$\frac{1}{3}\sum_{k=1}^{3}\sum_{m=1}^{M} h(\mathcal{G}_{km}^{M}|\bar{Y}_k) = (M-1)nR + n\ o(\log\rho) + o(n) \tag{89}$$

where $\frac{1}{3}\sum_{k=1}^{3}$ indicates averaging over user indices, and $\mathcal{G}_{km}^{M}$ denotes the m^{th} set of genie signals provided to Receiver k for the $(p, p+1)$ case. In this paper we also write $\mathcal{G}_{1m}^{M}$ as $\mathcal{G}_{m}^{M}$ or $\mathcal{G}_m$ with respect to Receiver 1 for simplicity if there is no ambiguity, and $\mathcal{G}_{2m}, \mathcal{G}_{3m}$ can be obtained by advancing user indices of $\mathcal{G}_m$. Therefore, we have the following inequality:

$$3MnR \leq MNn\log\rho + (M-1)nR + n\ o(\log\rho) + o(n) \tag{90}$$

which implies the DoF per user bound

$$d \leq \frac{MN}{2M+1} = \frac{MN}{M+N}. \tag{91}$$

The arguments we show above based on the onion peeling intuition are summarized in Table 1 where $\mathcal{G}_{m}^{M}$ denotes the m^{th} genie signals set provided to Receiver 1 for the $(M, N) = (p, p+1)$ case. From this table, it can be easily seen that for the $(M, M+1)$ case, the first $M-2$ genie signal sets, i.e., $\mathcal{G}_{m}^{M},\ m = 1, 2, \cdots, M-2$, are always equal to $\mathcal{G}_{m}^{M-2},\ m = 1, 2, \cdots, M-2$ plus $(X_{2a_L}^{n}, X_{3c_L}^{n})$. This observation builds the foundation of the recursive proof that will be shown in detail in Appendix A.

Table 1: Examples of Genie Signal Sets Provided to Receiver 1 for the $(M, M+1)$ Setting where $M = 2, 3, \cdots, 7$ (Note that in the previous analysis, for $L = 1$ cases we omit the layer indices associated with the antennas for simplicity.)

Layers	$L=1$	$L=2$	$L=3$	$\cdots$
M is even $M=2L$	$\mathcal{G}_1^2 = \{X_{2a_1}^n\}$ $\mathcal{G}_2^2 = \{X_{3c_1}^n\}$	$\mathcal{G}_1^4 = \{\mathcal{G}_1^2, X_{2a_2}^n, X_{3c_2}^n\}$ $\mathcal{G}_2^4 = \{\mathcal{G}_2^2, X_{2a_2}^n, X_{3c_2}^n\}$ $\mathcal{G}_3^4$ $\mathcal{G}_4^4$	$\mathcal{G}_1^6 = \{\mathcal{G}_1^4, X_{2a_3}^n, X_{3c_3}^n\}$ $\mathcal{G}_2^6 = \{\mathcal{G}_2^4, X_{2a_3}^n, X_{3c_3}^n\}$ $\mathcal{G}_3^6 = \{\mathcal{G}_3^4, X_{2a_3}^n, X_{3c_3}^n\}$ $\mathcal{G}_4^6 = \{\mathcal{G}_4^4, X_{2a_3}^n, X_{3c_3}^n\}$ $\mathcal{G}_5^6$ $\mathcal{G}_6^6$	$\cdots$
M is odd $M=2L+1$	$\mathcal{G}_1^3 = \{X_{2a_1}^n, X_{3c_1}^n\}$ $\mathcal{G}_2^3 = \{X_{2a_1}^n, X_{2b_1}^n\}$ $\mathcal{G}_3^3 = \{X_{3b_1}^n, X_{3c_1}^n\}$	$\mathcal{G}_1^5 = \{\mathcal{G}_1^3, X_{2a_2}^n, X_{3c_2}^n\}$ $\mathcal{G}_2^5 = \{\mathcal{G}_2^3, X_{2a_2}^n, X_{3c_2}^n\}$ $\mathcal{G}_3^5 = \{\mathcal{G}_3^3, X_{2a_2}^n, X_{3c_2}^n\}$ $\mathcal{G}_4^5$ $\mathcal{G}_5^5$	$\mathcal{G}_1^7 = \{\mathcal{G}_1^5, X_{2a_3}^n, X_{3c_3}^n\}$ $\mathcal{G}_2^7 = \{\mathcal{G}_2^5, X_{2a_3}^n, X_{3c_3}^n\}$ $\mathcal{G}_3^7 = \{\mathcal{G}_3^5, X_{2a_3}^n, X_{3c_3}^n\}$ $\mathcal{G}_4^7 = \{\mathcal{G}_4^5, X_{2a_3}^n, X_{3c_3}^n\}$ $\mathcal{G}_5^7 = \{\mathcal{G}_5^5, X_{2a_3}^n, X_{3c_3}^n\}$ $\mathcal{G}_6^7$ $\mathcal{G}_7^7$	$\cdots$

For general $(M, N) = (p, p+1)$ where $p > 4$ as well as $M/N \neq p/(p+1)$ cases, since the proofs are much more cumbersome but still follow from the intuition shown above, we defer all of them to Appendix A.

For the setting $M_T > M_R$, although the provided genie signal and the general information theoretic proofs are different from those for the setting $M_T < M_R$, they all follow from the onion peeling intuition introduced in Section 6. When presenting the proofs for the $M_T > M_R$ setting, we will begin with the proof for the $(M_T, M_R) = (3, 2)$ case first in Appendix B.1.1 as a motivating example, and then provide general proofs in the remainder of Appendix B.

8 DoF Achievability

The achievable schemes are based on linear beamforming at the transmitters and zero forcing at the receivers. Due to the reciprocity of the linear scheme, without loss of generality, we only provide achievable schemes for the case when $M_T < M_R$, so that $M_T = M$ and $M_R = N$.

As we explained in Section 5.1, when $M/N \leq 1/2$, interference alignment is not needed and zero-forcing at receivers is sufficient to achieve the optimal DoF. When $M/N > 1/2$, interference alignment is needed. In this regime, as mentioned in Section 5.3.1, only for $M/N \in \mathcal{B} = \{1/3, 3/5, 5/7, 7/9, \cdots\}$, neither M nor N contains any redundant dimensions. For all the other cases, either M or N can be reduced without losing DoF. Therefore, we can reduce M or N depending on their ratio such that the ratio becomes one of the elements in $\mathcal{B}$ to achieve the same DoF for the original M and N. As a consequence, we only need to provide achievable schemes for the case when $M/N \in \mathcal{B}$ and $M/N > 1/2$. The achievable schemes for all the other cases follow directly from these schemes.

8.1 Case: $M/N > 1/2$ and $M/N = (2p-1)/(2p+1)$

The cases $M/N > 1/2$ and $M/N \in \mathcal{B}$ correspond to the settings where each transmitter has $(2p-1)q$ antennas and each receiver has $(2p+1)q$ antennas, $\forall p, q \in \mathbb{Z}^+, p \geq 2$. We will show that each user can achieve pq DoF, for a total of $3pq$ DoF. Since an integer DoF value can be achieved, symbol extensions or spatial extensions are not needed.

The achievable scheme can be understood intuitively as follows. As explain in Section 5.2, the length of the subspace alignment chain is $\kappa_N = \lceil \frac{M}{N-M} \rceil = p$. Each subspace that participates in the chain is q dimensional and thus a total of pq DoF are sent for a chain, occupying a total of $(2p+1)q$ dimensions at all receivers. Since there are $(2p+1)q$ dimensions at each receiver, a total of three subspace alignment chains, each originating from one of three transmitters, can be packed for this channel, thus achieving $3pq$ DoF. Next, we will show how to design these three subspace alignment chains. Before we present the general achievable schemes, let us first consider two simple cases $p = 2, 3$.

8.1.1 Case: $(M, N) = (3q, 5q)$ $(p = 2)$

For the $3q \times 5q$ case, we will show each user can achieve $2q$ DoF. Specifically, Transmitter i sends $2q$ independent symbols using a $3q \times 2q$ beamforming matrix $\mathbf{V}_i$. First note that the length of the alignment chain is 2 in this case, so there are two signal spaces that participate in each alignment chain. Due to symmetry, there are also two signal spaces at each transmitter that participate in the other two alignment chains, one for each chain. In addition, since there are three alignment chains, we will use a superscript to denote the chain index to which the subspaces belong. Specifically, we use $\mathbf{V}_{i(s)}^k$ where $k, i \in \{1, 2, 3\}$ to denote the s^{th} q-dimensional subspace at Transmitter i that participates in the k^{th} chain, and in this case $s = 1$.

Let us start with the first alignment chain which originates from Transmitter 1:

$$\mathbf{V}^1_{1(1)} \overset{\text{Rx 2}}{\longleftrightarrow} \mathbf{V}^1_{3(1)}. \tag{92}$$

Mathematically, the alignment equation is

$$\mathbf{H}_{21}\mathbf{V}^1_{1(1)} = \mathbf{H}_{23}\mathbf{V}^1_{3(1)} \tag{93}$$

$$\Rightarrow \underbrace{[\mathbf{H}_{21} \quad -\mathbf{H}_{23}]_{5q\times 6q}}_{\mathbf{A}_p} \underbrace{\begin{bmatrix} \mathbf{V}^1_{1(1)} \\ \mathbf{V}^1_{3(1)} \end{bmatrix}}_{\mathbf{a}_p} = \mathbf{0}. \tag{94}$$

Since $\mathbf{A}_p$ is a generic $5q \times 6q$ matrix, and thus it has full row rank, $\mathbf{a}_p$ can be determined as q basis vectors of the null space of $\mathbf{A}_p$, and it can be represented explicitly as

$$\mathbf{a}_p = (\mathbf{I} - \mathbf{A}_p^H(\mathbf{A}_p\mathbf{A}_p^H)^{-1}\mathbf{A}_p)\det(\mathbf{A}_p\mathbf{A}_p^H)\mathbf{R}_\mathbf{a}$$

where $\mathbf{R}_\mathbf{a}$ is a randomly picked $6q \times q$ matrix. Thus, we have $\mathbf{V}^1_{1(1)} = \mathbf{a}_p(1:3q,:)$ and $\mathbf{V}^1_{3(1)} = \mathbf{a}_p(3q+1:6q,:)$.

Next, the second alignment chain which originates from Transmitter 2 is

$$\mathbf{V}^2_{2(1)} \overset{\text{Rx 3}}{\longleftrightarrow} \mathbf{V}^2_{1(1)}. \tag{95}$$

Mathematically, the second alignment equation can be written as

$$\mathbf{H}_{32}\mathbf{V}^2_{2(1)} = \mathbf{H}_{31}\mathbf{V}^2_{1(1)} \tag{96}$$

$$\Rightarrow \underbrace{[\mathbf{H}_{32} \quad -\mathbf{H}_{31}]}_{\mathbf{B}_p} \underbrace{\begin{bmatrix} \mathbf{V}^2_{2(1)} \\ \mathbf{V}^2_{1(1)} \end{bmatrix}}_{\mathbf{b}_p} = \mathbf{0}. \tag{97}$$

Again, since $\mathbf{B}_p$ is a generic $5q \times 6q$ matrix, $\mathbf{b}_p$ is chosen as q basis vectors of the null space of $\mathbf{B}_p$, which is given by $\mathbf{b}_p = (\mathbf{I} - \mathbf{B}_p^H(\mathbf{B}_p\mathbf{B}_p^H)^{-1}\mathbf{B}_p)\det(\mathbf{B}_p\mathbf{B}_p^H)\mathbf{R}_\mathbf{b}$ where $\mathbf{R}_\mathbf{b}$ is a randomly picked $6q \times q$ matrix. Thus, we obtain $\mathbf{V}^2_{2(1)} = \mathbf{b}_p(1:3q,:)$ and $\mathbf{V}^2_{1(1)} = \mathbf{b}_p(3q+1:6q,:)$.

The third alignment chain which originates from Transmitter 3 is

$$\mathbf{V}^3_{3(1)} \overset{\text{Rx 1}}{\longleftrightarrow} \mathbf{V}^3_{2(1)} \tag{98}$$

which produces the third alignment equation:

$$\mathbf{H}_{13}\mathbf{V}^3_{3(1)} = \mathbf{H}_{12}\mathbf{V}^3_{2(1)} \tag{99}$$

$$\Rightarrow \underbrace{[\mathbf{H}_{13} \quad -\mathbf{H}_{12}]}_{\mathbf{C}_p} \underbrace{\begin{bmatrix} \mathbf{V}^3_{3(1)} \\ \mathbf{V}^3_{2(1)} \end{bmatrix}}_{\mathbf{c}_p} = \mathbf{0}. \tag{100}$$

Here, $\mathbf{c}_p$ is chosen as q basis vectors of the null space of the generic matrix $\mathbf{C}_p$, which is given by $\mathbf{c}_p = (\mathbf{I} - \mathbf{C}_p^H(\mathbf{C}_p\mathbf{B}_p^H)^{-1}\mathbf{C}_p)\det(\mathbf{C}_p\mathbf{C}_p^H)\mathbf{R}_\mathbf{c}$ where $\mathbf{R}_\mathbf{c}$ is a randomly picked $6q \times q$ matrix. Thus, we have $\mathbf{V}^3_{3(1)} = \mathbf{c}_p(1:3q,:)$ and $\mathbf{V}^3_{2(1)} = \mathbf{c}_p(3q+1:6q,:)$.

Finally, the beamforming vectors of each transmitter can be written as follows.

$$\mathbf{V}_1 = [\mathbf{V}_{1(1)}^1 \;\; \mathbf{V}_{1(1)}^2], \quad \mathbf{V}_2 = [\mathbf{V}_{2(1)}^2 \;\; \mathbf{V}_{2(1)}^3], \quad \mathbf{V}_3 = [\mathbf{V}_{3(1)}^1 \;\; \mathbf{V}_{3(1)}^3].$$

After aligning interference, we ensure that the total number of dimensions occupied by the interference is small enough. To decode the desired signals, the desired signal space cannot overlap with the space of interference. This is guaranteed by an independent linear transformation of the desired signals from the transmitter to its corresponding receiver. Note that the direct channels are not used to align interference and they are generic channel matrices without special structure. Therefore, after going through an independent linear transformation, the desired signals do not overlap with the interference almost surely. The linear independence of the chosen beamforming vectors is shown at the end of Section 8.1.3 for general cases.

8.1.2 Case: $(M, N) = (5q, 7q)$ $(p = 3)$

Now let us consider the case $5q \times 7q$, for which every user can achieve $3q$ DoF. Again, we need to design three alignment chains each with length 3, each originating from one of three transmitters.

The first alignment chain which originates from Transmitter 1 is

$$\mathbf{V}_{1(1)}^1 \overset{\text{Rx 2}}{\longleftrightarrow} \mathbf{V}_{3(1)}^1 \overset{\text{Rx 1}}{\longleftrightarrow} \mathbf{V}_{2(1)}^1.$$

Mathematically, the alignment equations are

$$\begin{cases} \mathbf{H}_{21}\mathbf{V}_{1(1)}^1 &= \mathbf{H}_{23}\mathbf{V}_{3(1)}^1 \\ \mathbf{H}_{13}\mathbf{V}_{3(1)}^1 &= \mathbf{H}_{12}\mathbf{V}_{2(1)}^1 \end{cases} \tag{101}$$

$$\Rightarrow \underbrace{\begin{bmatrix} \mathbf{H}_{21} & -\mathbf{H}_{23} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{13} & -\mathbf{H}_{12} \end{bmatrix}_{14q \times 15q}}_{\mathbf{A}_p} \underbrace{\begin{bmatrix} \mathbf{V}_{1(1)}^1 \\ \mathbf{V}_{3(1)}^1 \\ \mathbf{V}_{2(1)}^1 \end{bmatrix}}_{\mathbf{a}_p} = \mathbf{0}. \tag{102}$$

Note that $\mathbf{A}_p$ is a $14q \times 15q$ matrix which has full row rank for generic channel realizations [28]. Therefore, $\mathbf{a}_p$ is chosen as the q basis of the null space of $\mathbf{A}_p$, i.e.,

$$\mathbf{a}_p = (\mathbf{I} - \mathbf{A}_p^H(\mathbf{A}_p\mathbf{A}_p^H)^{-1}\mathbf{A}_p)\det(\mathbf{A}_p\mathbf{A}_p^H)\mathbf{R}_\mathbf{a}$$

where $\mathbf{R}_\mathbf{a}$ is a randomly picked $15q \times q$ matrix. Thus, we obtain $\mathbf{V}_{1(1)}^1 = \mathbf{a}_p(1 : 5q, :)$, $\mathbf{V}_{2(1)}^1 = \mathbf{a}_p(5q + 1 : 10q, :)$, $\mathbf{V}_{3(1)}^1 = \mathbf{a}_p(10q + 1 : 15q, :)$.

Then, the second alignment chain which originates from Transmitter 2 is

$$\mathbf{V}_{2(1)}^2 \overset{\text{Rx 3}}{\longleftrightarrow} \mathbf{V}_{1(1)}^2 \overset{\text{Rx 2}}{\longleftrightarrow} \mathbf{V}_{3(1)}^2.$$

Mathematically, we have the alignment equations

$$\begin{cases} \mathbf{H}_{32}\mathbf{V}_{2(1)}^2 &= \mathbf{H}_{31}\mathbf{V}_{1(1)}^2 \\ \mathbf{H}_{21}\mathbf{V}_{1(1)}^2 &= \mathbf{H}_{23}\mathbf{V}_{3(1)}^2 \end{cases} \tag{103}$$

$$\Rightarrow \underbrace{\begin{bmatrix} \mathbf{H}_{32} & -\mathbf{H}_{31} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{21} & -\mathbf{H}_{23} \end{bmatrix}_{14q \times 15q}}_{\mathbf{B}_p} \underbrace{\begin{bmatrix} \mathbf{V}_{2(1)}^2 \\ \mathbf{V}_{1(1)}^2 \\ \mathbf{V}_{3(1)}^2 \end{bmatrix}}_{\mathbf{b}_p} = \mathbf{0}. \tag{104}$$

Similarly, $\mathbf{b}_p$ is chosen as the q basis of the null space of $\mathbf{B}_p$, which is given by $\mathbf{b}_p = (\mathbf{I} - \mathbf{B}_p^H(\mathbf{B}_p\mathbf{B}_p^H)^{-1}\mathbf{B}_p)\det(\mathbf{B}_p\mathbf{B}_p^H)\mathbf{R_b}$ where $\mathbf{R_b}$ is a randomly picked $15q \times q$ matrix. Thus, we obtain $\mathbf{V}_{2(1)}^2 = \mathbf{b}_p(1:5q,:)$, $\mathbf{V}_{1(1)}^2 = \mathbf{b}_p(5q+1:10q,:)$, $\mathbf{V}_{3(1)}^2 = \mathbf{b}_p(10q+1:15q,:)$.

The third alignment chain which originates from Transmitter 3 is

$$\mathbf{V}_{3(1)}^3 \overset{\text{Rx 1}}{\longleftrightarrow} \mathbf{V}_{2(1)}^3 \overset{\text{Rx 3}}{\longleftrightarrow} \mathbf{V}_{1(1)}^3.$$

This subspace alignment chain produces the following alignment equations:

$$\begin{cases} \mathbf{H}_{13}\mathbf{V}_{3(1)}^3 & = & \mathbf{H}_{12}\mathbf{V}_{2(1)}^3 \\ \mathbf{H}_{32}\mathbf{V}_{2(1)}^2 & = & \mathbf{H}_{31}\mathbf{V}_{1(1)}^3 \end{cases} \tag{105}$$

$$\Rightarrow \underbrace{\begin{bmatrix} \mathbf{H}_{13} & -\mathbf{H}_{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{32} & -\mathbf{H}_{31} \end{bmatrix}_{14q\times 15q}}_{\mathbf{C}_p} \underbrace{\begin{bmatrix} \mathbf{V}_{3(1)}^3 \\ \mathbf{V}_{2(1)}^3 \\ \mathbf{V}_{1(1)}^3 \end{bmatrix}}_{\mathbf{c}_p} = \mathbf{0}. \tag{106}$$

Thus, we can also determine $\mathbf{c}_p$ as the q basis of the null space of $\mathbf{C}_p$, which is given by $\mathbf{c}_p = (\mathbf{I} - \mathbf{C}_p^H(\mathbf{C}_p\mathbf{C}_p^H)^{-1}\mathbf{C}_p)\det(\mathbf{C}_p\mathbf{C}_p^H)\mathbf{R_c}$ where $\mathbf{R_c}$ is a randomly picked $15q \times q$ matrix. Thus, we have $\mathbf{V}_{3(1)}^3 = \mathbf{c}_p(1:5q,:)$, $\mathbf{V}_{2(1)}^3 = \mathbf{c}_p(5q+1:10q,:)$ and $\mathbf{V}_{1(1)}^3 = \mathbf{c}_p(10q+1:15q,:)$.

Finally, the beamforming vectors of each Transmitter i can be written as follows.

$$\mathbf{V}_i = [\mathbf{V}_{i(1)}^1 \ \ \mathbf{V}_{i(1)}^2 \ \ \mathbf{V}_{i(1)}^3], \qquad i = 1,2,3.$$

The linear independence of the chosen beamforming vectors is shown at the end of Section 8.1.3 for general cases.

8.1.3 Case: $(M,N) = ((2p-1)q, (2p+1)q)$

Now, consider the general $(2p-1)q \times (2p+1)q$ case. Again, we will design three q-dimensional subspace alignment chains with length p, each originating from one of three transmitters. Let $\mathbf{V}_{i(s)}^k$, $k,i \in \{1,2,3\}, s \in \{1,\cdots,\lceil p/3 \rceil\}$ denote the s^{th} q-dimensional subspace at Transmitter i that participates in the k^{th} subspace alignment chain. In addition, let $\underline{p} = (p \mod 3)$ and $\overline{p} = \lceil p/3 \rceil$. Next, we can write three subspace alignment chains.

We start with the first alignment chain which originates from Transmitter 1. Let $a(1) = 1, a(2) = 3, a(0) = 2$. Then the first alignment chain is

$$\mathbf{V}_{1(1)}^1 \overset{\text{Rx 2}}{\longleftrightarrow} \mathbf{V}_{3(1)}^1 \overset{\text{Rx 1}}{\longleftrightarrow} \mathbf{V}_{2(1)}^1 \overset{\text{Rx 3}}{\longleftrightarrow} \mathbf{V}_{1(2)}^1 \overset{\text{Rx 2}}{\longleftrightarrow} \mathbf{V}_{3(2)}^1 \overset{\text{Rx 1}}{\longleftrightarrow} \mathbf{V}_{2(2)}^1 \cdots \mathbf{V}_{a(\underline{p-1})(\overline{p-1})}^1 \overset{\text{Rx } a(\underline{p+1})}{\longleftrightarrow} \mathbf{V}_{a(\underline{p})(\overline{p})}^1. \tag{107}$$

Mathematically, we have the following alignment equation:

$$\mathbf{A}_p\mathbf{a}_p = \mathbf{0} \tag{108}$$

where

$$\mathbf{A}_p = \begin{bmatrix} \mathbf{H}_{21} & -\mathbf{H}_{23} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{13} & -\mathbf{H}_{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_{32} & -\mathbf{H}_{31} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H}_{a(\underline{p+1})a(\underline{p-1})} & -\mathbf{H}_{a(\underline{p+1})a(\underline{p})} \end{bmatrix}, \quad \mathbf{a}_p = \begin{bmatrix} \mathbf{V}_{1(1)}^1 \\ \mathbf{V}_{3(1)}^1 \\ \vdots \\ \mathbf{V}_{a(\underline{p})(\overline{p})}^1 \end{bmatrix}. \tag{109}$$

Note that $\mathbf{A}_p$ is a $(p-1)(2p+1)q \times p(2p-1)q$ matrix, shown to have full row rank for generic channel realizations [28]. Thus, q columns of $\mathbf{a}_p$ are chosen as the q basis vectors of the null space of $\mathbf{A}_p$, so that $\mathbf{a}_p = (\mathbf{I} - \mathbf{A}_p^H(\mathbf{A}_p\mathbf{A}_p^H)^{-1}\mathbf{A}_p)\det(\mathbf{A}_p\mathbf{A}_p^H)\mathbf{R_a}$ where $\mathbf{R_a}$ is a randomly picked $p(2p-1)q \times q$ matrix. From $\mathbf{a}_p$, we obtain $\mathbf{V}^1_{1(1)}, \cdots, \mathbf{V}^1_{a(\underline{p})(\overline{p})}$.

Similarly, let $b(1) = 2, b(2) = 1, b(0) = 3$. Then the second alignment chain which originates from Transmitter 2 is

$$\mathbf{V}^2_{2(1)} \xleftrightarrow{\text{Rx } 3} \mathbf{V}^2_{1(1)} \xleftrightarrow{\text{Rx } 2} \mathbf{V}^2_{3(1)} \xleftrightarrow{\text{Rx } 1} \mathbf{V}^2_{2(2)} \xleftrightarrow{\text{Rx } 3} \mathbf{V}^2_{1(2)} \xleftrightarrow{\text{Rx } 2} \mathbf{V}^2_{3(2)} \cdots \mathbf{V}^2_{b(\underline{p-1})(\overline{p-1})} \xleftrightarrow{\text{Rx } b(p+1)} \mathbf{V}^2_{b(\underline{p})(\overline{p})}. \tag{110}$$

Mathematically, we have the second alignment equation:

$$\mathbf{B}_p\mathbf{b}_p = \mathbf{0} \tag{111}$$

where

$$\mathbf{B}_p = \begin{bmatrix} \mathbf{H}_{32} & -\mathbf{H}_{31} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{21} & -\mathbf{H}_{23} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_{13} & -\mathbf{H}_{12} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H}_{b(\underline{p+1})b(\underline{p-1})} & -\mathbf{H}_{b(\underline{p+1})b(\underline{p})} \end{bmatrix}, \quad \mathbf{b}_p = \begin{bmatrix} \mathbf{V}^2_{2(1)} \\ \mathbf{V}^2_{1(1)} \\ \vdots \\ \mathbf{V}^2_{b(\underline{p})(\overline{p})} \end{bmatrix}. \tag{112}$$

Again, $\mathbf{B}_p$ is a $(p-1)(2p+1)q \times p(2p-1)q$ matrix with full row rank [28] for generic channel realizations. So the q columns of $\mathbf{b}_p$ are chosen as the q basis vectors of the null space of $\mathbf{B}_p$, which is given by $\mathbf{b}_p = (\mathbf{I} - \mathbf{B}_p^H(\mathbf{B}_p\mathbf{B}_p^H)^{-1}\mathbf{B}_p)\det(\mathbf{B}_p\mathbf{B}_p^H)\mathbf{R_b}$ where $\mathbf{R_b}$ is a randomly picked $p(2p-1)q \times q$ matrix. From $\mathbf{b}_p$, we obtain $\mathbf{V}^2_{2(1)}, \cdots, \mathbf{V}^2_{b(\underline{p})(\overline{p})}$.

Finally, let $c(1) = 3, c(2) = 2, c(0) = 1$. Then the third alignment chain which originates from Transmitter 3 is

$$\mathbf{V}^3_{3(1)} \xleftrightarrow{\text{Rx } 1} \mathbf{V}^3_{2(1)} \xleftrightarrow{\text{Rx } 3} \mathbf{V}^3_{1(1)} \xleftrightarrow{\text{Rx } 2} \mathbf{V}^3_{3(2)} \xleftrightarrow{\text{Rx } 1} \mathbf{V}^3_{2(2)} \xleftrightarrow{\text{Rx } 3} \mathbf{V}^3_{1(2)} \cdots \mathbf{V}^3_{c(\underline{p-1})(\overline{p-1})} \xleftrightarrow{\text{Rx } c(p+1)} \mathbf{V}^3_{c(\underline{p})(\overline{p})}. \tag{113}$$

Mathematically, this subspace alignment chain implies the following alignment equation:

$$\mathbf{C}_p\mathbf{c}_p = \mathbf{0} \tag{114}$$

where

$$\mathbf{C}_p = \begin{bmatrix} \mathbf{H}_{13} & -\mathbf{H}_{12} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{32} & -\mathbf{H}_{31} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_{21} & -\mathbf{H}_{23} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H}_{c(\underline{p+1})c(\underline{p-1})} & -\mathbf{H}_{c(\underline{p+1})c(\underline{p})} \end{bmatrix}, \quad \mathbf{c}_p = \begin{bmatrix} \mathbf{V}^3_{3(1)} \\ \mathbf{V}^3_{2(1)} \\ \vdots \\ \mathbf{V}^3_{b(\underline{p})(\overline{p})} \end{bmatrix}. \tag{115}$$

The q columns of $\mathbf{c}_p$ are chosen as the q basis vectors of the null space of $\mathbf{C}_p$ which has full row rank for generic channel realizations [28]. Thus, we have $\mathbf{c}_p = (\mathbf{I} - \mathbf{C}_p^H(\mathbf{C}_p\mathbf{C}_p^H)^{-1}\mathbf{C}_p)\det(\mathbf{C}_p\mathbf{C}_p^H)\mathbf{R_c}$ where $\mathbf{R_c}$ is a randomly picked $p(2p-1)q \times q$ matrix. From $\mathbf{c}_p$, we obtain $\mathbf{V}^3_{3(1)}, \cdots, \mathbf{V}^3_{b(\underline{p})(\overline{p})}$. By concatenating the beamforming vectors that we obtain for each transmitter through the three subspace alignment chains above, we obtain the entire beamforming matrix of each transmitter.

After the interference is aligned, we ensure that the dimension of the space spanned by the interference is small enough. To make sure each user can achieve pq DoF, it remains to show 1) the space spanned by the desired signal and that spanned by the interference do not overlap, and 2) the constructed beamforming vectors for each user using the proposed alignment scheme are linearly independent. For the first one, since the direct channels do not appear in the alignment equations and they are generic without any special structure, the desired signals and interference do not overlap each other almost surely. Next we prove the second one.

First note that all entries of the beamforming vectors are ratios of polynomial functions of the channel coefficients. To prove that the $(2p-1)q \times pq$ beamforming matrix $\mathbf{V}_i$ at transmitter i is full rank almost surely, it suffices to show that the determinant of the following square matrix is not equal to zero almost surely:

$$\mathbf{V}_i' = [\mathbf{V}_i \ \mathbf{U}_i] \tag{116}$$

where $\mathbf{U}_i$ is a randomly generated $(2p-1)q \times (p-1)q$ matrix. Now the determinant of the above matrix is a polynomial function of its entries, i.e., the channel coefficients $\{\mathbf{H}_{ji}\}$ and the randomly picked projection matrices $\mathbf{R_a}$, $\mathbf{R_b}$ and $\mathbf{R_c}$. This polynomial is either a zero polynomial or not equal to zero almost surely, since the solutions of a non-zero polynomial equation have measure zero. Next, we will show the polynomial is not a zero polynomial. To do that, we only need to find one specific set of realization of channel coefficients and picked matrices such that the polynomial is not equal to zero [28]. For brevity, we next construct the channels for the case when $q = 1$. The construction for arbitrary q follows in a straightforward manner.

Suppose the channel matrix $\mathbf{H}_{(i-1)i}$ is

$$\mathbf{H}_{(i-1)i} = \begin{bmatrix} \mathbf{I}_{(p-1)\times(p-1)} & \mathbf{0}_{(p-1)\times p} \\ \mathbf{0}_{1\times(p-1)} & \mathbf{0}_{1\times p} \\ \mathbf{0}_{p\times(p-1)} & \mathbf{I}_{p\times p} \\ \mathbf{0}_{1\times(p-1)} & \mathbf{0}_{1\times p} \end{bmatrix}, \forall i \in \{1,2,3\}. \tag{117}$$

And the channel matrix $\mathbf{H}_{(i+1)i}$ is

$$\mathbf{H}_{(i+1)i} = \begin{bmatrix} \mathbf{0}_{1\times(p-1)} & \mathbf{0}_{1\times p} \\ \mathbf{I}_{(p-1)\times(p-1)} & \mathbf{0}_{(p-1)\times p} \\ \mathbf{0}_{1\times(p-1)} & \mathbf{0}_{1\times p} \\ \mathbf{0}_{p\times(p-1)} & \mathbf{I}_{p\times p} \end{bmatrix}, \forall i \in \{1,2,3\}. \tag{118}$$

It can be verified that the alignment schemes proposed before produce the following beamforming matrix for each transmitter:

$$\mathbf{V}_i = \begin{bmatrix} \mathbf{0}_{(p-1)\times p} \\ \mathbf{I}_{p\times p} \end{bmatrix} \tag{119}$$

With this full rank beamforming matrix, it can be easily seen that $\mathbf{V}_i'$ has full rank and its determinant is not equal to zero almost surely. Therefore, the beamforming vectors of each user designed by the alignment schemes before are linear independent almost surely.

8.2 Achievability for General Cases with only Space Extension

With the understanding of the achievable schemes for the settings $M/N = (2p-1)/(2p+1)$ for $p = 2, 3, \cdots$, we can consider the achievable schemes for all other cases. Recall that the optimal

DoF given by Theorem 1 are as follows:

$$\overline{\text{DoF}} = \begin{cases} \frac{p}{2p-1}M, & \frac{p-1}{p} \leq \frac{M}{N} \leq \frac{2p-1}{2p+1} \\ \frac{p}{2p+1}N, & \frac{2p-1}{2p+1} \leq \frac{M}{N} \leq \frac{p}{p+1} \end{cases} \qquad p = 2, 3, \cdots . \tag{120}$$

We will prove that for a finite integer q such that qd is an integer, the $qM \times qN$ interference channel achieves qd DoF per user. Therefore, the spatially-normalized DoF are d.

Let us first consider the case $\frac{p-1}{p} \leq \frac{M}{N} \leq \frac{2p-1}{2p+1}$. In this regime, the DoF only depend on M, i.e., $d = \frac{p}{2p-1}M$. In other words, the bottleneck of the DoF in this regime is the value of M. If we fix M, that DoF value does not change even if N changes as long as the ratio remains in that regime. If M is an integer multiple of $2p-1$, we can reduce the number of antennas N such that the ratio becomes $\frac{2p-1}{2p+1}$. Once the ratio is equal to $\frac{2p-1}{2p+1}$, the achievable scheme proposed in last subsection can be directly applied. On the other hand, if M is *not* an integer multiple of $2p-1$, we scale M and N by a factor of $2p-1$ to obtain a $(2p-1)M \times (2p-1)N$ interference channel which is a spatially scaled version of the original channel. Again for this channel, we reduce the number of antennas at the receiver from $(2p-1)N$ to $(2p+1)M$ such that the ratio of the number of transmit and receive antennas is equal to $\frac{2p-1}{2p+1}$. By applying the achievable scheme in last subsection, we can achieve pM DoF per user. Therefore, the spatially-normalized DoF of $\frac{p}{2p-1}M$ per user can be achieved for the original $M \times N$ interference channel.

Now consider the other case $\frac{2p-1}{2p+1} \leq \frac{M}{N} \leq \frac{p}{p+1}$, where the optimal DoF per user, $\frac{p}{2p+1}N$, only depends on N. Similar to the previous case, if N is an integer multiple of $2p+1$, we can simply reduce the number of transmit antennas (i.e., some transmit antennas are redundant), such that the ratio becomes $\frac{2p-1}{2p+1}$. Therefore, the achievable scheme proposed before can be applied here. If N is not an integer $2p+1$, we can scale M and N by a factor of $2p+1$. By reducing the number of transmit antennas for the spatially scaled version of the original channel, we can achieve pN DoF per user, thus spatially-normalized $\frac{p}{2p+1}N$ DoF for the original channel.

8.3 Achievability with Symbol Extensions in Time/Frequency Domains

While we have shown the achievability for all cases (with spatial normalization for the cases where fractional DoF are optimal), one question that remains unresolved is whether the outer bound can be achieved as well if we restrict to only symbol extensions over time/frequency, i.e., without spatial normalization. The difficulty is to deal with the added complexity of block diagonal channel structure that would result from channel extensions. In other words, the channels are not generic with symbol extensions. As a consequence, although the alignment schemes presented before can still be applied to align interference, it is not clear whether the desired signals remain distinguishable from interference after going through the direct channel due to the special channel structure. In this subsection, we first explore some simple cases and show that the outer bound is still achievable for these simple settings with linear schemes using symbol extensions. Then we will construct a linear scheme with symbol extensions for general cases. While in many cases simple symbol extensions in time over constant channels are sufficient to achieve the information theoretic DoF outer bound, there are also cases where either time-variations/frequency-selectivity of the channel or asymmetric complex signaling over constant channels [27] may be needed along with the linear interference alignment schemes that we introduce in this paper. The feasibility of such schemes can be resolved (with high probability) through a numerical test.

Let us start with some settings where simple symbol extensions over constant channels are *not* sufficient for the linear schemes introduced in this paper[5], to achieve the outer bound, i.e., $(M, N) = (p, p+1)$ where $p \geq 2$.

8.3.1 Case: $(M, N) = (p, p+1)$

We will first consider the 2×3 setting and analytically prove that over the *constant* channels, the linear scheme introduced in this paper does not apply to this setting. Such analysis can be generalized to show that for $(M, N) = (p, p+1)$ cases where $p \geq 2$, linear schemes with symbol extensions over constant channels are not sufficient to achieve the DoF outer bound. Then we will introduce time-variation/frequency-selectivity to the channel and show that in this case, linear schemes with symbol extensions can achieve the DoF outer bound.

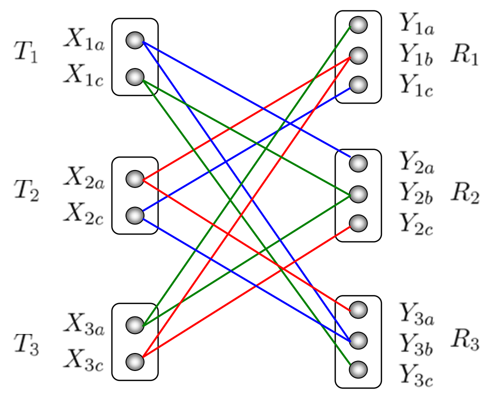


Figure 16: Normalizing the Interference-carrying Links of the 2×3 Setting to Identity Matrices

Let us start with the 2×3 setting. First let us recall the invertible linear transformation introduced in Fig. 8 in Section 6. It can be seen that at Receiver k after subtracting the signal carrying its desired message, the first antenna only sees $X_{(k-1)a}$ and the third antenna only sees $X_{(k+1)a}$. Therefore, Receiver k is able to subtract $X_{(k-1)a}, X_{(k+1)a}$ from S_{kb}, such that the resulting channel connectivity becomes the one shown in Fig. 16. In Fig. 16 there are three open chains, denoted as blue, green and red colors, each implying a subspace alignment chain with length 2. Because they are open loop, we can normalize the channel coefficient of each segment to be one. As a result, we can write the cross channel matrices as follows,

$$\mathbf{H}_{k(k+1)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{H}_{k(k-1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{121}$$

For simplicity, we still use $\mathbf{H}_{kk}$ to denote the direct channel matrix of User k after the change of basis operations. Now Suppose we can achieve $6/5$ DoF per user, which can be done by achieving 6 DoF per user over 5 time slots. With 5 symbol extensions, the effective channel matrix becomes

$$\bar{\mathbf{H}}_{ji} = \mathbf{I}_5 \otimes \mathbf{H}_{ji} \tag{122}$$

where $\mathbf{I}_5$ is the 5×5 identity matrix. To ensure each user can separate 6 desired signal vectors from the interference in its 15-dimensional signal space, the dimension of the space spanned by 12 interference vectors cannot be more than 9. Therefore, at each receiver, we need to align

[5] Asymmetric complex signaling [15] is useful for these cases, as explored in [27].

3 interference vectors. Note that in the signal space at each receiver, there is a 5-dimensional subspace that can be accessed by two interferers. Then we can randomly choose 3-dimensional subspace in this common subspace as the subspace where the 3 interference vectors align. Mapping these 3-dimensional subspace back to the interferers determines the beamforming matrix at the transmitter. Since each transmitter interferes with 2 receivers, each of two unintended receivers will determine 3 beamforming vectors, for a total of 6 beamforming vectors.

With this intuitive understanding, on the symbol extended channel, we write the 10×6 beamforming matrix of user i, $\bar{\mathbf{V}}_i$, as $\bar{\mathbf{V}}_i = \left[\bar{\mathbf{V}}_{i,1} \; \bar{\mathbf{V}}_{i,2}\right]$ where $\bar{\mathbf{V}}_{i,1}$ and $\bar{\mathbf{V}}_{i,2}$ are 10×3 matrices. Then at Receiver 1, we align the first 3 beams from Transmitter 2 with those from Transmitter 3. Mathematically, we have

$$\bar{\mathbf{H}}_{13}\bar{\mathbf{V}}_{3,1} = \bar{\mathbf{H}}_{12}\bar{\mathbf{V}}_{2,1} \tag{123}$$

$$\Rightarrow \quad \underbrace{\left[\bar{\mathbf{H}}_{13} \quad -\bar{\mathbf{H}}_{12}\right]}_{\bar{\mathbf{A}}} \underbrace{\begin{bmatrix} \bar{\mathbf{V}}_{3,1} \\ \bar{\mathbf{V}}_{2,1} \end{bmatrix}}_{\bar{\mathbf{a}}} = \mathbf{0} \tag{124}$$

Since $\bar{\mathbf{A}}$ is a 15×20 matrix, $\bar{\mathbf{a}}$ can be obtained as 3 linearly independent vectors in the 5-dimensional null space of $\bar{\mathbf{A}}$. Notice that the five basis vectors of the null space of $\bar{\mathbf{A}}$ are columns of the matrix $\mathbf{I}_5 \otimes \mathbf{V}_1$ where $\mathbf{V}_1$ is the unique vector in the null space of the 3×4 matrix $[\mathbf{H}_{13} \quad -\mathbf{H}_{12}]$, i.e.,

$$[\mathbf{H}_{13} \quad -\mathbf{H}_{12}]\, \mathbf{V}_1 = \mathbf{0} \tag{125}$$

where

$$[\mathbf{H}_{13} \;\; -\mathbf{H}_{12}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{126}$$

Therefore, the direction of $\mathbf{V}_1$ is uniquely determined and can be written as:

$$\mathbf{V}_1 = [0 \;\; 1 \;\; 1 \;\; 0]^T. \tag{127}$$

Then, we obtain three columns of $\bar{\mathbf{a}}$ as

$$\bar{\mathbf{a}} = (\mathbf{I}_5 \otimes \mathbf{V}_1)\mathbf{a} \tag{128}$$

where $\mathbf{a} = (a_{ij})$ is a 5×3 combining matrix with i.i.d. randomly generated entries.

Similarly, at Receiver 2, we align the first 3 beams from Transmitter 1 with the last 3 beams from Transmitter 3. Mathematically,

$$\bar{\mathbf{H}}_{21}\bar{\mathbf{V}}_{1,1} = \bar{\mathbf{H}}_{23}\bar{\mathbf{V}}_{3,2} \tag{129}$$

$$\Rightarrow \quad \left[\bar{\mathbf{H}}_{21} \quad -\bar{\mathbf{H}}_{23}\right] \underbrace{\begin{bmatrix} \bar{\mathbf{V}}_{1,1} \\ \bar{\mathbf{V}}_{3,2} \end{bmatrix}}_{\bar{\mathbf{b}}} = \mathbf{0} \tag{130}$$

Then we can choose $\bar{\mathbf{b}}$ as

$$\bar{\mathbf{b}} = (\mathbf{I}_5 \otimes \mathbf{V}_2)\mathbf{b} \tag{131}$$

where $\mathbf{V}_2$ is the unique vector in the null space of the 3×4 matrix $[\mathbf{H}_{21} \quad -\mathbf{H}_{23}]$, i.e., $\mathbf{V}_2 = \mathbf{V}_1$, and $\mathbf{b} = (b_{ij})$ is a 5×3 matrix with i.i.d. randomly generated entries.

At Receiver 3, we align the last 3 beams from Transmitter 1 with those from Transmitter 2, whereby we have the equation:

$$\bar{\mathbf{H}}_{32}\bar{\mathbf{V}}_{2,2} = \bar{\mathbf{H}}_{31}\bar{\mathbf{V}}_{1,2} \tag{132}$$

$$\Rightarrow \quad \left[\bar{\mathbf{H}}_{32} \quad -\bar{\mathbf{H}}_{31}\right] \underbrace{\left[\begin{array}{c} \bar{\mathbf{V}}_{2,2} \\ \bar{\mathbf{V}}_{1,2} \end{array}\right]}_{\bar{\mathbf{c}}} = \mathbf{0} \tag{133}$$

Then we can choose $\bar{\mathbf{c}}$ as

$$\bar{\mathbf{c}} = (\mathbf{I}_5 \otimes \mathbf{V}_3)\mathbf{c} \tag{134}$$

where $\mathbf{V}_3$ is the unique vector in the null space of the 3×4 matrix $[\mathbf{H}_{32} \quad -\mathbf{H}_{31}]$, i.e., $\mathbf{V}_3 = \mathbf{V}_1$, and $\mathbf{c}$ is a 5×3 matrix with i.i.d. randomly generated entries.

After aligning interference, we ensure that the dimension of the space spanned by interference is small enough. In order for each receiver to decode the desired message, it remains to see if the desired signals and interference do not overlap at each receiver. Specifically, we need to see if the 15×15 matrix consisting of 6 desired signal vectors and 9 effective interference vectors has full rank. Due to the symmetry of the signaling, let us consider Receiver 1. The 15×15 matrix at Receiver 1 is given by:

$$\mathbf{G} = \left[\bar{\mathbf{H}}_{11}\bar{\mathbf{V}}_{1,1} \quad \bar{\mathbf{H}}_{11}\bar{\mathbf{V}}_{1,2} \quad \bar{\mathbf{H}}_{12}\bar{\mathbf{V}}_{2,1} \quad \bar{\mathbf{H}}_{12}\bar{\mathbf{V}}_{2,2} \quad \bar{\mathbf{H}}_{13}\bar{\mathbf{V}}_{3,2}\right] \tag{135}$$

where $\bar{\mathbf{H}}_{13}\bar{\mathbf{V}}_{3,1}$ does not appear because it aligns with $\bar{\mathbf{H}}_{12}\bar{\mathbf{V}}_{2,1}$ at Receiver 1. Now let us substitute the channels as well as beamforming matrices into the equation above, and rearrange the rows and columns, then we obtain:

$$\mathbf{G} = \left[\begin{array}{ccccc} h_{21}\mathbf{a} & h_{11}\mathbf{b} & \mathbf{0} & \mathbf{0} & \mathbf{a} \\ h_{22}\mathbf{a} & h_{12}\mathbf{b} & \mathbf{0} & \mathbf{c} & \mathbf{0} \\ h_{23}\mathbf{a} & h_{13}\mathbf{b} & \mathbf{b} & \mathbf{0} & \mathbf{0} \end{array}\right] \tag{136}$$

where $h_{ij} = [\bar{\mathbf{H}}_{11}]_{ij}$ and $\mathbf{0}$ is a 5×3 zero matrix. Furthermore, through an invertible linear transformation, $\mathbf{G}$ becomes:

$$\mathbf{G}' = \left[\begin{array}{ccccc} \mathbf{0} & \mathbf{b} & \mathbf{0} & \mathbf{0} & \mathbf{a} \\ \mathbf{a} & \mathbf{b} & \mathbf{0} & \mathbf{c} & \mathbf{0} \\ \mathbf{a} & \mathbf{0} & \mathbf{b} & \mathbf{0} & \mathbf{0} \end{array}\right] \tag{137}$$

In order to see if $\mathbf{G}'$ has full rank, we only need to see if the following equation has non-zero solution:

$$\mathbf{G}' \left[\lambda_1^T \quad \lambda_2^T \quad \lambda_3^T \quad \lambda_4^T \quad \lambda_5^T\right]^T = \mathbf{0} \tag{138}$$

where $\lambda_l \in \mathbb{C}^{3\times 1},\ l = 1, 2, \cdots, 5$. The equation above implies that

$$\left\{\begin{array}{r} \mathbf{a}\lambda_5 + \mathbf{b}\lambda_2 = \mathbf{0} \\ \mathbf{a}\lambda_1 + \mathbf{b}\lambda_2 + \mathbf{c}\lambda_4 = \mathbf{0} \\ \mathbf{a}\lambda_1 + \mathbf{b}\lambda_3 = \mathbf{0}. \end{array}\right. \tag{139}$$

If we let $\lambda_5 = \lambda_1$, $\lambda_3 = \lambda_2$ and $\lambda_4 = \mathbf{0}$, then (139) can be equivalently simplified as:

$$\mathbf{a}\lambda_1 + \mathbf{b}\lambda_2 = \mathbf{0}. \tag{140}$$

Now it can be easily seen that $\mathbf{G}'$ is rank deficient because $[\mathbf{a} \;\; \mathbf{b}]$ is a 5×6 matrix which has at least one-dimensional null space, and thus λ_1, λ_2 can be non-zero. Therefore, we cannot achieve 6/5 DoF per user using the linear beamforming schemes with symbol extensions over the constant channels.

Remark: While we only show the infeasibility of linear beamforming schemes for the 2×3 setting using time extensions over the constant channels, in fact for all $(M, N) = (p, p+1)$ cases where $p \geq 2$, we can take similar invertible linear transformations at each transmitter and receiver, such that the resulting interference-carrying links are identity matrices. The proof of the infeasibility is similar to that for the 2×3 setting, and thus omitted here. On the other hand, although linear schemes introduced in this paper cannot be applied to $(M_T, M_R) = (p, p+1)$ cases where $p \geq 2$ over the constant channels, the DoF outer bounds can still be achieved by rational alignment proposed in [10] or by using the linear scheme with asymmetric complex signaling, as introduced in [27].

While the linear scheme with time extensions over constant channels fails to achieve the DoF outer bound for the setting 2×3, it does not imply that linear scheme is not optimal for this setting in general. In fact, the linear scheme is still sufficient to achieve the optimal DoF if the channel is time-varying or frequency-selective. To see this, let us consider Fig. 16 again. Note that we can always obtain the resulting channel connectivity no matter if the channels are constant or time-varying/ frequency-selective. Therefore, the interference-carrying links are identical to that in (121), and $\mathbf{H}_{kk}(t)$ stands for the direct channel matrices at the t^{th} time slot. Again, consider 5 symbol extensions. Then the effective cross channel matrices become

$$\bar{\mathbf{H}}_{ji} = \mathbf{I}_5 \otimes \mathbf{H}_{ji}, \qquad j \neq i. \tag{141}$$

Note that the direct channel cannot be expressed in the same manner since it varies over time or frequency although it is still a block-diagonal matrix. Then we can express the matrix $\mathbf{G}$ presented in (136) as

$$\mathbf{G} = \begin{bmatrix} \mathbf{R}_{21}^h\mathbf{a} & \mathbf{R}_{11}^h\mathbf{b} & \mathbf{0} & \mathbf{0} & \mathbf{a} \\ \mathbf{R}_{22}^h\mathbf{a} & \mathbf{R}_{12}^h\mathbf{b} & \mathbf{0} & \mathbf{c} & \mathbf{0} \\ \mathbf{R}_{23}^h\mathbf{a} & \mathbf{R}_{13}^h\mathbf{b} & \mathbf{b} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{142}$$

where $\mathbf{R}_{nm}^h,\ n \in \{1, 2, 3\},\ m \in \{1, 2\}$ is given by

$$\mathbf{R}_{nm}^h = \mathrm{diag}\{[h_{nm}(1) \;\; h_{nm}(2) \;\; h_{nm}(3) \;\; h_{nm}(4) \;\; h_{nm}(5)]\} \tag{143}$$

and $h_{nm}(t) = [\mathbf{H}_{11}(t)]_{nm}$. Note that each entry of the diagonal matrix $\mathbf{R}_{nm}^h$ is generic, and thus the rank deficient argument for the matrix $\mathbf{G}$ used for constant channel no long holds here. Moreover, through numerical tests, we can easily verify that $\mathbf{G}$ has full rank almost surely if we randomly pick the entries of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Thus, 6/5 DoF per user can be achieved almost surely by using this non-asymptotical linear scheme if the channel coefficients are time-varying/ frequency-selective.

Remark: It should be noted that the rational alignment scheme proposed for the constant channel in [10] can also be translated to the linear scheme for time-varying/frequency-selective channels to achieve the DoF outer bound. Such scheme does not explore antenna cooperations and achieves the outer bound asymptotically by using *infinite* symbol extensions. In contrast, the achievable schemes proposed above which explore antenna cooperation achieve the outer bound exactly with only a *finite* number of symbol extensions.

8.3.2 A Linear Scheme with Symbol Extensions in Time/Frequency for General Cases

In this subsection, we will construct a linear scheme with symbol extensions for general cases. The feasibility of such scheme can be settled in every case through a numerical test. Since the DoF value depends on either M or N, we consider them separately.

(1) $d(M,N)=\frac{p}{2p-1}M$ **if** $\frac{p-1}{p}\leq\frac{M}{N}\leq\frac{2p-1}{2p+1}$

Since the DoF outer bound is a fractional value, we take $2p-1$ symbol extensions to make the DoF value an integer, i.e., pM. Then over the extended channel, our goal is to achieve pM DoF. Note that after the symbol extensions, each transmitter has $(2p-1)M$ dimensions and each receiver has $(2p-1)N$ dimensions. The DoF value depending on M implies that the receiver includes redundant dimensions. Therefore, we randomly generate a $(2p+1)M\times(2p-1)N$ matrix at each receiver independently and multiply it to the channel seen at each *receiver*, such that each receiver effectively has a $(2p+1)M$ dimensional space. As a consequence, we have an effective $((2p-1)M,(2p+1)N)$ setting, for which the interference alignment scheme introduced in Section 8.1.3 can be applied such that the dimension of the space spanned by interference is small enough. What remains to be shown is that the desired symbols are resolvable both from each other and from the interference. This is shown by proving that the pM desired symbols and the $(p+1)M$ interfering symbols together span the entire $(2p+1)M$ dimensional signal space. While we do not prove this statement in general, we provide the recipe for completing the proof subject to a numerical test that can determine feasibility with high probability. The independence of the desired signal space from the interference space is equivalent to the condition that the matrix containing the interference vectors and the desired signal vectors has full rank, i.e., the determinant of the matrix is non-zero. Since the determinant is a polynomial with the channel coefficients and the elements of the random projection matrix as the variables, there are only two possibilities. Either it is the zero polynomial or it is not. To prove that it is not the zero polynomial, it suffices to substitute any numerical values for all variables and show that the polynomial evaluates to a non-zero value. Then, since the polynomial is not the zero polynomial, it must almost surely be non-zero for generic values of channel coefficients and the projection matrix, thus completing the proof (subject to the condition that the numerical evaluation produces a non-zero value).

(2) $d(M,N)=\frac{p}{2p+1}N$ **if** $\frac{2p-1}{2p+1}\leq\frac{M}{N}\leq\frac{p}{p+1}$

Similar to the previous case, consider $2p+1$ symbol extensions. Our goal is to achieve pN DoF over the extended channel. After the symbol extensions, each transmitter has $(2p+1)M$ dimensions and each receiver has $(2p+1)N$ dimensions. Now we randomly generate a $(2p+1)M\times(2p-1)N$ matrix at each *transmitter*, and multiply it to the channel seen at each transmitter, such that each transmitter effectively has a $(2p-1)N$-dimensional space. As a result, we end up with an effective $((2p-1)N,(2p+1)N)$ setting, for which interference alignment schemes introduced in Subsection 8.1.3 can be applied. Again we can apply numerical tests to verify if the desired signal overlaps with the interference.

While we do not prove that the numerical test will produce a non-zero value for all M,N, we have carried out the numerical test to complete the proof for all (M,N) values upto $M,N\leq 10$ as shown in Fig. 2. As we can see, linear schemes with symbol extensions over constant channels are sufficient to achieve the information theoretic DoF outer bounds for all (M,N) values upto $M,N\leq 10$ except for the $(q,q+1)$ cases where time-variations/frequency-selectivity are needed as shown in Section 8.3.1 or asymmetric complex signaling for constant channels [27].

8.4 Feasibility of Linear Interference Alignment without any Symbol Extensions in Time/Frequency/Space

In this section, we consider the feasibility of linear interference alignment for the 3-user $M_T \times M_R$ MIMO interference channel. As stated in Theorem 2, the DoF demand per user, d, is feasible with linear interference alignment if and only if $d \leq \lfloor \text{DoF}^\star \rfloor$. The outer bound follows directly from Lemma 1. Next, we provide a proof of Lemma 2 to show that $d = \lfloor \text{DoF}^\star \rfloor$ per user is achievable.

Since $\text{DoF}^\star$ is limited by either the M-bound or the N-bound depending on M/N, we consider the two cases separately. We begin with the setting when $\frac{2p-1}{2p+1} \leq \frac{M}{N} \leq \frac{p}{p+1}$ and show that $\lfloor \frac{p}{2p+1} N \rfloor$ DoF per user is achievable using linear interference alignment *without* the need for symbol extensions in time/frequency/space.

8.4.1 Case: $\frac{2p-1}{2p+1} \leq \frac{M}{N} \leq \frac{p}{p+1} \Rightarrow d = \lfloor \frac{p}{2p+1} N \rfloor$:

For this case, we will provide the achievable scheme for the case $M_T = M$ and $M_R = N$. Due to the reciprocity of linear schemes, such schemes can be applied to the case $M_T = N$ and $M_R = M$ as well. Let us first consider the case when N is an integer multiple of $2p+1$. In this case, $d = \lfloor \frac{p}{2p+1} N \rfloor = \frac{p}{2p+1} N$ which is an integer. As proved in Section 8.1, for this case, no symbol extensions are needed to achieve the information theoretic DoF outer bound.

Now let us consider the remaining cases when N is not an integer multiple of $2p+1$. We first provide an intuition from the linear dimension counting perspective, and then show the rigorous proof through constructing specific channels as in Section 8.1. As shown previously, for $\frac{2p-1}{2p+1} \leq \frac{M}{N} \leq \frac{p}{p+1}$, the maximum length of the alignment chain is $\kappa_N = \lceil \frac{M}{N-M} \rceil = p$. Since the alignment chain with the maximum length corresponds to the most efficient alignment scheme, we would like the dimensions of the subspaces that participate in such chains to be as large as possible. Recall that if the dimension of the subspace is d_0, then the total number of dimensions transmitted from all the transmitters is pd_0 and the total number of dimensions occupied at all the receivers is $(2p+1)d_0$. Due to symmetric antenna configurations, we can pack three such chains, each originating from one of three transmitters, as shown in (107), (110) and (113). Then each transmitter achieves pd_0 DoF, and $(2p+1)d_0$ dimensions are occupied at each receiver. Since the number of dimensions at each receiver is N, $(2p+1)d_0$ cannot exceed N, implying that $d_0 \leq \lfloor \frac{N}{2p+1} \rfloor$. Thus, d_0 is set to be equal to $\lfloor \frac{N}{2p+1} \rfloor$. After this, each transmitter still needs to achieve $\lfloor \frac{p}{2p+1} N \rfloor - p \lfloor \frac{N}{2p+1} \rfloor$ DoF. This can be done if we can pack three more subspace alignment chains with length $p' = \lfloor \frac{p}{2p+1} N \rfloor - p \lfloor \frac{N}{2p+1} \rfloor$, each originating from one transmitter, and the dimension of the subspaces that participate in each chain is one. These symbols will occupy $2p'+1$ dimensions at each receiver. Note that this is possible if the total number of dimensions occupied by these six alignment chains is less than N. In order to see this is true, we need to check that $2p'+1 = 2(\lfloor \frac{p}{2p+1} N \rfloor - p \lfloor \frac{N}{2p+1} \rfloor) + 1$ does not exceed the remaining dimensions at each receiver after accommodating the first three alignment chains with length p, i.e., $N - \lfloor \frac{N}{2p+1} \rfloor (2p+1)$.

$$\begin{aligned}
& \left[2\left(\left\lfloor \frac{p}{2p+1}N \right\rfloor - \left\lfloor \frac{N}{2p+1} \right\rfloor p\right) + 1\right] - \left[N - \left\lfloor \frac{N}{2p+1} \right\rfloor (2p+1)\right] \\
= \;& 2\left\lfloor \frac{p}{2p+1}N \right\rfloor + \left\lfloor \frac{N}{2p+1} \right\rfloor + 1 - N \qquad (144) \\
< \;& 2\frac{p}{2p+1}N + \frac{N}{2p+1} + 1 - N \qquad (145) \\
= \;& 1 \qquad (146)
\end{aligned}$$

where (145) is obtained because N is not an integer multiple of $2p+1$. Because $2p'+1$ and $(N - \lfloor \frac{N}{2p+1} \rfloor (2p+1))$ are both integers, (146) implies that $2p'+1 \leq (N - \lfloor \frac{N}{2p+1} \rfloor (2p+1))$. As a result, each user can send $\lfloor \frac{p}{2p+1}N \rfloor$ DoF and interference occupies small enough dimensions. Intuitively, the $\lfloor \frac{p}{2p+1}N \rfloor$ vectors of each user should be linearly independent, almost surely, because the channels are generic and these vectors do not align among themselves unless they have to. Next we provide the proof to show this is true through constructing specific channels.

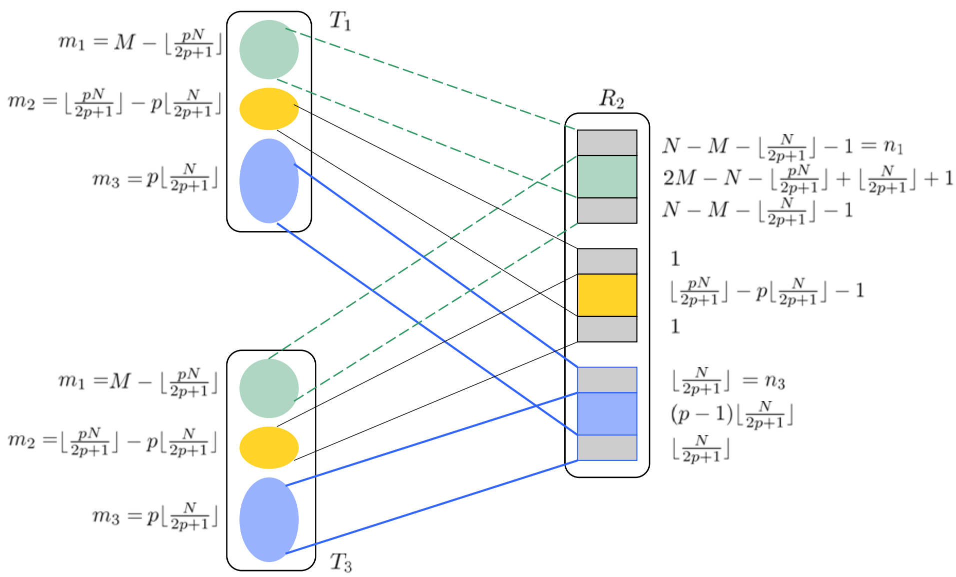


Figure 17: Linear Dimension Counting of Subspaces Participating in the Interference Alignment for the $M \times N$ setting (the values denote the dimensions of each corresponding subspace, and it is easy to verify each value is non-negative.)

The linear dimension counting argument shown above is characterized in Fig. 17 where we consider the signal subspaces at Transmitter 1, Transmitter 3 and Receiver 2. Specifically, we need to design six subspace alignment chains. Three of them are p-length $\lfloor \frac{N}{2p+1} \rfloor$ dimensional, for a total of $p\lfloor \frac{N}{2p+1} \rfloor$ dimensions at each transmitter, denoted as the blue color at Transmitter 1 and Transmitter 3. Since three p-length $\lfloor \frac{N}{2p+1} \rfloor$ dimensional chains will occupy $(2p+1)\lfloor \frac{N}{2p+1} \rfloor$ dimensions at each receiver, we need to align $(p-1)\lfloor \frac{N}{2p+1} \rfloor$ dimensions, i.e., the blue color box at Receiver 2, and the non-aligned $\lfloor \frac{N}{2p+1} \rfloor$ dimensional subspace coming from each transmitter is denoted as the adjacent grey color region. The other three chains are p'-length one dimensional, denoted as the yellow color region at Transmitter 1 and Transmitter 3. Again, since these three

p'-length one dimensional chains occupy $2p'+1$ dimensions at each receiver, we need to align $p'-1$ dimensions, i.e., the yellow color box at Receiver 2, and the non-aligned one dimensional subspace from each transmitter is denoted as the adjacent grey color region. Last, since Transmitter 1 and Transmitter 3 have a total of $2M-N$ common dimensions at Receiver 2 almost surely, the remaining two $M - \lfloor \frac{pN}{2p+1} \rfloor$ dimensional subspaces (denoted as the green color) from these two transmitters have to have the remaining $2M - N - \lfloor \frac{pN}{2p+1} \rfloor + \lfloor \frac{N}{2p+1} \rfloor + 1$ dimensional intersection at Receiver 2 (denoted as the green color), and the non-aligned $N - M - \lfloor \frac{N}{2p+1} \rfloor - 1$ dimensional subspace from each transmitter is denoted as the adjacent grey color box.

Let us first design the three p-length subspace alignment chains, which are shown in (107), (110) and (113). These three chains yield the three alignment equations shown in (108), (111) and (114), where each of three matrices $\mathbf{A}_p, \mathbf{B}_p, \mathbf{C}_p$ is a $(p-1)N \times pM$ matrix. Since null spaces of $\mathbf{A}_p, \mathbf{B}_p, \mathbf{C}_p$ all have $pM-(p-1)N$ dimensions, and also because $pM - (p-1)N \geq \frac{N}{2p+1} > \lfloor \frac{N}{2p+1} \rfloor$, we can choose $\mathbf{a}_p, \mathbf{b}_p, \mathbf{c}_p$ as the $\lfloor \frac{N}{2p+1} \rfloor$ basis vectors in the $pM-(p-1)N$ dimensional null spaces of $\mathbf{A}_p, \mathbf{B}_p, \mathbf{C}_p$, respectively. That is, assuming $\mathbf{Q}_{k,p}$ are three $(pM-(p-1)N) \times \lfloor \frac{N}{2p+1} \rfloor$ randomly generated matrices, then we have

$$
\begin{aligned}
\mathbf{a}_p &= (\mathbf{I} - \mathbf{A}_p^H(\mathbf{A}_p\mathbf{A}_p^H)^{-1}\mathbf{A}_p)\det(\mathbf{A}_p\mathbf{A}_p^H)\mathbf{Q}_{1,p}, && (147)\\
\mathbf{b}_p &= (\mathbf{I} - \mathbf{B}_p^H(\mathbf{B}_p\mathbf{B}_p^H)^{-1}\mathbf{B}_p)\det(\mathbf{B}_p\mathbf{B}_p^H)\mathbf{Q}_{2,p}, && (148)\\
\mathbf{c}_p &= (\mathbf{I} - \mathbf{C}_p^H(\mathbf{C}_p\mathbf{C}_p^H)^{-1}\mathbf{C}_p)\det(\mathbf{C}_p\mathbf{C}_p^H)\mathbf{Q}_{3,p}. && (149)
\end{aligned}
$$

Next we design the three p'-length one-dimensional subspace alignment chains, which are still identical to (107), (110) and (113) by replacing p with p'. These three chains yield the three alignment equations shown in (108), (111) and (114) by replacing p with p', where each of three matrices $\mathbf{A}_{p'}, \mathbf{B}_{p'}, \mathbf{C}_{p'}$ is a $(p'-1)N \times p'M$ matrix. Since null spaces of $\mathbf{A}_{p'}, \mathbf{B}_{p'}, \mathbf{C}_{p'}$ all have $p'M-(p'-1)N$ dimensions, which is always no less than one, we can choose $\mathbf{a}_{p'}, \mathbf{b}_{p'}, \mathbf{c}_{p'}$ as the basis vector in the $p'M-(p'-1)N$ dimensional null spaces of $\mathbf{A}_{p'}, \mathbf{B}_{p'}, \mathbf{C}_{p'}$, respectively. Assuming $\mathbf{Q}_{k,p'}$ are three $(p'M-(p'-1)N) \times 1$ randomly generated vectors, then we have

$$
\begin{aligned}
\mathbf{a}_{p'} &= (\mathbf{I} - \mathbf{A}_{p'}^H(\mathbf{A}_{p'}\mathbf{A}_{p'}^H)^{-1}\mathbf{A}_{p'})\det(\mathbf{A}_{p'}\mathbf{A}_{p'}^H)\mathbf{Q}_{1,p'}, && (150)\\
\mathbf{b}_{p'} &= (\mathbf{I} - \mathbf{B}_{p'}^H(\mathbf{B}_{p'}\mathbf{B}_{p'}^H)^{-1}\mathbf{B}_{p'})\det(\mathbf{B}_{p'}\mathbf{B}_{p'}^H)\mathbf{Q}_{2,p'}, && (151)\\
\mathbf{c}_{p'} &= (\mathbf{I} - \mathbf{C}_{p'}^H(\mathbf{C}_{p'}\mathbf{C}_{p'}^H)^{-1}\mathbf{C}_{p'})\det(\mathbf{C}_{p'}\mathbf{C}_{p'}^H)\mathbf{Q}_{3,p'}. && (152)
\end{aligned}
$$

Consider the square matrix of each receiver which consists of the desired signal and interference vectors. Now its determinant is a polynomial function of the channel coefficients and entries of $\mathbf{Q}_{k,p}, \mathbf{Q}_{k,p'}, k=1,2,3$, and this polynomial is an either zero or non-zero polynomial almost surely. As in Section 8.1, we only need to construct specific channels as well as $\mathbf{Q}_{k,p}, \mathbf{Q}_{k,p'}$, and show the beamforming vectors of each user are linearly independent. We choose the channel matrices as follows:

$$
\mathbf{H}_{(i-1)i} = \begin{bmatrix}
\mathbf{0}_{n_1\times m_1} & \mathbf{0}_{n_1\times m_2} & \mathbf{0}_{n_1\times m_3}\\
\mathbf{I}_{m_1\times m_1} & \mathbf{0}_{m_1\times m_2} & \mathbf{0}_{m_1\times m_3}\\
\mathbf{0}_{1\times m_1} & \mathbf{0}_{1\times m_2} & \mathbf{0}_{1\times m_3}\\
\mathbf{0}_{m_2\times m_1} & \mathbf{I}_{m_2\times m_2} & \mathbf{0}_{m_2\times m_3}\\
\mathbf{0}_{n_3\times m_1} & \mathbf{0}_{n_3\times m_2} & \mathbf{0}_{n_3\times m_3}\\
\mathbf{0}_{m_3\times m_1} & \mathbf{0}_{m_3\times m_2} & \mathbf{I}_{m_3\times m_3}
\end{bmatrix}, \quad
\mathbf{H}_{(i+1)i} = \begin{bmatrix}
\mathbf{I}_{m_1\times m_1} & \mathbf{0}_{m_1\times m_2} & \mathbf{0}_{m_1\times m_3}\\
\mathbf{0}_{n_1\times m_1} & \mathbf{0}_{n_1\times m_2} & \mathbf{0}_{n_1\times m_3}\\
\mathbf{0}_{m_2\times m_1} & \mathbf{I}_{m_2\times m_2} & \mathbf{0}_{m_2\times m_3}\\
\mathbf{0}_{1\times m_1} & \mathbf{0}_{1\times m_2} & \mathbf{0}_{1\times m_3}\\
\mathbf{0}_{m_3\times m_1} & \mathbf{0}_{m_3\times m_2} & \mathbf{I}_{m_3\times m_3}\\
\mathbf{0}_{n_3\times m_1} & \mathbf{0}_{n_3\times m_2} & \mathbf{0}_{n_3\times m_3}
\end{bmatrix} \quad (153)
$$

where $\forall i \in \{1, 2, 3\}$ and the values of m_i, n_i are shown in Fig. 17. It can be easily verified that the alignment schemes we describe above produce one solution of the beamforming matrix for each transmitter as follows:

$$\mathbf{V}_i = [\mathbf{V}_{i,p} \;\; \mathbf{V}_{i,p'}], \qquad \mathbf{V}_{i,p} = \begin{bmatrix} \mathbf{0}_{m_1 \times m_3} \\ \mathbf{0}_{m_2 \times m_3} \\ \mathbf{I}_{m_3 \times m_3} \end{bmatrix}, \qquad \mathbf{V}_{i,p'} = \begin{bmatrix} \mathbf{0}_{m_1 \times m_2} \\ \mathbf{I}_{m_2 \times m_2} \\ \mathbf{0}_{m_3 \times m_2} \end{bmatrix} \tag{154}$$

where $\mathbf{V}_{i,p}$ and $\mathbf{V}_{i,p'}$ are obtained from the p-length $\lfloor \frac{N}{2p+1} \rfloor$ dimensional chains and p'-length one dimensional chains, respectively. Clearly, since the column spaces of $\mathbf{V}_{i,p}$ and $\mathbf{V}_{i,p'}$ are orthogonal to each other, the rank of $\mathbf{V}_i$ is $m_2 + m_3 = \lfloor \frac{p}{2p+1} N \rfloor$. Therefore, the beamforming vectors of each user are linearly independent among themselves, almost surely. The linear independence established through this specific example proves that the polynomial (comprised of channel variables and $\mathbf{Q}$ variables) representing the determinant of a square matrix containing the beamforming vectors originating at a transmitter, is not the zero-polynomial. Since it is not the zero-polynomial, it is almost surely not zero for random choices of channel coefficients and $\mathbf{Q}$ matrices, i.e., the linear interference alignment solution exists almost surely.

8.4.2 Case: $\frac{p-1}{p} \leq \frac{M}{N} \leq \frac{2p-1}{2p+1} \Rightarrow d = \lfloor \frac{p}{2p-1} M \rfloor$:

For this case, we provide the achievable schemes for the setting $M_T = N$ and $M_R = M$. The proof for this case is similar as the previous case with $2p+1$ and N in the previous argument replaced by $2p-1$ and M, respectively. The detailed proof followed is omitted due to the similarity. ■

9 Conclusion

In this paper we characterize both the spatially-normalized information-theoretic DoF as well as the DoF achievable through linear beamforming schemes without channel extensions in time/frequency/space, for the symmetric three-user $M_T \times M_R$ Gaussian MIMO interference channel where each transmitter has M_T antennas and each receiver has M_R receive antennas. In order to establish this result, we derive information theoretic DoF outer bounds for arbitrarily values of M_T and M_R. The information theoretic DoF outer bounds are facilitated by a change of basis operation at both transmitter and receiver sides which helps identify the projections of the transmitted signal space to be provided as genie signals for the outer bound arguments. One of the main ideas involved in this problem is the notion of *subspace alignment chains*, the length of which indicates the limitations of aligning interference symbols. Achievable schemes take advantage of the linear-dimension counting arguments and generic property of channels, both of which are captured by subspace alignment chains.

Several interesting observations follow as a byproduct of our analysis. First we precisely identify settings with redundant dimensions at the transmitters, receivers, both or neither. The maximally redundant settings correspond to $M/N = 1/2, 2/3, 3/4, \cdots$ and contain redundant dimensions at both the transmitters and receivers. The DoF outer bounds essentially boil down to these settings. These settings are also the only ones where there is no DoF benefit of joint processing across the multiple antennas located at any transmitter or receiver node. The minimally redundant settings correspond to $M/N = 3/5, 5/7, 7/9, \cdots$ and contain no redundant dimensions at either the transmitters or receivers. The achievability results essentially boil own to these settings. These settings are also the only ones where proper systems are guaranteed to be feasible.

We also note the recent work in [26] where partial achievability results are independently obtained and shown to be optimal in some cases with respect to linear interference alignment. To the best of our understanding, all the results of [26] can also be recovered as special cases of the results presented in this paper.

Finally, we conclude with a pointer to the open problem of characterizing the DoF of asymmetric settings, where each transmitter and receiver may have different number of antennas. DoF results for certain asymmetric settings can be obtained directly from our results whenever the asymmetric setting differs from the symmetric case only in redundant dimensions. However, in general, the complexity of the asymmetric problem may be understood in light of the difficulty of packing subspace alignment chains within the space constraints imposed by the number of antennas at each node, which may be tight or redundant. Both outer bounds and inner bounds are challenging for this general setting. Dimension counting arguments, motivated by subspace alignment chains may be a good starting point for this general setting.

Appendix

A Information Theoretic DoF Outer Bound for $M_T < M_R$

We will show the proofs in the following two subsections. In the first, we investigate the cases of $M/N = p/(p+1)$ for which we only need to consider $(M,N) = (p,p+1)$ as we mentioned before. In the second subsection, we investigate the $M/N \neq p/(p+1)$ cases.

A.1 Cases: $(M,N) = (p,p+1) \Rightarrow$ DoF $\leq \frac{MN}{M+N}$

A.1.1 M is an Even Number

We consider M is even in this subsection. We first take recursive invertible linear transformations introduced in Section 6 at each user. The transmitted signals from each antenna and what transmitted signals that each receiver antenna can see are specified in Fig. 18. Note that again we use $S_{k(\cdot)}$ to denote the received signal at the antenna $(\cdot)$ of Receiver k after subtracting the signal carrying its desired message. Thus, $S_{k(\cdot)}$ consists of linearly independent combinations of the interference symbols at the antenna $(\cdot)$ of Receiver k.

For the general $(M,N) = (p,p+1)$ setting, as we have shown in Section 7, we need a total of M sets of genie signals and each set has $M-1$ dimensions. We denote $\mathcal{G}_{km}^{M},\ k \in \{1,2,3\},\ m \in \{1,\cdots,M\}$ as the M sets of genie signals provided to Receiver k for the $(M,N) = (p,p+1)$ network. If $k=1$, $\mathcal{G}_{1m}^{M}$ or directly $\mathcal{G}_{m}^{M}$ if without ambiguity is given by:

$$\begin{aligned} \text{for} \ \ M=2, \qquad & \mathcal{G}_1^M = \{X_{2a_1}^n\} \\ & \mathcal{G}_2^M = \{X_{3c_1}^n\} \\ \text{then for} \ \ M>2, \qquad & \mathcal{G}_m^M = \{X_{2a_L}^n, X_{3c_L}^n, \mathcal{G}_m^{M-2}\} \qquad m = \{1,2,\cdots,M-2\}, \\ \text{and} \qquad & \mathcal{G}_{M-1}^M = \{X_{2a_L}^n, \{X_{2a_l}^n, X_{2c_l}^n | 1 \leq l \leq L-1\}\} \\ & \mathcal{G}_M^M = \{X_{3c_L}^n, \{X_{3a_l}^n, X_{3c_l}^n | 1 \leq l \leq L-1\}\} \end{aligned} \tag{155}$$

and $\mathcal{G}_{2m}^{M}$, $\mathcal{G}_{3m}^{M}$ can be obtained by advancing the user indices. It is easy to check that the cardinality of each set is $M-1$. Also, note that if Receiver 1 knows the two signals $(X_{2a_L}^n, X_{3c_L}^n)$ in the L^{th} layer, then after removing $(X_{2a_L}^n, X_{3c_L}^n)$ from $\mathcal{G}_m^M,\ m \in \{1,\cdots,M-2\}$ in (155), we obtain

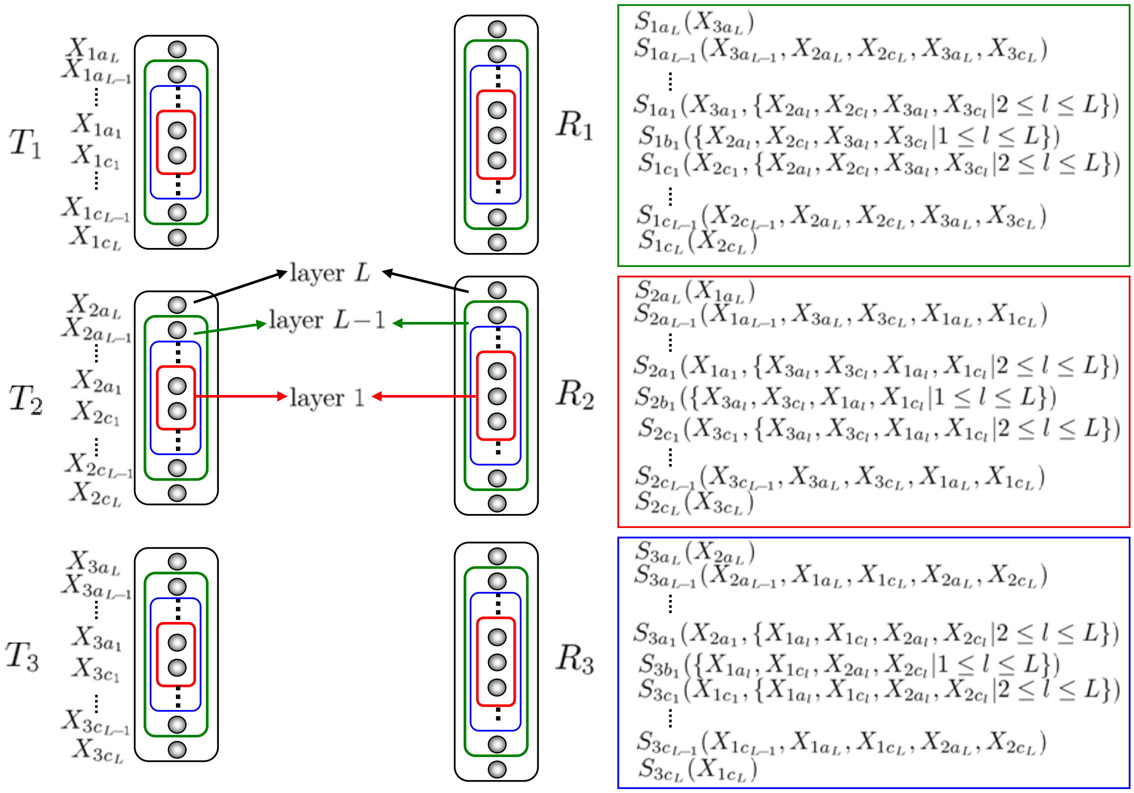


Figure 18: Intuition of Onion Peeling if M is even

$\mathcal{G}_m^{M-2}$, $m \in \{1, \cdots, M-2\}$. This fact is essential to why we can design the genie signals in an iterative way, which help us recursively derive the DoF outer bound based on mathematical induction. Consider an $(M, M+1)$ interference network, which has a total of $L = M/2$ layers, with two transmit signals $(X_{ka_l}^n, X_{kc_l}^n)$ at the l^{th} layer where $l = \{1, 2, \cdots, L\}$. The transmit signals in the l^{th} layer will be heard by all antennas in all $l' \leq l$ layers. Now let us peel the L^{th} layer at both sides, i.e., a genie provides the two symbols $(X_{(k+1)a_L}^n, X_{(k-1)c_L}^n)$ to Receiver k. Since the peeled $S_{ka_L}^n$ and $S_{kc_L}^n$ at Receiver k are already noisy versions of $X_{(k-1)a_L}^n$ and $X_{(k+1)c_L}^n$ respectively, Receiver 1 can recover $(X_{(k+1)a_L}^n, X_{(k+1)c_L}^n, X_{(k-1)a_L}^n, X_{(k-1)c_L}^n)$ subject to noise distortion, and thus it can subtract them from the received signals at the other $M-1$ antennas. Therefore, we obtain an embedded $(M-2, M-1)$ interference network.

The proof is based on mathematical induction. Since we have already finished the proof for $(M, M+1) = (2, 3)$ and $(4, 5)$ previously, now let us assume it works for the $(M-2, M-1)$ case. That is to say, by providing genie signals $\mathcal{G}_m^{M-2}$, $m \in \{1, \cdots, M-2\}$ to Receiver 1, we obtain a total of $M-2$ sum rate inequalities, each obtained by averaging over user indices. If we add up all these $(M-2)$-inequalities, at the left-hand side we have $(M-2)nR$; at the right-hand side we have $(M-2)N \log \rho + (M-3)nR + n\ o(\log \rho) + o(n)$. In other words, we can bound the sum of differential entropy of all genie signals provided to three receivers by:

$$\frac{1}{3} \sum_{m=1}^{M-2} \sum_{k=1}^{3} h(\mathcal{G}_{km}^{M-2} | \bar{Y}_k^n) \leq (M-3)nR_\Sigma + n\ o(\log \rho) + o(n) \tag{156}$$

where $\frac{1}{3} \sum_{k=1}^{3}$ stands for averaging over user indices.

Next let consider the $(M, M+1)$ case, i.e., adding one layer to the previous network. Since by providing $(X^n_{(k+1)a_L}, X^n_{(k-1)c_L})$ to Receiver k, we can boil down the $(M, M+1)$ setting to the $(M-2, M-1)$ setting. By our assumptions, we can easily obtain:

$$\frac{1}{3}\sum_{m=1}^{M-2}\sum_{k=1}^{3} h(\mathcal{G}^M_{km} \setminus \{X^n_{(k+1)a_L}, X^n_{(k-1)c_L}\}|\bar{Y}^n_k, X^n_{(k+1)a_L}, X^n_{(k-1)c_L}) \leq$$

$$(M-3)\frac{1}{3}\sum_{k=1}^{3} h(\{X^n_{ka_l}, X^n_{kc_l}|1 \leq l \leq L-1\}|X^n_{ka_L}, X^n_{kc_L}) + n\ o(\log\rho) + o(n). \tag{157}$$

For the $(M, M+1)$ case, we consider the sum differential entropy of all genie signals provided to all receivers in the first $M-2$ sets conditioning on the observations of each receiver, respectively. That is,

$$\sum_{m=1}^{M-2}\sum_{k=1}^{3} h(\mathcal{G}^M_{km}|\bar{Y}^n_k)$$

$$= \sum_{m=1}^{M-2}\sum_{k=1}^{3} \Big(h(X^n_{(k+1)a_L}, X^n_{(k-1)c_L}|\bar{Y}^n_k) + h(\mathcal{G}^M_{km} \setminus \{X^n_{(k+1)a_L}, X^n_{(k-1)c_L}\}|\bar{Y}^n_k, X^n_{(k+1)a_L}, X^n_{(k-1)c_L}) \Big) \tag{158}$$

$$= (M-2)\sum_{k=1}^{3} h(X^n_{(k+1)a_L}, X^n_{(k-1)c_L}|\bar{Y}^n_k) + \sum_{m=1}^{M-2}\sum_{k=1}^{3} h(\mathcal{G}^M_{km} \setminus \{X^n_{(k+1)a_L}, X^n_{(k-1)c_L}\}|\bar{Y}^n_k, X^n_{(k+1)a_L}, X^n_{(k-1)c_L}) \tag{159}$$

$$\leq \sum_{k=1}^{3} h(X^n_{(k+1)a_L}, X^n_{(k-1)c_L}) + (M-3)\sum_{k=1}^{3} \Big(h(X^n_{(k-1)c_L}) + h(X^n_{(k+1)a_L}|X^n_{(k+1)c_L}) \Big)$$
$$+ \sum_{m=1}^{M-2}\sum_{k=1}^{3} h(\mathcal{G}^M_{km} \setminus \{X^n_{(k+1)a_L}, X^n_{(k-1)c_L}\}|\bar{Y}^n_k, X^n_{(k+1)a_L}, X^n_{(k-1)c_L}) \tag{160}$$

$$= \sum_{k=1}^{3} h(X^n_{(k+1)a_L}, X^n_{(k-1)c_L}) + (M-3)\sum_{k=1}^{3} \Big(h(X^n_{(k-1)c_L}) - h(X^n_{(k+1)c_L}) + h(X^n_{(k+1)a_L}, X^n_{(k+1)c_L}) \Big)$$
$$+ (M-3)n\sum_{k=1}^{3} h(\{X^n_{ka_l}, X^n_{kc_l}|1 \leq l \leq L-1\}|X^n_{ka_L}, X^n_{kc_L}) + n\ o(\log\rho) + o(n) \tag{161}$$

$$= \sum_{k=1}^{3} h(X^n_{(k+1)a_L}, X^n_{(k-1)c_L}) + n\ o(\log\rho) + o(n)$$
$$+ (M-3)\sum_{k=1}^{3} \Big(h(X^n_{(k+1)a_L}, X^n_{(k+1)c_L}) + h(\{X^n_{ka_l}, X^n_{kc_l}|1 \leq l \leq L-1\}|X^n_{ka_L}, X^n_{kc_L}) \Big) \tag{162}$$

$$\leq \sum_{k=1}^{3} \Big(h(X^n_{(k+1)a_L}) + h(X^n_{(k-1)c_L}) \Big) + (M-3)\sum_{k=1}^{3} h(\{X^n_{ka_l}, X^n_{kc_l}|1 \leq l \leq L\}) + n\ o(\log\rho) + o(n) \tag{163}$$

$$= \sum_{k=1}^{3} \big(h(X^n_{ka_L}) + h(X^n_{kc_L}) \big) + (M-3)nR_\Sigma + n\ o(\log\rho) + o(n). \tag{164}$$

Therefore, the summation of the $M-2$ sum rate inequalities produces the following inequality:

$$3(M-2)nR \leq (M-2)Nn\log\rho + (M-3)nR$$
$$+ \frac{1}{3}\sum_{k=1}^{3} \big(h(X^n_{ka_L}) + h(X^n_{kc_L}) \big) + n\ o(\log\rho) + o(n).$$

Or equivalently it can be rewritten as:

$$3(M-2)nR \leq (M-2)Nn\log\rho + (M-3)nR + \big(h(X_{a_L}^n) + h(X_{c_L}^n)\big) + n\ o(\log\rho) + o(n). \quad (165)$$

For the remaining last two genie signal sets, if a genie provides $\mathcal{G}_{M-1}^M = \{X_{2a_L}^n, \{X_{2a_l}^n, X_{2c_l}^n | 1 \leq l \leq L-1\}\}$ to Receiver 1, we have:

$$\begin{aligned}
nR_\Sigma &\leq I(W_1, W_2, W_3; \bar{Y}_1^n, \mathcal{G}_{M-1}^M) + n\ o(\log\rho) + o(n) && (166)\\
&\leq Nn\log\rho + h(\mathcal{G}_{M-1}^M | \bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (167)\\
&\leq Nn\log\rho + h(\mathcal{G}_{M-1}^M | X_{2c_L}^n) + n\ o(\log\rho) + o(n) && (168)\\
&\leq Nn\log\rho + nR_2 - h(X_{2c_L}^n) + n\ o(\log\rho) + o(n) && (169)
\end{aligned}$$

and thus by averaging over user indices, we have the $(M-1)^{th}$ inequality:

$$3nR \leq Nn\log\rho + nR - h(X_{c_L}^n) + n\ o(\log\rho) + o(n). \quad (170)$$

With similar analysis if a genie provides $\mathcal{G}_{M-1}^M = \{X_{3c_L}^n, \{X_{3a_l}^n, X_{3c_l}^n | 1 \leq l \leq L-1\}\}$ to Receiver 1, we have the M^{th} inequality:

$$3nR \leq Nn\log\rho + nR - h(X_{a_L}^n) + n\ o(\log\rho) + o(n). \quad (171)$$

Add up the inequalities of (165), (170) and (171), we eventually obtain

$$3MnR \leq MN\log\rho + (M-1)nR + n\ o(\log\rho) + o(n), \quad (172)$$

which implies that the DoF per user outer bound

$$d \leq \frac{MN}{2M+1} \leq \frac{MN}{M+N}. \quad (173)$$

Therefore, we establish the outer bound results. ■

A.1.2 M is an Odd Number

For the $(M, M+1)$ case where M is odd, the proof for the DoF outer bound can be carried out in a similar way as what we have shown if M is even. Specifically, the total number of layers is still equal to L where $M = 2L+1$. By peeling out the layers one by one, we eventually obtain the 3×4 core. The interference signallings comprised at the received signal at each antenna after the linear transformations is shown in Fig.19.

For this case, we will provide M-sets of genie signals to each receiver, each set with $M-1$ dimensional signals. Again, we show the genie provided to receiver as follows, and the genie signals provided to Receiver 2 and 3 can be obtained by advancing the user indices.

$$\begin{aligned}
\text{for}\quad M = 3, \qquad & \mathcal{G}_1^M = \{X_{2a_1}^n, X_{3c_1}^n\}\\
& \mathcal{G}_2^M = \{X_{2a_1}^n, X_{2b_1}^n\}\\
& \mathcal{G}_3^M = \{X_{3b_1}^n, X_{3c_1}^n\}\\
\text{then for}\quad M > 3, \qquad & \mathcal{G}_m^M = \{X_{2a_L}^n, X_{3c_L}^n, \mathcal{G}_m^{M-2}\} \qquad m = \{1, 2, \cdots, M-2\},\\
\text{and}\qquad & \mathcal{G}_{M-1}^M = \{X_{2a_L}^n, \{X_{2a_l}^n, X_{2c_l}^n | 1 \leq l \leq L-1\}, X_{2b_1}^n\}\\
& \mathcal{G}_M^M = \{X_{3c_L}^n, \{X_{3a_l}^n, X_{3c_l}^n | 1 \leq l \leq L-1\}, X_{3b_1}^n\} \qquad (174)
\end{aligned}$$

The remaining proof is the similar as that if M is even by replacing the genie signals $\mathcal{G}_m^M$ in (155) with that shown in (174).

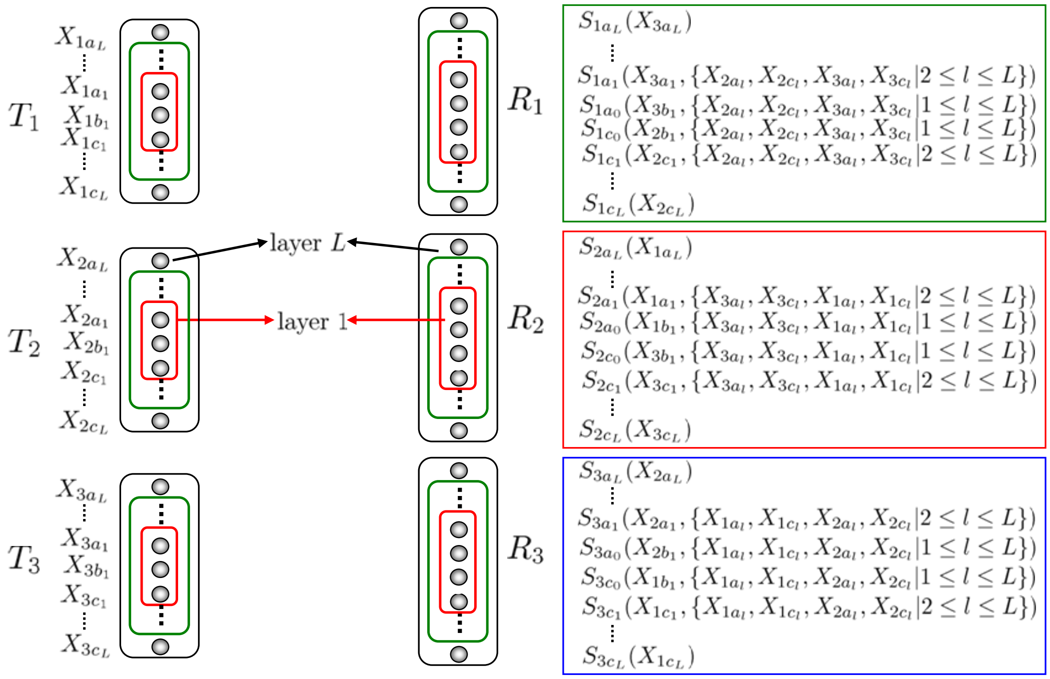


Figure 19: Intuition of Onion Peeling if M is odd

A.2 DoF Outer Bound for $\frac{M}{N} \neq \frac{p}{p+1}$

In this subsection, we only show the information-theoretic DoF outer bound proofs for $\frac{M}{N} \neq \frac{p}{p+1}$. In fact, all the derivations follow directly from what we have shown for $M/N = p/(p+1)$. We will first show some examples and then extend the analysis to the general cases.

A.2.1 Case: Any $M/N \geq 2/3 \Rightarrow$ DoF $\leq \frac{3M}{5}$

In this section, consider arbitrary (M, N) values such that $M/N \geq 2/3$. For all such values of (M, N), the DoF outer bound that holds is $d \leq \frac{3M}{5}$. In order to keep the presentation clean and simple, from now on we will use tables instead of figures to show the channel connectivity. Note that the linear transformation still follows the similar manner as that introduced in Section 6.

$\lvert X_{1a}\rvert = (N-M)$	X_{1a}	$\circ$
$\lvert X_{1b}\rvert = 3M-2N \geq 0$	X_{1b}	$\circ$
$\lvert X_{1c}\rvert = (N-M)$	X_{1c}	$\circ$

$\circ$	$S_{1a}(X_{3a})$	$\lvert S_{1a}\rvert = (N-M)$
$\circ$	$S_{1b}(X_{2a}, X_{2b}, X_{2c}, X_{3a}, X_{3b}, X_{3c})$	$\lvert S_{1b}\rvert = 2M-N \geq (N-M)$
$\circ$	$S_{1c}(X_{2c})$	$\lvert S_{1c}\rvert = (N-M)$

$\lvert X_{2a}\rvert = (N-M)$	X_{2a}	$\circ$
$\lvert X_{2b}\rvert = 3M-2N \geq 0$	X_{2b}	$\circ$
$\lvert X_{2c}\rvert = (N-M)$	X_{2c}	$\circ$

$\circ$	$S_{2a}(X_{1a})$	$\lvert S_{2a}\rvert = (N-M)$
$\circ$	$S_{2b}(X_{3a}, X_{3b}, X_{3c}, X_{1a}, X_{1b}, X_{1c})$	$\lvert S_{2b}\rvert = 2M-N \geq (N-M)$
$\circ$	$S_{2c}(X_{3c})$	$\lvert S_{2c}\rvert = (N-M)$

$\lvert X_{3a}\rvert = (N-M)$	X_{3a}	$\circ$
$\lvert X_{3b}\rvert = 3M-2N \geq 0$	X_{3b}	$\circ$
$\lvert X_{3c}\rvert = (N-M)$	X_{3c}	$\circ$

$\circ$	$S_{3a}(X_{2a})$	$\lvert S_{3a}\rvert = (N-M)$
$\circ$	$S_{3b}(X_{1a}, X_{1b}, X_{1c}, X_{2a}, X_{2b}, X_{2c})$	$\lvert S_{3b}\rvert = 2M-N \geq (N-M)$
$\circ$	$S_{3c}(X_{1c})$	$\lvert S_{3c}\rvert = (N-M)$

In the table above, notice that one circle could represent more than one antennas. For simplicity, hereinafter we still use X to denote the signal vector at corresponding antennas if it would cause no ambiguity.

First a genie provides $\mathcal{G}_1 = \{X_{2a}^n, X_{2b}^n\}$ to receiver 1. Then the total number of dimensions available to receiver 1 (including those provided by the genie) is equal to:

$$N + |\mathcal{G}_1| = N + |X_{2a}^n| + |X_{2b}^n| = N + (N - M) + (3M - 2N) = 2M$$

With these (at least) $2M$ dimensions, it is easy to see that receiver 1 is able to resolve both interfering signals and thus it can decode all three messages. Therefore, we have:

$$\begin{aligned} nR_\Sigma &\leq Nn\log\rho + h(X_{2a}^n, X_{2b}^n|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) & (175)\\ &\leq Nn\log\rho + h(X_{2a}^n, X_{2b}^n|X_{2c}^n) + n\ o(\log\rho) + o(n) & (176)\\ &= Nn\log\rho + nR_2 - h(X_{2c}^n) + n\ o(\log\rho) + o(n) & (177)\end{aligned}$$

By advancing user indices, therefore we have:

$$3nR \leq Nn\log\rho + nR - h(X_c^n) + n\ o(\log\rho) + o(n). \tag{178}$$

Second a genie provides $\mathcal{G}_2 = \{X_{3b}^n, X_{3c}^n\}$ to receiver 1. Then the total number of dimensions available to receiver 1 (including those provided by the genie) is equal to:

$$N + |\mathcal{G}_2| = N + |X_{3b}^n| + |X_{3c}^n| = N + (3M - 2N) + (N - M) = 2M$$

Thus, receiver 1 is able to resolve both interfering signals. Therefore, once again, we have:

$$\begin{aligned} nR_\Sigma &\leq Nn\log\rho + h(X_{3b}^n, X_{3c}^n|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) & (179)\\ &\leq Nn\log\rho + h(X_{3b}^n) + h(X_{3c}^n) + n\ o(\log\rho) + o(n) & (180)\\ &\leq Nn\log\rho + (3M - 2N)n\log\rho + h(X_{3c}^n) + n\ o(\log\rho) + o(n) & (181)\end{aligned}$$

and therefore we have:

$$3nR \leq Nn\log\rho + (3M - 2N)n\log\rho + h(X_c^n) + n\ o(\log\rho) + o(n) \tag{182}$$

Adding up the inequalities in (178) and (182) we obtain:

$$6nR \leq 3Mn\log\rho + nR + n\ o(\log\rho) + o(n)$$

which implies that

$$d \leq \frac{3M}{5}.$$

A.2.2 Case: Any $M/N \in [1/2, 2/3] \Rightarrow$ DoF $\leq \frac{2N}{5}$

In this section, consider arbitrary (M, N) values such that $M/N \in [1/2, 2/3]$. For all such values of (M, N), the DoF outer bound that holds is $d \leq \frac{2N}{5}$.

We still take the linear transformation introduced in Section 4 first at the receiver side. In addition, we take linear transformation at the transmitter side by multiplying an $M \times M$ square matrix at transmitter k which is the inverse of the channel matrix from X_k to $(Y_{(k-1)b}, Y_{(k-1)c})$.

This operation will force $Y_{(k-1)c}$ to only see the signals of $(X_2^n)_{[(2M-N+1):M]}$ where $(X)_{[m:n]}$ denotes the m^{th} to n^{th} entries of the signal vector X. The resulting network and corresponding connectivity are present in the following table.

$\lvert X_1\rvert = M$	X_1	$\circ$

$\circ$	$S_{1a}(X_3)$	$\lvert S_{1a}\rvert = (N-M)$
$\circ$	$S_{1b}(X_2, X_3)$	$\lvert S_{1b}\rvert = 2M-N \leq (N-M)$
$\circ$	$S_{1c}(X_2)$	$\lvert S_{1c}\rvert = (N-M)$

$\lvert X_2\rvert = M$	X_2	$\circ$

$\circ$	$S_{2a}(X_1)$	$\lvert S_{2a}\rvert = (N-M)$
$\circ$	$S_{2b}(X_3, X_1)$	$\lvert S_{2b}\rvert = 2M-N \leq (N-M)$
$\circ$	$S_{2c}(X_3)$	$\lvert S_{2c}\rvert = (N-M)$

$\lvert X_3\rvert = M$	X_3	$\circ$

$\circ$	$S_{3a}(X_2)$	$\lvert S_{3a}\rvert = (N-M)$
$\circ$	$S_{3b}(X_1, X_2)$	$\lvert S_{3b}\rvert = 2M-N \leq (N-M)$
$\circ$	$S_{3c}(X_1)$	$\lvert S_{3c}\rvert = (N-M)$

First a genie provides $\mathcal{G}_1 = (X_2^n)_{[1:(N-M)]}$ to receiver 1. Then the total number of dimensions available to receiver 1 (including those provided by the genie) is equal to:

$$N + |\mathcal{G}_1| = N + (N-M) \geq 3M - M = 2M$$

With at least $2M$ dimensions among them, it is easy to see that receiver 1 is able to resolve both interfering signals. Therefore, we have:

$$\begin{aligned} nR_\Sigma &\leq Nn\log\rho + h((X_2^n)_{[1:(N-M)]}|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (183)\\ &\leq Nn\log\rho + h((X_2^n)_{[1:(N-M)]}|(X_2^n)_{[(N-M+1):M]}) + n\ o(\log\rho) + o(n) && (184)\\ &= Nn\log\rho + nR_2 - h((X_2^n)_{[(N-M+1):M]}) + n\ o(\log\rho) + o(n) && (185)\end{aligned}$$

where (184) follows from the fact that dropping condition terms does not decrease the entropy. Specifically, $Y_{(k-1)c}$ only see the signals of $(X_2^n)_{[(2M-N+1):M]}$. Since we are considering $\frac{1}{2} \leq \frac{M}{N} \leq \frac{2}{3}$ which implies that $3M \leq 2N$, we have $(2M-N+1) \leq (N-M+1)$. In other words, we only keep the last $2M-N$ entries of $\bar{Y}_1$ in the condition and drop the other terms. By advancing user indices, therefore we have:

$$3nR \leq Nn\log\rho + nR - h((X^n)_{[(N-M+1):M]}) + n\ o(\log\rho) + o(n). \quad (186)$$

Second a genie provides $\mathcal{G}_2 = (X_3^n)_{[(N-M+1):M]}$ to receiver 1. Then the total number of dimensions available to receiver 1 including those provided by the genie is equal to:

$$N + |\mathcal{G}_2| = N + (M - (N-M+1) + 1) = 2M$$

With these $2M$ dimensions, receiver 1 once again is able to resolve both interfering signals. Therefore, we have:

$$\begin{aligned} nR_\Sigma &\leq Nn\log\rho + h((X_3^n)_{[(N-M+1):M]}|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (187)\\ &\leq Nn\log\rho + h((X_3^n)_{[(N-M+1):M]}) + n\ o(\log\rho) + o(n) && (188)\end{aligned}$$

By advancing user indices, therefore we have:

$$3nR \leq Nn\log\rho + h((X^n)_{[(N-M+1):M]}) + n\ o(\log\rho) + o(n). \tag{189}$$

Adding up the inequalities in (186) and (189) we obtain:

$$6nR \leq 2Nn\log\rho + nR + n\ o(\log\rho) + o(n)$$

which implies that

$$d \leq \frac{2N}{5}.$$

A.2.3 Case: Any $M/N \geq 4/5 \ \Rightarrow$ DoF $\leq \frac{5M}{9}$

In this section, consider arbitrary (M,N) values such that $M/N \geq 4/5$. For all such values of (M,N), the DoF outer bound that holds is $d \leq \frac{5M}{9}$.

After the linear transformation that we introduce in Section 4, the resulting network and channel connectivity are shown in the following table.

$\lvert X_{1a_2}\rvert = N-M$	X_{1a_2}	$\circ$	$\circ$	$S_{1a_2}(X_{3a_2})$	$\lvert S_{1a_1}\rvert = N-M$
$\lvert X_{1a_1}\rvert = N-M$	X_{1a}	$\circ$	$\circ$	$S_{1a_1}(X_{3a_1}, X_{3a_2}, X_{3c_2}, X_{2a_2}, X_{2c_2})$	$\lvert S_{1a}\rvert = N-M$
$\lvert X_{1b_1}\rvert = 5M-4N$	X_{1b_1}	$\circ$	$\circ$	$S_{1b_1}(X_2, X_3)$	$\lvert S_{1b}\rvert = 4M-3N \geq (N-M)$
$\lvert X_{1c_1}\rvert = N-M$	X_{1c_1}	$\circ$	$\circ$	$S_{1c_1}(X_{2c_1}, X_{3a_2}, X_{3c_2}, X_{2a_2}, X_{2c_2})$	$\lvert S_{1c}\rvert = N-M$
$\lvert X_{1c_2}\rvert = N-M$	X_{1c_2}	$\circ$	$\circ$	$S_{1c_2}(X_{2c_2})$	$\lvert S_{1c_1}\rvert = N-M$

X_{2a_2}	$\circ$	$\circ$	$S_{2a_2}(X_{1a_2})$
X_{2a_1}	$\circ$	$\circ$	$S_{2a_1}(X_{1a_1}, X_{1a_2}, X_{1c_2}, X_{3a_2}, X_{3c_2})$
X_{2b_1}	$\circ$	$\circ$	$S_{2b_1}(X_3, X_1)$
X_{2c_1}	$\circ$	$\circ$	$S_{2c_1}(X_{3c_1}, X_{1a_2}, X_{1c_2}, X_{3a_2}, X_{3c_2})$
X_{2c_2}	$\circ$	$\circ$	$S_{2c_2}(X_{3c_2})$

X_{3a_2}	$\circ$	$\circ$	$S_{3a_2}(X_{2a_2})$
X_{3a_1}	$\circ$	$\circ$	$S_{3a_1}(X_{2a_1}, X_{2a_2}, X_{2c_2}, X_{1a_2}, X_{1c_2})$
X_{3b_1}	$\circ$	$\circ$	$S_{3b_1}(X_1, X_2)$
X_{3c_1}	$\circ$	$\circ$	$S_{3c_1}(X_{1c_1}, X_{2a_2}, X_{2c_2}, X_{1a_2}, X_{1c_2})$
X_{3c_2}	$\circ$	$\circ$	$S_{3c_2}(X_{1c_2})$

First a genie provides $\mathcal{G}_1 = \{X^n_{2a_2}, X^n_{3c_2}, X^n_{2a_1}, X^n_{2b_1}\}$ to receiver 1. Then the total number of dimensions available to receiver 1 (including those provided by the genie) is equal to:

$$\begin{aligned} N + |\mathcal{G}_1| &= N + |X^n_{2a_2}| + |X^n_{3c_2}| + |X^n_{2a_1}| + |X^n_{2b_1}| \\ &= N + (N-M) + (N-M) + (N-M) + (5M-4N) = 2M \end{aligned}$$

With these $2M$ dimensions, receiver 1 is able to resolve both interfering signals. Therefore, we have:

$$\begin{aligned} nR_\Sigma &\leq Nn\log\rho + h(X^n_{2a_2}, X^n_{3c_2}, X^n_{2a_1}, X^n_{2b_1}|\bar{Y}^n_1) + n\ o(\log\rho) + o(n) && (190)\\ &\leq Nn\log\rho + h(X^n_{3c_2}|\bar{Y}^n_1) + h(X^n_{2a_2}|\bar{Y}^n_1) + h(X^n_{2a_1}, X^n_{2b_1}|\bar{Y}^n_1, X^n_{2a_2}, X^n_{3c_2}) + n\ o(\log\rho) + o(n) && (191)\\ &\leq Nn\log\rho + h(X^n_{3c_2}) + h(X^n_{2a_2}|X^n_{2c_2}) + h(X^n_{2a_1}, X^n_{2b_1}|X^n_{2c_1}, X^n_{2a_2}, X^n_{3c_2}) + n\ o(\log\rho) + o(n) && (192)\\ &= Nn\log\rho + h(X^n_{3c_2}) + nR_2 - h(X^n_{2c_2}) - h(X^n_{2c_1}|X^n_{2a_2}, X^n_{2c_2}) + n\ o(\log\rho) + o(n) && (193) \end{aligned}$$

and by advancing user indices, therefore we have:

$$3nR \leq Nn\log\rho + nR - h(X^n_{c_1}|X^n_{a_2}, X^n_{c_2}) + n\ o(\log\rho) + o(n). \tag{194}$$

Second a genie provides $\mathcal{G}_2 = \{X^n_{2a_2}, X^n_{3c_2}, X^n_{3b_1}, X^n_{3c_1}\}$ to receiver 1. Then the total number of dimensions available to receiver 1 (including those provided by the genie) is equal to:

$$\begin{aligned} N + |\mathcal{G}_1| &= N + |X^n_{2a_2}| + |X^n_{3c_2}| + |X^n_{3b_1}| + |X^n_{3c_1}| \\ &= N + (N - M) + (N - M) + (5M - 4N) + (N - M) = 2M \end{aligned}$$

With these $2M$ dimensions, receiver 1 is able to resolve both interfering signals. Therefore, we have:

$$\begin{aligned} nR_\Sigma &\leq Nn\log\rho + h(X^n_{2a_2}, X^n_{3c_2}, X^n_{3b_1}, X^n_{3c_1}|\bar{Y}^n_1) + n\ o(\log\rho) + o(n) && (195) \\ &\leq Nn\log\rho + h(X^n_{2a_2}) + h(X^n_{3c_2}) + h(X^n_{3b_1}) + h(X^n_{3c_1}|X^n_{3a_2}, X^n_{3c_2}) + n\ o(\log\rho) + o(n) && (196) \\ &\leq Nn\log\rho + h(X^n_{2a_2}) + h(X^n_{3c_2}) + (5M-4N)n\log\rho + h(X^n_{3c_1}|X^n_{3a_2}, X^n_{3c_2}) + n\ o(\log\rho) + o(n) && (197) \end{aligned}$$

and by advancing user indices, therefore we have:

$$3nR \leq (5M - 3N)n\log\rho + nR + h(X^n_{a_2}) + h(X^n_{c_2}) - h(X^n_{c_1}|X^n_{a_2}, X^n_{c_2}) + n\ o(\log\rho) + o(n). \tag{198}$$

Third a genie provides $\mathcal{G}_3 = \{X^n_{2a_2}, X^n_{2a_1}, X^n_{2b_1}, X^n_{2c_1}\}$ to receiver 1. Then the total number of dimensions available to receiver 1 (including those provided by the genie) is equal to:

$$N + |\mathcal{G}_1| = N + (N - M) + (N - M) + (5M - 4N) + (N - M) = 2M$$

With these $2M$ dimensions, receiver 1 is able to resolve both interfering signals. Therefore, we have:

$$\begin{aligned} nR_\Sigma &\leq Nn\log\rho + h(X^n_{2a_2}, X^n_{2a_1}, X^n_{2b_1}, X^n_{2c_1}|\bar{Y}^n_1) + n\ o(\log\rho) + o(n) && (199) \\ &\leq Nn\log\rho + h(X^n_{2a_2}, X^n_{2a_1}, X^n_{2b_1}, X^n_{2c_1}|X^n_{2c_2}) + n\ o(\log\rho) + o(n) && (200) \\ &= Nn\log\rho + nR_2 - h(X^n_{2c_2}) + n\ o(\log\rho) + o(n). && (201) \end{aligned}$$

By advancing user indices, therefore we have:

$$3nR \leq Nn\log\rho + nR - h(X^n_{c_2}) + n\ o(\log\rho) + o(n). \tag{202}$$

Last a genie provides $\mathcal{G}_4 = \{X^n_{3c_2}, X^n_{3a_1}, X^n_{3b_1}, X^n_{3c_1}\}$ to receiver 1. With the similar analysis shown above, we obtain

$$3nR \leq Nn\log\rho + nR - h(X^n_{a_2}) + n\ o(\log\rho) + o(n). \tag{203}$$

Adding up the four inequalities in (194), (198), (202) and (203) we obtain:

$$12nR \leq 5Mn\log\rho + 3nR + n\ o(\log\rho) + o(n)$$

which implies that

$$d \leq \frac{5M}{9}.$$

A.2.4 Case: Any $M/N \in [3/4, 4/5] \Rightarrow$ DoF $\leq \frac{4N}{9}$

In this section, consider arbitrary (M, N) values such that $M/N \in [3/4, 4/5]$. For all such values of (M, N), the DoF outer bound that holds is $d \leq \frac{4N}{9}$.

With the linear transformation at both transmitter and receiver sides, we have the following table representing the network and corresponding connectivity.

$\lvert X_{1a_2}\rvert = N - M$	X_{1a_2}	$\circ$
$\lvert X_{1b}\rvert = 3M - 2N > (N - M)$	X_{1b}	$\circ$
$\lvert X_{1c_2}\rvert \leq N - M$	X_{1c_2}	$\circ$

$\circ$	$S_{1a_2}(X_{3a_2})$	$\lvert S_{1a_2}\rvert = N - M$
$\circ$	$S_{1a_1}(X_{3b}, X_{2a_2}, X_{2c_2}, X_{3a_2}, X_{3c_2})$	$\lvert S_{1a_1}\rvert = N - M$
$\circ$	$S_{1b_1}(X_2, X_3)$	$\lvert S_{1b}\rvert = 4M - 3N$
$\circ$	$S_{1c_1}(X_{2b}, X_{2a_2}, X_{2c_2}, X_{3a_2}, X_{3c_2})$	$\lvert S_{1c_1}\rvert = N - M$
$\circ$	$S_{1c_2}(X_{2c_2})$	$\lvert S_{1c_2}\rvert = N - M$

X_{2a_2}	$\circ$
X_{2b}	$\circ$
X_{2c_2}	$\circ$

$\circ$	$S_{2a_2}(X_{1a_2})$
$\circ$	$S_{2a_1}(X_{1b}, X_{3a_2}, X_{3c_2}, X_{1a_2}, X_{1c_2})$
$\circ$	$S_{2b_1}(X_3, X_1)$
$\circ$	$S_{2c_1}(X_{3b}, X_{3a_2}, X_{3c_2}, X_{1a_2}, X_{1c_2})$
$\circ$	$S_{2a_2}(X_{3c_2})$

X_{3a_2}	$\circ$
X_{3b}	$\circ$
X_{3c_2}	$\circ$

$\circ$	$S_{3a_2}(X_{2a_2})$
$\circ$	$S_{3a_1}(X_{2b}, X_{1a_2}, X_{1c_2}, X_{2a_2}, X_{2c_2})$
$\circ$	$S_{3b_1}(X_1, X_2)$
$\circ$	$S_{3c_1}(X_{1b}, X_{1a_2}, X_{1c_2}, X_{2a_2}, X_{2c_2})$
$\circ$	$S_{3c_2}(X_{1c_2})$

Similar to $M/N \in [1/2, 2/3]$ case, we at last multiply at the transmitter k a square matrix which is inverse of the channel matrix from X_{kb} to $(X_{(k-1)b_1}, X_{(k-1)c_1})$ for $k \in \{1, 2, 3\}$. This operation will help us only keep the necessary condition terms when we bound the sum rate of three messages, as what we have shown in $M/N \in [1/2, 2/3]$ case.

First a genie provides $\mathcal{G}_1 = \{X_{2a_2}^n, X_{3c_2}^n, (X_{2b}^n)_{[1:(N-M)]}\}$ to receiver 1. Then the total number of dimensions available to receiver 1 is equal to:

$$N + |\mathcal{G}_1| = N + (N - M) + (N - M) + (N - M) = 4N - 3M \geq 2M$$

With at least $2M$ dimensions among them, receiver 1 is able to resolve both interfering signals. Therefore, we have:

$$\begin{aligned}
nR_\Sigma &\leq Nn\log\rho + h(X_{2a_2}^n, X_{3c_2}^n, (X_{2b}^n)_{[1:(N-M)]}|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (204)\\
&\leq Nn\log\rho + h(X_{3c_2}^n) + h(X_{2a_2}^n|X_{2c_2}^n) + h((X_{2b}^n)_{[1:(N-M)]}|\bar{Y}_1^n, X_{2a_2}^n, X_{3c_2}^n) + n\ o(\log\rho) + o(n) && (205)\\
&\leq Nn\log\rho + h(X_{3c_2}^n) + h(X_{2a_2}^n|X_{2c_2}^n) + h((X_{2b}^n)_{[1:(N-M)]}|(X_{2b}^n)_{[(N-M+1):M]}, X_{2a_2}^n, X_{2c_2}^n) \\
&\quad + n\ o(\log\rho) + o(n) && (206)\\
&= Nn\log\rho + h(X_{3c_2}^n) + nR_2 - h(X_{2c_2}^n) - h((X_{2b}^n)_{[(N-M+1):(3M-2N)]}|X_{2a_2}^n, X_{2c_2}^n) \\
&\quad + n\ o(\log\rho) + o(n) && (207)
\end{aligned}$$

where (206) follows from the fact that dropping condition terms does not decrease the entropy. Specifically, with $X_{2a_2}^n, X_{3c_2}^n$ and $S_{1a_2}(X_{3a_2}^n), S_{1c_2}(X_{2c_2}^n)$, we can subtract them from S_{1c_1} and thus only leave a clean X_{2b}^n. Since we have inverted the associated channel at the transmitter 2, we can only keep the term $(X_{2b}^n)_{[(N-M+1):M]}$ in the condition. By advancing user indices, therefore we have:

$$3nR \leq Nn\log\rho + nR - h((X_b^n)_{[(N-M+1):(3M-2N)]}|X_{a_2}^n, X_{c_2}^n) + n\ o(\log\rho) + o(n). \tag{208}$$

Second a genie provides $\mathcal{G}_2 = \{X_{2a_2}^n, X_{3c_2}^n, (X_{3b}^n)_{[(N-M+1):(3M-2N)]}\}$ to receiver 1. Then the total number of dimensions available to receiver 1 is equal to:

$$N + |\mathcal{G}_2| = N + (N-M) + (N-M) + ((3M-2N) - (N-M+1) + 1) = 2M$$

With these $2M$ dimensions, receiver 1 is able to resolve both interfering signals. Therefore, we have:

$$\begin{aligned} nR_\Sigma &\leq Nn\log\rho + h(X_{2a_2}^n, X_{3c_2}^n, (X_{3b}^n)_{[(N-M+1):(3M-2N)]}|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (209)\\ &\leq Nn\log\rho + h(X_{2a_2}^n, X_{3c_2}^n) + h((X_{3b}^n)_{[(N-M+1):(3M-2N)]}|\bar{Y}_1^n, X_{2a_2}^n, X_{3c_2}^n) + n\ o(\log\rho) + o(n) && (210)\\ &\leq Nn\log\rho + h(X_{2a_2}^n) + h(X_{3c_2}^n) + h((X_{3b}^n)_{[(N-M+1):(3M-2N)]}|X_{3a_2}^n, X_{3c_2}^n) + n\ o(\log\rho) + o(n) && (211)\end{aligned}$$

and by advancing user indices, therefore we have:

$$3nR \leq Nn\log\rho + h(X_{a_2}^n) + h(X_{c_2}^n) + h(X_b^n)_{[(N-M+1):(3M-2N)]}|X_{a_2}^n, X_{c_2}^n) + n\ o(\log\rho) + o(n). \tag{212}$$

Third, if a genie provides $\mathcal{G}_3 = \{X_{2a_2}^n, X_{2b}^n\}$ to receiver 1. Then the total number of dimensions available to receiver 1 (including those provided by the genie) is equal to:

$$N + |\mathcal{G}_1| = N + (N-M) + (3M-2N) = 2M$$

With these $2M$ dimensions, receiver 1 is able to resolve both interfering signals. Therefore, we have:

$$\begin{aligned} nR_\Sigma &\leq Nn\log\rho + h(X_{2a_2}^n, X_{2b}^n|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (213)\\ &\leq Nn\log\rho + nR_2 - h(X_{2c_2}^n) + n\ o(\log\rho) + o(n). && (214)\end{aligned}$$

By advancing user indices, therefore we have:

$$3nR \leq Nn\log\rho + nR - h(X_{c_2}^n) + n\ o(\log\rho) + o(n). \tag{215}$$

Lastly a genie provides $\mathcal{G}_4 = \{X_{3b}^n, X_{3c_2}^n\}$ to receiver 1. With the similar analysis shown above, we obtain

$$3nR \leq Nn\log\rho + nR - h(X_{a_2}^n) + n\ o(\log\rho) + o(n). \tag{216}$$

Adding up the inequalities in (208), (212), (215) and (216) we obtain:

$$12nR \leq 4Nn\log\rho + 3nR + n\ o(\log\rho) + o(n)$$

which implies that

$$d \leq \frac{4N}{9}.$$

A.2.5 Case: Any $M/N \geq \frac{2L}{2L+1} \Rightarrow$ DoF $\leq \frac{2L+1}{4L+1}M$

For this case, the resulting network and channel connectivity after the linear transformation are shown in the following table.

X_{1a_L}	$\circ$	$\circ$	$S_{1a_L}(X_{3a_L})$
$\vdots$	$\circ$	$\circ$	$\vdots$
X_{1a_1}	$\circ$	$\circ$	$S_{1a_1}(X_{3a_1}, \{X_{2a_l}, X_{2c_l}, X_{3a_l}, X_{3c_l} \mid 2 \leq l \leq L\})$
X_{1b_1}	$\circ$	$\circ$	$S_{1b_1}(X_2, X_3)$
X_{1c_1}	$\circ$	$\circ$	$S_{1c_1}(X_{2c_1}, \{X_{2a_l}, X_{2c_l}, X_{3a_l}, X_{3c_l} \mid 2 \leq l \leq L\})$
$\vdots$	$\circ$	$\circ$	$\vdots$
X_{1c_L}	$\circ$	$\circ$	$S_{1c_L}(X_{2c_L})$

X_{2a_L}	$\circ$	$\circ$	$S_{2a_L}(X_{1a_L})$
$\vdots$	$\circ$	$\circ$	$\vdots$
X_{2a_1}	$\circ$	$\circ$	$S_{2a_1}(X_{1a_1}, \{X_{3a_l}, X_{3c_l}, X_{1a_l}, X_{1c_l} \mid 2 \leq l \leq L\})$
X_{2b_1}	$\circ$	$\circ$	$S_{2b_1}(X_3, X_1)$
X_{2c_1}	$\circ$	$\circ$	$S_{2c_1}(X_{3c_1}, \{X_{3a_l}, X_{3c_l}, X_{1a_l}, X_{1c_l} \mid 2 \leq l \leq L\})$
$\vdots$	$\circ$	$\circ$	$\vdots$
X_{2c_L}	$\circ$	$\circ$	$S_{2c_L}(X_{3c_L})$

X_{3a_L}	$\circ$	$\circ$	S_{3a_L}
$\vdots$	$\circ$	$\circ$	$\vdots$
X_{3a_1}	$\circ$	$\circ$	S_{3a_1}
X_{3b_1}	$\circ$	$\circ$	S_{3b_1}
X_{3c_1}	$\circ$	$\circ$	S_{3c_1}
$\vdots$	$\circ$	$\circ$	$\vdots$
X_{3c_L}	$\circ$	$\circ$	S_{3c_L}

We denote $\mathcal{G}_m^{2L},\ m \in \{1, \cdots, 2L\}$ as the the $2L$ sets of the genie signals provided to receiver 1 for the cases we consider, which can be written in an iterative form as follows: for $m' = \{1, 2, \cdots, L\}$ where L is the total number of layers,

$$\begin{aligned} \mathcal{G}_{2m'-1}^{2L} &= \{\{X_{2a_l}^n, X_{3c_l}^n \mid m'+1 \leq l \leq L\}, X_{2a_{m'}}^n, \{X_{2a_l}^n, X_{2c_l}^n \mid 1 \leq l \leq m'-1\}, X_{2b_1}^n\} \\ \mathcal{G}_{2m'}^{2L} &= \{\{X_{2a_l}^n, X_{3c_l}^n \mid m'+1 \leq l \leq L\}, X_{3c_{m'}}^n, \{X_{3a_l}^n, X_{3c_l}^n \mid 1 \leq l \leq m'-1\}, X_{3b_1}^n\} \end{aligned} \tag{217}$$

For each set shown above, the total number of dimensions available to receiver 1 including the dimensions provided by the genie is equal to:

$$N + |\mathcal{G}_m^{2L}| = N + (N-M)[2(L-1)+1] + [(2L+1)M - 2LN] = 2M.$$

Similar to the proof shown in Appendix A.1, we still use mathematical induction to prove the general result. Due to the similarities between the proofs in this subsection and in Appendix A.1, we will only highlight their difference.

With the intuition of onion peeling, we have a total of L layers at each user in this case. For the l^{th} layer where $2 \leq l \leq L$, there are two groups of signals $(X^n_{ka_l}, X^n_{kc_l})$ for user k with the cardinality $|X^n_{ka_l}| = |X^n_{kc_l}| = N - M$, while in the first layer, there are three groups of signals $(X^n_{ka_1}, X^n_{kb_1}, X^n_{kc_1})$ with the cardinality $|X^n_{ka_1}| = |X^n_{kc_1}| = N - M$ and $|X^n_{kb_1}| = (2L+1)M - 2LN$. Since we have proved $L = 1,\ 2$ cases previously, now let us assume it works for the $L-1$ case. That is to say, by providing genie signals $\mathcal{G}^{2(L-1)}_m$, $m \in \{1, \cdots, L-1\}$ to receiver 1 and by advancing user indices, we obtain a total of $(2L-2)$-rate per user inequalities. If we add up all these inequalities, at the left-hand side we have $3(2L-2)nR$; at the right-hand side we have $(2L-2)Nn\log\rho + [(2L+1)M - 2LN]n\log\rho + (2L-3)nR + n\ o(\log\rho) + o(n)$. In other words, we can bound the sum differential entropy of all provided genie signals to three users above by:

$$\sum_{m=1}^{2L-2}\sum_{k=1}^{3} h(\mathcal{G}^{2L-2}_{km}|\bar{Y}^n_k) \leq [(2L+1)M - 2LN]n\log\rho + (2L-3)nR + n\ o(\log\rho) + o(n). \tag{218}$$

Thus, if we add the L^{th} layer, the resulting summation of rate outer bounds provided by the first $2L-2$ inequalities, as shown in (165), in this case becomes:

$$\begin{aligned} 3(2L-2)nR \leq\ & (2L-2)Nn\log\rho + [(2L+1)M - 2LN]n\log\rho + (2L-3)nR \\ & + h(X^n_{a_L}) + h(X^n_{c_L}) + n\ o(\log\rho) + o(n) \qquad (219)\\ =\ & [(2L+1)M - 2N]n\log\rho + (2L-3)nR + h(X^n_{ka_L}) + h(X^n_{kc_L}) + n\ o(\log\rho) + o(n). \qquad (220) \end{aligned}$$

Next, if the genie provides the last two sets of signals $\mathcal{G}^{2L}_{2L-1}$ and $\mathcal{G}^{2L}_{2L}$ to receiver 1, respectively, we can easily obtain the last two inequalities as:

$$3nR \leq Nn\log\rho + nR - h(X^n_{c_L}) + n\ o(\log\rho) + o(n) \tag{221}$$
$$3nR \leq Nn\log\rho + nR - h(X^n_{a_L}) + n\ o(\log\rho) + o(n). \tag{222}$$

Add up inequalities in (220), (221) and (222), we eventually obtain

$$6LnR \leq (2L+1)Mn\log\rho + (2L-1)nR + n\ o(\log\rho) + o(n), \tag{223}$$

which implies that the sum DoF outer bound

$$d \leq \frac{2L+1}{4L+1}M. \tag{224}$$

A.2.6 Case: Any $M/N \in [\frac{2L-1}{2L}, \frac{2L}{2L+1}] \Rightarrow$ DoF $\leq \frac{2L}{4L+1}N$

X_{1a_L}	$\circ$
$\vdots$	$\circ$
X_{1a_2}	$\circ$
X_{1b}	$\circ$
X_{1c_2}	$\circ$
$\vdots$	$\circ$
X_{1c_L}	$\circ$

$\circ$	$S_{1a_L}(X_{3a_L})$
$\circ$	$\vdots$
$\circ$	$S_{1a_2}(X_{3a_2}, \{X_{2a_l}, X_{2c_l}, X_{3a_l}, X_{3c_l} \vert 3 \leq l \leq L\})$
$\circ$	$S_{1a_1}(X_{3b}, \{X_{2a_l}, X_{2c_l}, X_{3a_l}, X_{3c_l} \vert 2 \leq l \leq L\})$
$\circ$	$S_{1b_1}(X_2, X_3)$
$\circ$	$S_{1c_1}(X_{2b}, \{X_{2a_l}, X_{2c_l}, X_{3a_l}, X_{3c_l} \vert 2 \leq l \leq L\})$
$\circ$	$S_{1c_2}(X_{2c_2}, \{X_{2a_l}, X_{2c_l}, X_{3a_l}, X_{3c_l} \Vert 3 \leq l \leq L\})$
$\circ$	$\vdots$
$\circ$	$S_{1c_L}(X_{2c_L})$

X_{2a_L}	$\circ$
$\vdots$	$\circ$
X_{2a_2}	$\circ$
X_{2b}	$\circ$
X_{2c_2}	$\circ$
$\vdots$	$\circ$
X_{2c_2}	$\circ$

$\circ$	S_{2a_L}
$\circ$	$\vdots$
$\circ$	S_{2a_2}
$\circ$	S_{2a_1}
$\circ$	S_{2b_1}
$\circ$	S_{2c_1}
$\circ$	S_{2c_2}
$\circ$	$\vdots$
$\circ$	S_{2a_L}

X_{3a_L}	$\circ$
$\vdots$	$\circ$
X_{3a_2}	$\circ$
X_{3b}	$\circ$
X_{3c_2}	$\circ$
$\vdots$	$\circ$
X_{3c_2}	$\circ$

$\circ$	S_{3a_L}
$\vdots$	$\circ$
$\circ$	S_{3a_2}
$\circ$	S_{3a_1}
$\circ$	S_{3b_1}
$\circ$	S_{3c_1}
$\circ$	S_{3c_2}
$\vdots$	$\circ$
$\circ$	S_{3c_L}

With the intuition of onion peeling, we have a total of L layers at each user in this case. For the l^{th} layer where $2 \leq l \leq L$, there are two groups of signals $(X^n_{ka_l}, X^n_{kc_l})$ for user k with the cardinality $|X^n_{ka_l}| = |X^n_{kc_l}| = N - M$, while in the first layer, we only have $X^n_{kb_1}$ with the cardinality $|X^n_{kb}| = (2L-1)M - (2L-2)N$.

We denote $\mathcal{G}^{2L}_m,\ m \in \{1, \cdots, 2L\}$ as the the $2L$ sets of the provided genie signals to receiver 1, which can be written in an iterative form as follows:

$$
\begin{aligned}
&\text{for} \quad m' = 1, \\
&\mathcal{G}^{2L}_{2m'-1} = \{\{X^n_{2a_l}, X^n_{3c_l} | m'+1 \leq l \leq L\}, (X^n_{2b})_{[1:(N-M)]}\} \\
&\quad \mathcal{G}^{2L}_{2m'} = \{\{X^n_{2a_l}, X^n_{3c_l} | m'+1 \leq l \leq L\}, (X^n_{3b})_{[(N-M+1):((2L-1)M-(2L-2)N)]}\} \\
&\text{for} \quad m' = \{2, 3, \cdots, L\}, \\
&\mathcal{G}^{2L}_{2m'-1} = \{\{X^n_{2a_l}, X^n_{2c_l} | m'+1 \leq l \leq L\}, \{X^n_{2a_l}, X^n_{2c_l} | 2 \leq l \leq m'-1\}, X^n_{2a_{m'}}, X^n_{2b}\} \\
&\quad \mathcal{G}^{2L}_{2m'} = \{\{X^n_{3a_l}, X^n_{3c_l} | m'+1 \leq l \leq L\}, \{X^n_{3a_l}, X^n_{3c_l} | 2 \leq l \leq m'-1\}, X^n_{3c_{m'}}, X^n_{3b}\}
\end{aligned} \tag{225}
$$

For each set shown above, the total number of dimensions available to receiver 1 including the dimensions provided by the genie is equal to:

$$
\begin{aligned}
&N + |\mathcal{G}_1^{2L}| = N + (N - M)2(L - 1) + (N - M) \geq 2M \\
&N + |\mathcal{G}_2^{2L}| = N + (N - M)2(L - 1) + [((2L - 1)M - (2L - 2)N) - (N - M + 1) + 1] = 2M \\
&N + |\mathcal{G}_m^{2L}| = N + (N - M)2(L - 2) + (N - M) + [(2L - 1)M - (2L - 2)N] = 2M \quad \text{for } m \geq 3.
\end{aligned}
$$

The rigorous proof is still based on mathematical induction. Since we have shown the proof for $L = 1,\ 2$, the remaining part directly follows from what we have shown in Appendix A.1 by replacing the genie signals sets with that in (225).

A.2.7 Case: Any $M/N \geq \frac{2L+1}{2L+2} \Rightarrow$ DoF $\leq \frac{2L+2}{4L+3}M$

X_{1a_L}	$\circ$	$\circ$	$S_{1a_L}(X_{3a_L})$
$\vdots$	$\circ$	$\circ$	$\vdots$
X_{1a_1}	$\circ$	$\circ$	$S_{1a_1}(X_{3a_1}, \{X_{2a_l}, X_{2c_l}, X_{3a_l}, X_{3c_l} \mid 2 \leq l \leq L\})$
X_{1b_1}	$\circ$	$\circ$	$S_{1b_1}(X_2, X_3)$
X_{1c_1}	$\circ$	$\circ$	$S_{1c_1}(X_{2c_1}, \{X_{2a_l}, X_{2c_l}, X_{3a_l}, X_{3c_l} \mid 2 \leq l \leq L\})$
$\vdots$	$\circ$	$\circ$	$\vdots$
X_{1c_2}	$\circ$	$\circ$	$S_{1c_L}(X_{2c_L})$

X_{2a_L}	$\circ$	$\circ$	S_{2a_L}
$\vdots$	$\circ$	$\circ$	$\vdots$
X_{2a_1}	$\circ$	$\circ$	S_{2a_1}
X_{2b_1}	$\circ$	$\circ$	S_{2b_1}
X_{2c_1}	$\circ$	$\circ$	S_{2c_1}
$\vdots$	$\circ$	$\circ$	$\vdots$
X_{2c_L}	$\circ$	$\circ$	S_{2a_L}

X_{3a_L}	$\circ$	$\circ$	S_{3a_L}
$\vdots$	$\circ$	$\vdots$	$\circ$
X_{3a_2}	$\circ$	$\circ$	S_{3a_1}
X_{3b}	$\circ$	$\circ$	S_{3b_1}
X_{3c_2}	$\circ$	$\circ$	S_{3c_1}
$\vdots$	$\circ$	$\vdots$	$\circ$
X_{3c_2}	$\circ$	$\circ$	S_{3c_L}

The proof for this case still directly follows what we have shown in Appendix A.2 and B.1.1. The only difference is the first set of signals provided by the genie. Specifically, we write out the genie signal sets as follows:

$$\begin{aligned}
\mathcal{G}_1^{2L+1} &= \{\{X_{2a_l}^n, X_{3c_l}^n | 1 \le l \le L\}, (X_{3b_1}^n)_{[1:((2L+2)M-(2L+1)N)]}\} \\
\text{and for} \quad m' &= \{2, 3, \cdots, L\}, \\
\mathcal{G}_{2m'}^{2L+1} &= \{\{X_{2a_l}^n, X_{2c_l}^n | m'+1 \le l \le L\}, \{X_{2a_l}^n, X_{2c_l}^n | 2 \le l \le m'-1\}, X_{2a_{m'}}^n, X_{2b}^n\} \\
\mathcal{G}_{2m'+1}^{2L+1} &= \{\{X_{3a_l}^n, X_{3c_l}^n | m'+1 \le l \le L\}, \{X_{3a_l}^n, X_{3c_l}^n | 2 \le l \le m'-1\}, X_{3c_{m'}}^n, X_{3b}^n\} \quad (226)
\end{aligned}$$

Note that if we peel out all the l^{th} layers where $l \ge 2$, the aim of the first set of signals provided by the genie, e.g., the example $(M, N) = (3, 4)$ in Section 4.1.2 and Appendix A.2, is to produce the following terms

$$Nn \log \rho + h(X_{2a_1}^n | \{X_{2a_l}^n | 2 \le l \le L\}) + h(X_{3c_1}^n | \{X_{3c_l}^n | 2 \le l \le L\}) + n\ o(\log \rho) + o(n) \qquad (227)$$

on the right-hand side of the inequality, where the intermediate terms $h(X_{2a_1}^n | \{X_{2a_l}^n | 2 \le l \le L\})$ and $h(X_{3c_1}^n | \{X_{3c_l}^n | 2 \le l \le L\})$ can be be canceled by the summation of the remaining $2L$ inequalities (by providing $\mathcal{G}_m^{2L+1},\ 2 \le m \le 2L+1$) on the right-hand side. In this case, however, because $\mathcal{G}_1^{2L+1}$ includes an extra signal $(X_{3b_1}^n)_{[1:((2L+2)M-(2L+1)N)]}$ which will produce an extra term $((2L+2)M - (2L+1)N)n \log \rho$ in (227). Therefore, following the idea of previous proofs, by adding up the total of $2L+1$-inequalities, we can obtain:

$$3(2L+1)nR \le (2L+1)Nn \log \rho + ((2L+2)M - (2L+1)N)n \log \rho + 2LnR + n\ o(\log \rho) + o(n)$$

which implies that

$$d \le \frac{2L+2}{4L+3} M.$$

A.2.8 Case: Any $M/N \in [\frac{2L}{2L+1}, \frac{2L+1}{2L+2}] \Rightarrow$ DoF $\le \frac{2L+1}{4L+3} N$

The proof for this case is identical to that in Appendix A.1.2 by only noting $|X_{ka_l}| = |X_{kc_l}| = N - M$ where $l = 1, 2, \cdots, L$ and $|X_{kb_1}| = (2L+1)M - 2LN$.

B Information Theoretic DoF Outer Bound for $M_T > M_R$

In this section we consider the DoF outer bound for $M_T > M_R$, i.e., the $N \times M$ setting. Note that the linear transformation for the (N, M) case is still identical to that for the (M, N) case that we introduce in Section 6 but in a reciprocal manner. Therefore, we directly show the table representing each resulting network and corresponding channel connectivity for each case without specifying the linear transformation process in this section.

B.1 Outerbound Proofs around M/N ratios $2/3, 4/5, 6/7, \cdots$

Consider 3 user $N \times M$ MIMO interference channel, i.e., with N transmit antennas and M receive antennas for each user. We assume $M \le N$ throughout.

B.1.1 Case $(N,M)=(3,2) \Rightarrow$ DoF $\leq \frac{6}{5}$

X_{1a}	$\circ$
X_{1b}	$\circ$
X_{1c}	$\circ$

$\circ$	$S_{1a}(X_{2a}, X_{2b}, X_{3b})$
$\circ$	$S_{1c}(X_{2b}, X_{3b}, X_{3c})$

X_{2a}	$\circ$
X_{2b}	$\circ$
X_{2c}	$\circ$

$\circ$	$S_{2a}(X_{3a}, X_{3b}, X_{1b})$
$\circ$	$S_{2c}(X_{3b}, X_{1b}, X_{1c})$

X_{3a}	$\circ$
X_{3b}	$\circ$
X_{3c}	$\circ$

$\circ$	$S_{3a}(X_{1a}, X_{1b}, X_{2b})$
$\circ$	$S_{3c}(X_{1b}, X_{2b}, X_{2c})$

First, a genie provides $\mathcal{G}_1 = (X_{3a}^n, X_{3b}^n, X_{2b}^n, X_{2c}^n)$ to receiver 1:

$$\begin{aligned}
nR_\Sigma &\leq I(W_1, W_2, W_3; \bar{Y}_1^n, \mathcal{G}_1) + n\ o(\log\rho) + o(n) && (228)\\
&= I(W_1, W_2, W_3; \bar{Y}_1^n) + I(W_1, W_2, W_3; \mathcal{G}_1|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (229)\\
&\leq Mn\log\rho + h(X_{3a}^n, X_{3b}^n) + h(X_{2b}^n, X_{2c}^n) + n\ o(\log\rho) + o(n) && (230)\\
\hookrightarrow 3nR &\leq Mn\log\rho + h(X_a^n, X_b^n) + h(X_b^n, X_c^n) + n\ o(\log\rho) + o(n) && (231)
\end{aligned}$$

The symbol $\hookrightarrow$ stands for "follows from symmetry" that we have applied in previous section.

Second, a genie provides $\mathcal{G}_2 = (W_2, X_{3a}^n) \equiv (X_{2a}^n, X_{2b}^n, X_{2c}^n, X_{3a}^n)$ to receiver 1:

$$\begin{aligned}
nR_\Sigma &\leq I(W_1, W_2, W_3; \bar{Y}_1^n, \mathcal{G}_2) + n\ o(\log\rho) + o(n) && (232)\\
&\leq Mn\log\rho + h(\mathcal{G}_2|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (233)\\
&\leq Mn\log\rho + nR_2 + h(X_{3a}^n|X_{3b}^n, X_{3c}^n) + n\ o(\log\rho) + o(n) && (234)\\
\hookrightarrow 3nR &\leq Mn\log\rho + nR + h(X_a^n|X_b^n, X_c^n) + n\ o(\log\rho) + o(n) && (235)
\end{aligned}$$

Similarly if a genie provides $\mathcal{G}_3 = (W_3, X_{2c}^n) \equiv (X_{3a}^n, X_{3b}^n, X_{3c}^n, X_{2c}^n)$ to receiver 1, we obtain

$$3nR \leq Mn\log\rho + nR + h(X_c^n|X_b^n, X_a^n) + n\ o(\log\rho) + o(n) \qquad (236)$$

Adding (231), (235), (236) we have:

$$9nR \leq 3Mn\log\rho + 4nR + n\ o(\log\rho) + o(n) \qquad (237)$$

which implies that

$$d \leq \frac{3M}{5} = \frac{6}{5} \qquad (238)$$

B.1.2 Case: Any $M/N \geq 2/3 \Rightarrow$ DoF $\leq \frac{3M}{5}$

In this section, consider arbitrary M, N values such that $M/N \geq 2/3$. For all such values of M, N, the DoF outer bound that holds is DoF $\leq 3M/5$.

$\|X_{1a}\| = (N-M)$	X_{1a}	$\circ$	$\circ$	$S_{1a}(X_{2a}, X_{2b}, X_{3b})$	$\|S_{1a}\| = (N-M)$
$\|X_{1b}\| = N-2(N-M) \geq (N-M)$	X_{1b}	$\circ$	$\circ$	$S_{1b}(X_{2b}, X_{3b})$	$\|S_{1b}\| = M-2(N-M) \geq 0$
$\|X_{1c}\| = (N-M)$	X_{1c}	$\circ$	$\circ$	$S_{1c}(X_{2b}, X_{3b}, X_{3c})$	$\|S_{1c}\| = (N-M)$

$\|X_{2a}\| = (N-M)$	X_{2a}	$\circ$	$\circ$	$S_{2a}(X_{3a}, X_{3b}, X_{1b})$	$\|S_{2a}\| = (N-M)$
$\|X_{2b}\| = N-2(N-M) \geq (N-M)$	X_{2b}	$\circ$	$\circ$	$S_{2b}(X_{3b}, X_{1b})$	$\|S_{2b}\| = M-2(N-M) \geq 0$
$\|X_{2c}\| = (N-M)$	X_{2c}	$\circ$	$\circ$	$S_{2c}(X_{3b}, X_{1b}, X_{1c})$	$\|S_{2c}\| = (N-M)$

$\|X_{3a}\| = (N-M)$	X_{3a}	$\circ$	$\circ$	$S_{3a}(X_{1a}, X_{1b}, X_{2b})$	$\|S_{3a}\| = (N-M)$
$\|X_{3b}\| = N-2(N-M) \geq (N-M)$	X_{3b}	$\circ$	$\circ$	$S_{3b}(X_{1b}, X_{2b})$	$\|S_{3b}\| = M-2(N-M) \geq 0$
$\|X_{3c}\| = (N-M)$	X_{3c}	$\circ$	$\circ$	$S_{3c}(X_{1b}, X_{2b}, X_{2c})$	$\|S_{3c}\| = (N-M)$

Note that the size of $|X_b| \geq N - M$ and the size of $|S_b| \geq 0$ as indicated above, only because $M/N \geq 2/3$.

First, a genie provides $\mathcal{G}_1 = (X_{3a}, X_{3b}, X_{2b}, X_{2c})$ to receiver 1. Thus, the total number of dimensions available to the receiver 1 (including those provided by the genie) is:

$$M + |G| = M + |X_a| + 2|X_b| + |X_c| = M + N + |X_b| \geq M + N + (N - M) = 2N$$

With these (at least) $2N$ dimensions, the receiver is able to resolve both interfering signals. Therefore, once again, we have

$$\begin{aligned}
nR_\Sigma &\leq I(W_1, W_2, W_3; \bar{Y}_1^n, \mathcal{G}_1) + n\ o(\log\rho) + o(n) && (239)\\
&\leq Mn\log\rho + h(\mathcal{G}_1|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (240)\\
&\leq Mn\log\rho + h(X_{3a}^n, X_{3b}^n) + h(X_{2b}^n, X_{2c}^n) + n\ o(\log\rho) + o(n) && (241)\\
\hookrightarrow 3nR &\leq Mn\log\rho + h(X_a^n, X_b^n) + h(X_b^n, X_c^n) + n\ o(\log\rho) + o(n) && (242)
\end{aligned}$$

Second, a genie provides $\mathcal{G}_2 = (W_2, X_{3a}^n) \equiv (X_{2a}^n, X_{2b}^n, X_{2c}^n, X_{3a}^n)$ to receiver 1, then the total number of dimensions available to the receiver 1 is:

$$M + |G| = M + N + |X_a| = M + N + N - M = 2N$$

So, $2N$ dimensions are available, and all interference can be resolved.

$$\begin{aligned}
nR_\Sigma &\leq I(W_1, W_2, W_3; \bar{Y}_1^n, \mathcal{G}_2) + n\ o(\log\rho) + o(n) && (243)\\
&\leq Mn\log\rho + h(\mathcal{G}_2|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (244)\\
&\leq Mn\log\rho + nR_2 + h(X_{3a}^n|X_{3b}^n, X_{3c}^n) + n\ o(\log\rho) + o(n) && (245)\\
\hookrightarrow 3nR &\leq Mn\log\rho + nR + h(X_a^n|X_b^n, X_c^n) + n\ o(\log\rho) + o(n) && (246)
\end{aligned}$$

Similarly if a genie provides $\mathcal{G}_3 = (W_3, X_{2c}^n) \equiv (X_{3a}^n, X_{3b}^n, X_{3c}^n, X_{2c}^n)$ to receiver 1, we obtain

$$3nR \leq Mn\log\rho + nR + h(X_c^n|X_b^n, X_a^n) + n\ o(\log\rho) + o(n) \quad (247)$$

Adding (242), (246), (247) we have:

$$9nR \leq 3Mn\log\rho + 4nR + n\ o(\log\rho) + o(n) \tag{248}$$

which implies that

$$d \leq \frac{3M}{5}. \tag{249}$$

B.1.3 Case: Any $M/N \in [1/2, 2/3] \Rightarrow$ DoF $\leq \frac{2N}{5}$

In this section, consider arbitrary M, N values such that $M/N \leq 2/3$. For all such values of M, N, the DoF outer bound that holds is DoF $\leq 2N/5$.

$\lvert X_{1a}\rvert = (N-M)$	X_{1a}	$\circ$
$\lvert X_{1b}\rvert = N-2(N-M) \leq (N-M)$	X_{1b}	$\circ$
$\lvert X_{1c}\rvert = (N-M)$	X_{1c}	$\circ$

$\circ$	$S_1(X_{2a}, X_{2b}, X_{3b}, X_{3c})$	$\lvert S_1\rvert = M \geq N-M$

$\lvert X_{2a}\rvert = (N-M)$	X_{2a}	$\circ$
$\lvert X_{2b}\rvert = N-2(N-M) \leq (N-M)$	X_{2b}	$\circ$
$\lvert X_{2c}\rvert = (N-M)$	X_{2c}	$\circ$

$\circ$	$S_2(X_{3a}, X_{3b}, X_{1b}, X_{1c})$	$\lvert S_2\rvert = M \geq N-M$

$\lvert X_{3a}\rvert = (N-M)$	X_{3a}	$\circ$
$\lvert X_{3b}\rvert = N-2(N-M) \leq (N-M)$	X_{3b}	$\circ$
$\lvert X_{3c}\rvert = (N-M)$	X_{3c}	$\circ$

$\circ$	$S_3(X_{1a}, X_{1b}, X_{2b}, X_{2c})$	$\lvert S_3\rvert = M \geq N-M$

Note that the size of $|X_b| \leq N-M$ as indicated above, because $M/N \leq 2/3$.

First, a genie provides $\mathcal{G}_1 = (X_{3a}^n, X_{3b}^n, (X_{2a}^n, X_{3c}^n)_{[2N-3M]}, X_{2b}^n, X_{2c}^n)$ to receiver 1. Note that $(X_{2a}^n, X_{3c}^n)_{[\min(2N-3M, N-M)]}$ represents the extra dimensions provided to the receiver so that the total number of dimensions is at least $2N$ (the minimum necessary to resolve all interference). This is verified as follows.

The receiver already has M dimensions. The usual genie information $X_{3a}^n, X_{3b}^n, X_{2b}^n, X_{2c}^n$ constitutes another $N + |X_b^n| = N + 2M - N = 2M$ dimensions, for a total of $3M$ dimensions. This is still less than $2N$ because $M/N \leq 2/3$. So we need another $2N-3M$ dimensions. So we provide any $2N-3M$ elements from X_{2a}^n, X_{3c}^n, which is the *extra* genie information indicated as $(X_{2a}^n, X_{3c}^n)_{[2N-3M]}$. Thus the total number of dimensions available to the receiver (including those provided by the genie) is $2N$, and all interference can be resolved. In the outer bound, the extra dimensions are directly accounted in the prelog factor and dropped from the conditioning terms (which can only increase the entropy terms).

$$\begin{aligned} nR_\Sigma &\leq I(W_1, W_2, W_3; \bar{Y}_1^n, \mathcal{G}_1) + n\ o(\log\rho) + o(n) && (250)\\ &\leq Mn\log\rho + h(\mathcal{G}_1|\bar{Y}_1^n) + n\ o(\log\rho) + n\ o(\log\rho) + o(n) && (251)\\ &\leq Mn\log\rho + (2N-3M)n\log\rho + h(X_{3a}^n, X_{3b}^n) + h(X_{2b}^n, X_{2c}^n) + n\ o(\log\rho) + o(n) && (252)\\ \hookrightarrow 3nR &\leq (M + 2N - 3M)n\log\rho + h(X_a^n, X_b^n) + h(X_b^n, X_c^n) + n\ o(\log\rho) + o(n) && (253) \end{aligned}$$

Second, a genie provides $\mathcal{G}_2 = (W_2, X_{3a}^n) \equiv (X_{2a}^n, X_{2b}^n, X_{2c}^n, X_{3a}^n)$ to receiver 1, then the total number of dimensions available to receiver 1 is:

$$M + |G| = M + N + |X_a| = M + N + N - M = 2N$$

So, $2N$ dimensions are available, no *extra* genie information is needed, and all interference can be resolved.

$$\begin{aligned}
nR_\Sigma &\leq I(W_1, W_2, W_3; \bar{Y}_1^n, \mathcal{G}_2) + n\ o(\log\rho) + o(n) && (254)\\
&\leq Mn\log\rho + h(\mathcal{G}_2|\bar{Y}_1^n) + n\ o(\log\rho) + n\ o(\log\rho) + o(n) && (255)\\
&\leq Mn\log\rho + nR_2 + h(X_{3a}^n|X_{3b}^n, X_{3c}^n) + n\ o(\log\rho) + o(n) && (256)\\
\hookrightarrow 3nR &\leq M + nR + h(X_a^n|X_b^n, X_c^n) && (257)
\end{aligned}$$

Similarly if a genie provides $\mathcal{G}_3 = (W_3, X_{2c}^n) \equiv (X_{3a}^n, X_{3b}^n, X_{3c}^n, X_{2c}^n)$ to receiver 1, we obtain

$$3nR \leq Mn\log\rho + nR + h(X_c^n|X_b^n, X_a^n) + n\ o(\log\rho) + o(n) \qquad (258)$$

Adding (253), (257), (258) we have

$$\begin{aligned}
9nR &\leq (3M + 2N - 3M)n\log\rho + 2nR + 2nR + n\ o(\log\rho) + o(n) && (259)\\
\Rightarrow d &\leq \frac{2N}{5} && (260)
\end{aligned}$$

B.1.4 Case $(N, M) = (5, 4) \Rightarrow$ DoF $\leq \frac{20}{9}$

X_{1a_1}	$\circ$
X_{1a}	$\circ$
X_{1b}	$\circ$
X_{1c}	$\circ$
X_{1c_1}	$\circ$

$\circ$	$S_{1a_1}(X_{2a_1}, X_{2a}, X_{2b}, X_{2c}, X_{3a}, X_{3b}, X_{3c})$
$\circ$	$S_{1a}(X_{2a}, X_{2b}, X_{3b})$
$\circ$	$S_{1c}(X_{2b}, X_{3b}, X_{3c})$
$\circ$	$S_{1c_1}(X_{2a}, X_{2b}, X_{2c}, X_{3a}, X_{3b}, X_{3c}, X_{3c_1})$

X_{2a_1}	$\circ$
X_{2a}	$\circ$
X_{2b}	$\circ$
X_{2c}	$\circ$
X_{2c_1}	$\circ$

$\circ$	$S_{2a_1}(X_{3a_1}, X_{3a}, X_{3b}, X_{3c}, X_{1a}, X_{1b}, X_{1c})$
$\circ$	$S_{2a}(X_{3a}, X_{3b}, X_{1b})$
$\circ$	$S_{2c}(X_{3b}, X_{1b}, X_{1c})$
$\circ$	$S_{2c_1}(X_{3a}, X_{3b}, X_{3c}, X_{1a}, X_{1b}, X_{1c}, X_{1c_1})$

X_{3a_1}	$\circ$
X_{3a}	$\circ$
X_{3b}	$\circ$
X_{3c}	$\circ$
X_{3c_1}	$\circ$

$\circ$	$S_{3a_1}(X_{1a_1}, X_{1a}, X_{1b}, X_{1c}, X_{2a}, X_{2b}, X_{2c})$
$\circ$	$S_{3a}(X_{1a}, X_{1b}, X_{2b})$
$\circ$	$S_{3c}(X_{1b}, X_{2b}, X_{2c})$
$\circ$	$S_{3c_1}(X_{1a}, X_{1b}, X_{1c}, X_{2a}, X_{2b}, X_{2c}, X_{2c_1})$

First, a genie provides $\mathcal{G}_1 = (X_{3a_1}^n, X_{3a}^n, X_{3b}^n, X_{2b}^n, X_{2c}^n, X_{2c_1}^n)$ to receiver 1, then we have:

$$\begin{aligned}
nR_\Sigma &\le I(W_1, W_2, W_3; \bar{Y}_1^n, \mathcal{G}_1) + n\ o(\log\rho) + o(n) && (261)\\
&\le Mn\log\rho + h(\mathcal{G}_1|\bar{Y}_1^n) + n\ o(\log\rho) + n\ o(\log\rho) + o(n) && (262)\\
&\le Mn\log\rho + h(X_{3a}^n, X_{3b}^n) + h(X_{2b}^n, X_{2c}^n) + n\ o(\log\rho) + o(n)\\
&\quad + h(X_{3a_1}^n|X_{3a}^n, X_{3b}^n, X_{3c}^n, X_{3c_1}^n) + h(X_{2c_1}^n|X_{2a_1}^n, X_{2a}^n, X_{2b}^n, X_{2c}^n) && (263)\\
\hookrightarrow 3nR &\le Mn\log\rho + h(X_a^n, X_b^n) + h(X_b^n, X_c^n) + n\ o(\log\rho) + o(n)\\
&\quad + h(X_{a_1}^n|X_a^n, X_b^n, X_c^n, X_{c_1}^n) + h(X_{c_1}^n|X_{a_1}^n, X_a^n, X_b^n, X_c^n) && (264)\\
\Rightarrow 3nR &\le Mn\log\rho + h(X_a^n, X_b^n) + h(X_b^n, X_c^n) + h(X_{a_1}^n, X_{c_1}^n|X_a^n, X_b^n, X_c^n) + n\ o(\log\rho) + o(n) && (265)
\end{aligned}$$

Second, if a genie provides $\mathcal{G}_2 = (X_{3a_1}^n, X_{3a}^n, X_{2a}^n, X_{2b}^n, X_{2c}^n, X_{2c_1}^n)$ to receiver 1, then we have:

$$\begin{aligned}
nR_\Sigma &\le I(W_1, W_2, W_3; \bar{Y}_1^n, \mathcal{G}_2) + n\ o(\log\rho) + o(n) && (266)\\
&\le Mn\log\rho + h(\mathcal{G}_2|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (267)\\
&\le Mn\log\rho + h(X_{2a}^n, X_{2b}^n, X_{2c}^n, X_{2c_1}^n) + h(X_{3a}^n|X_{3b}^n, X_{3c}^n) + h(X_{3a_1}^n|X_{3a}^n, X_{3b}^n, X_{3c}^n, X_{3c_1}^n)\\
&\quad + n\ o(\log\rho) + o(n) && (268)\\
\hookrightarrow 3nR &\le Mn\log\rho + h(X_a^n, X_b^n, X_c^n, X_{c_1}^n) + h(X_a^n|X_b^n, X_c^n) + h(X_{a_1}^n|X_a^n, X_b^n, X_c^n, X_{c_1}^n)\\
&\quad + n\ o(\log\rho) + o(n) && (269)\\
\Rightarrow 3nR &\le Mn\log\rho + nR + h(X_a^n|X_b^n, X_c^n) + n\ o(\log\rho) + o(n) && (270)
\end{aligned}$$

Similarly if a genie provides $\mathcal{G}_3 = (X_{2c_1}^n, X_{2c}^n, X_{3c}^n, X_{3b}^n, X_{3a}^n, X_{3a_1}^n)$ to receiver 1, then we obtain:

$$3nR \le Mn\log\rho + nR + h(X_c^n|X_a^n, X_b^n) + n\ o(\log\rho) + o(n) \qquad (271)$$

Last a genie provides $\mathcal{G}_4 = (X_{3a_1}^n, \underbrace{X_{2a_1}^n, X_{2a}^n, X_{2b}^n, X_{2c}^n, X_{2c_1}^n}_{W_2})$ to receiver 1, we have:

$$\begin{aligned}
nR_\Sigma &\le I(W_1, W_2, W_3; \bar{Y}_1^n, \mathcal{G}_4) + n\ o(\log\rho) + o(n) && (272)\\
&\le Mn\log\rho + h(\mathcal{G}_4|\bar{Y}_1^n) + n\ o(\log\rho) + o(n) && (273)\\
&\le Mn\log\rho + nR_2 + h(X_{3a_1}^n|X_{3a}^n, X_{3b}^n, X_{3c}^n, X_{3c_1}^n) + n\ o(\log\rho) + o(n) && (274)\\
\hookrightarrow 3nR &\le Mn\log\rho + nR + h(X_{a_1}^n|X_a^n, X_b^n, X_c^n, X_{c_1}^n) + n\ o(\log\rho) + o(n) && (275)
\end{aligned}$$

and similarly if a genie provides $\mathcal{G}_5 = (X_{2c_1}^n, \underbrace{X_{2c_1}^n, X_{3a}^n, X_{3b}^n, X_{3c}^n, X_{3c_1}^n}_{W_3})$ to receiver 1 then we have:

$$3nR \le Mn\log\rho + nR + h(X_{c_1}^n|X_{a_1}^n, X_a^n, X_b^n, X_c^n) + n\ o(\log\rho) + o(n) \qquad (276)$$

Finally, adding (265), (270), (271), (275), (276), we obtain

$$\begin{aligned}
15nR &\le 6nR + 5Mn\log\rho + n\ o(\log\rho) + o(n) && (277)\\
\Rightarrow d &\le \frac{5M}{9} = \frac{20}{9} && (278)
\end{aligned}$$

B.1.5 Case: Any $M/N \geq 4/5 \Rightarrow$ DoF $\leq \frac{5M}{9}$

$\lvert X_{1a_1}\rvert = N-M$	X_{1a_1}	$\circ$
$\lvert X_{1a}\rvert = N-M$	X_{1a}	$\circ$
$\lvert X_{1b}\rvert \geq N-M$	X_{1b}	$\circ$
$\lvert X_{1c}\rvert = N-M$	X_{1c}	$\circ$
$\lvert X_{1c_1}\rvert = N-M$	X_{1c_1}	$\circ$

$\circ$	$S_{1a_1}(X_{2a_1}, X_{2a}, X_{2b}, X_{2c}, X_{3a}, X_{3b}, X_{3c})$	$\lvert S_{1a_1}\rvert = N-M$
$\circ$	$S_{1a}(X_{2a}, X_{2b}, X_{3b})$	$\lvert S_{1a}\rvert = N-M$
$\circ$	$S_{1b}(X_{2a_1}, X_{2a}, X_{2b}, X_{2c}, X_{3a}, X_{3b}, X_{3c}, X_{3c_1})$	$\lvert S_{1b}\rvert \geq 0$
$\circ$	$S_{1c}(X_{2b}, X_{3b}, X_{3c})$	$\lvert S_{1c}\rvert = N-M$
$\circ$	$S_{1c_1}(X_{2a}, X_{2b}, X_{2c}, X_{3a}, X_{3b}, X_{3c}, X_{3c_1})$	$\lvert S_{1c_1}\rvert = N-M$

X_{2a_1}	$\circ$
X_{2a}	$\circ$
X_{2b}	$\circ$
X_{2c}	$\circ$
X_{2c_1}	$\circ$

$\circ$	$S_{2a_1}(X_{3a_1}, X_{3a}, X_{3b}, X_{3c}, X_{1a}, X_{1b}, X_{1c})$
$\circ$	$S_{2a}(X_{3a}, X_{3b}, X_{1b})$
$\circ$	S_{2b}
$\circ$	$S_{2c}(X_{3b}, X_{1b}, X_{1c})$
$\circ$	$S_{2c_1}(X_{3a}, X_{3b}, X_{3c}, X_{1a}, X_{1b}, X_{1c}, X_{1c_1})$

X_{3a_1}	$\circ$
X_{3a}	$\circ$
X_{3b}	$\circ$
X_{3c}	$\circ$
X_{3c_1}	$\circ$

$\circ$	$S_{3a_1}(X_{1a_1}, X_{1a}, X_{1b}, X_{1c}, X_{2a}, X_{2b}, X_{2c})$
$\circ$	$S_{3a}(X_{1a}, X_{1b}, X_{2b})$
$\circ$	S_{3b}
$\circ$	$S_{3c}(X_{1b}, X_{2b}, X_{2c})$
$\circ$	$S_{3c_1}(X_{1a}, X_{1b}, X_{1c}, X_{2a}, X_{2b}, X_{2c}, X_{2c_1})$

All genies work exactly as in the $(N, M) = (5, 4)$ case. Re-write the same equations, substituting M instead of 4 in each of (265), (270), (271), (275), (276), so that we obtain the bound:

$$R \;\leq\; \frac{5M}{9}$$

B.1.6 Case: Any $M/N \in [3/4, 4/5] \Rightarrow$ DoF $\leq \frac{4N}{9}$

$\lvert X_{1a_1}\rvert = N-M$	X_{1a_1}	$\circ$
$\lvert X_{1a}\rvert = N-M$	X_{1a}	$\circ$
$\lvert X_{1b}\rvert \leq N-M$	X_{1b}	$\circ$
$\lvert X_{1c}\rvert = N-M$	X_{1c}	$\circ$
$\lvert X_{1c_1}\rvert = N-M$	X_{1c_1}	$\circ$

$\circ$	$S_{1a_1}(X_{2a_1}, X_{2a}, X_{2b}, X_{2c}, X_{3a}, X_{3b}, X_{3c})$	$\lvert S_{1a_1}\rvert = N-M$
$\circ$	$S_1(X_{2a}, X_{2b}, X_{2c}, X_{3a}, X_{3b}, X_{3c})$	$\lvert S_1\rvert \geq N-M$
$\circ$	$S_{1c_1}(X_{2a}, X_{2b}, X_{2c}, X_{3a}, X_{3b}, X_{3c}, X_{3c_1})$	$\lvert S_{1c_1}\rvert = N-M$

X_{2a_1}	$\circ$
X_{2a}	$\circ$
X_{2b}	$\circ$
X_{2c}	$\circ$
X_{2c_1}	$\circ$

$\circ$	$S_{2a_1}(X_{3a_1}, X_{3a}, X_{3b}, X_{3c}, X_{1a}, X_{1b}, X_{1c})$
$\circ$	S_2
$\circ$	$S_{2c_1}(X_{3a}, X_{3b}, X_{3c}, X_{1a}, X_{1b}, X_{1c}, X_{1c_1})$

X_{3a_1}	$\circ$
X_{3a}	$\circ$
X_{3b}	$\circ$
X_{3c}	$\circ$
X_{3c_1}	$\circ$

$\circ$	$S_{3a_1}(X_{1a_1}, X_{1a}, X_{1b}, X_{1c}, X_{2a}, X_{2b}, X_{2c})$
$\circ$	S_3
$\circ$	$S_{3c_1}(X_{1a}, X_{1b}, X_{1c}, X_{2a}, X_{2b}, X_{2c}, X_{2c_1})$

Once again, only Genie 1 needs *extra* dimensions. Precisely an additional $4N-5M$ extra dimensions are needed, so that in place of (265) we get

$$\begin{aligned} 3nR &\leq (M+4N-5M)n\log\rho + h(X_a^n, X_b^n) + h(X_b^n, X_c^n) + h(X_{a_1}^n, X_{c_1}^n | X_a^n, X_b^n, X_c^n) \\ &\quad +n\ o(\log\rho) + o(n) \end{aligned} \tag{279}$$

The remaining genie bounds remain unaffected. Adding the corresponding bounds as before we obtain:

$$15nR \leq (4N-5M+5M)n\log\rho + 6nR + n\ o(\log\rho) + o(n) \tag{280}$$

$$\Rightarrow d \leq \frac{4N}{9} \tag{281}$$

B.1.7 Case: $(N,M) = (2L+3, 2L+2) \Rightarrow$ DoF $\leq \frac{MN}{M+N}$

X_{1a_L}	$\circ$
$\vdots$	$\circ$
X_{1a_1}	$\circ$
X_{1a_0}	$\circ$
X_{1b_0}	$\circ$
X_{1c_0}	$\circ$
X_{1c_1}	$\circ$
$\vdots$	$\circ$
X_{1c_L}	$\circ$

$\circ$	$S_{1a_L}(X_{2a_L}, \cdots, X_{2c_{L-1}}, X_{3a_{L-1}}, \cdots, X_{3c_{L-1}})$
$\circ$	$\vdots$
$\circ$	$S_{1a_1}(X_{2a_1}, X_{2a_0}, X_{2b_0}, X_{2c_0}, X_{3a_0}, X_{3b_0}, X_{3c_0})$
$\circ$	$S_{1a_0}(X_{2a_0}, X_{2b_0}, X_{3b_0})$
$\circ$	$S_{1c_0}(X_{2b_0}, X_{3b_0}, X_{3c_0})$
$\circ$	$S_{1c_1}(X_{2a_0}, X_{2b_0}, X_{2c_0}, X_{3a_0}, X_{3b_0}, X_{3c_0}, X_{3c_1})$
$\circ$	$\vdots$
$\circ$	$S_{1c_L}(X_{2a_{L-1}}, \cdots, X_{2c_{L-1}}, X_{3a_{L-1}}, \cdots X_{3c_L})$

X_{2a_L}	$\circ$
$\vdots$	$\circ$
X_{2a_1}	$\circ$
X_{2a_0}	$\circ$
X_{2b_0}	$\circ$
X_{2c_0}	$\circ$
X_{2c_1}	$\circ$
$\vdots$	$\circ$
X_{2c_L}	$\circ$

$\circ$	$S_{2a_L}(X_{3a_L}, \cdots, X_{3c_{L-1}}, X_{1a_{L-1}}, \cdots, X_{1c_{L-1}})$
$\circ$	$\vdots$
$\circ$	$S_{2a_1}(X_{3a_1}, X_{3a_0}, X_{3b_0}, X_{3c_0}, X_{1a_0}, X_{1b_0}, X_{1c_0})$
$\circ$	$S_{2a_0}(X_{3a_0}, X_{3b_0}, X_{1b_0})$
$\circ$	$S_{2c_0}(X_{3b_0}, X_{1b_0}, X_{1c_0})$
$\circ$	$S_{2c_1}(X_{3a_0}, X_{3b_0}, X_{3c_0}, X_{1a_0}, X_{1b_0}, X_{1c_0}, X_{1c_1})$
$\circ$	$\vdots$
$\circ$	$S_{2c_L}(X_{3a_{L-1}}, \cdots, X_{3c_{L-1}}, X_{1a_{L-1}}, \cdots X_{1c_L})$

X_{3a_L}	$\circ$
$\vdots$	$\circ$
X_{3a_1}	$\circ$
X_{3a_0}	$\circ$
X_{3b_0}	$\circ$
X_{3c_0}	$\circ$
X_{3c_1}	$\circ$
$\vdots$	$\circ$
X_{3c_L}	$\circ$

$\circ$	S_{3a_L}
$\circ$	$\vdots$
$\circ$	S_{3a_1}
$\circ$	S_{3a_0}
$\circ$	S_{3c_0}
$\circ$	S_{3c_1}
$\circ$	$\vdots$
$\circ$	S_{3c_L}

The rigorous proof for the general case is similar to that shown in Section 5. For brevity, we only provide the genie signals and show the bounds that we eventually obtain, and all of them are easily to check.

First, a genie provides $\mathcal{G}_1 = (X^n_{3a_L}, \cdots, X^n_{3b_0}, X^n_{2b_0}, \cdots, X^n_{2c_L})$ to receiver 1, and we obtain:

$$\begin{aligned} 3nR &\leq Mn\log\rho + h(X^n_{a_0}, X^n_{b_0}) + h(X^n_{b_0}, X^n_{c_0}) + \sum_{i=1}^{L} h(X^n_{a_i}, X^n_{c_i} | X_{a^n_{i-1}}, \cdots, X^n_{c_{i-1}}) \\ &\quad + n\ o(\log\rho) + o(n). \end{aligned} \tag{282}$$

Second, if a genie provides $\mathcal{G}_2 = (X^n_{3a_L}, \cdots, X^n_{3a_0}, X^n_{2a_0}, \cdots, X^n_{2c_L})$ and $\mathcal{G}_3 = (X^n_{2c_L}, \cdots, X^n_{2c_0}, X^n_{3c_0}, \cdots, X^n_{3a_L})$ to receiver 1, respectively, we eventually obtain:

$$6nR \leq 2Mn\log\rho + 2nR + h(X^n_{a_0}|X^n_{b_0}, X^n_{c_0}) + h(X^n_{c_0}|X^n_{a_0}, X^n_{b_0}) + n\ o(\log\rho) + o(n). \tag{283}$$

Third, if a genie provides $\mathcal{G}_4 = (X^n_{3a_L}, \cdots, X^n_{3a_1}, X^n_{2a_1}, \cdots, X^n_{2c_L})$ and $\mathcal{G}_5 = (X^n_{2c_L}, \cdots, X^n_{2c_1}, X^n_{3c_1}, \cdots, X^n_{3a_L})$ to receiver 1, respectively, we eventually obtain:

$$6nR \leq 2Mn\log\rho + 2nR + h(X^n_{a_1}, X^n_{c_1}|X^n_{a_0}, X^n_{b_0}, X^n_{c_0}) + n\ o(\log\rho) + o(n). \tag{284}$$

$\vdots$

Last, if a genie provides $\mathcal{G}_{2L+2} = (X^n_{3a_L}, X^n_{2a_L}, \cdots, X^n_{2c_L})$ and $\mathcal{G}_{2L+3} = (X^n_{2c_L}, X^n_{3c_L}, \cdots, X^n_{3a_L})$ to receiver 1, respectively, we eventually obtain:

$$6nR \leq 2Mn\log\rho + 2nR + h(X^n_{a_L}, X^n_{c_L}|X^n_{a_{L-1}}, \cdots, X^n_{c_{L-1}}) + n\ o(\log\rho) + o(n). \tag{285}$$

Adding all the outer bounds, we have:

$$\begin{aligned} 3(2L+3)nR &\leq (2L+3)Mn\log\rho + (2L+4)nR + n\ o(\log\rho) + o(n) && (286) \\ \Rightarrow d &\leq \frac{(2L+3)M}{4L+5} && (287) \\ \Rightarrow d &\leq \frac{MN}{M+N} && (288) \end{aligned}$$

And the bounds for $M/N \leq \frac{2L+2}{2L+3}$ and $M/N \geq \frac{2L+2}{2L+3}$ follow similarly as well.

B.2 Outerbound Proofs around M/N ratios $3/4, 5/6, \cdots$

B.2.1 Case $(N, M) = (4, 3) \Rightarrow$ DoF $\leq \frac{12}{7}$

X_{1a_1}	$\circ$
X_{1a_0}	$\circ$
X_{1c_0}	$\circ$
X_{1c_1}	$\circ$

$\circ$	$S_{1a_0}(X_{2a_1}, X_{2a_0}, X_{2c_0}, X_{3a_0}, X_{3c_0})$
$\circ$	$S_{1b_0}(X_{2a_0}, X_{3c_0})$
$\circ$	$S_{1c_0}(X_{2a_0}, X_{2c_0}, X_{3a_0}, X_{3c_0}, X_{3c_1})$

X_{2a_1}	$\circ$
X_{2a_0}	$\circ$
X_{2c_0}	$\circ$
X_{2c_1}	$\circ$

$\circ$	$S_{2a_0}(X_{3a_1}, X_{3a_0}, X_{3c_0}, X_{1a_0}, X_{1c_0})$
$\circ$	$S_{2b_0}(X_{3a_0}, X_{1c_0})$
$\circ$	$S_{2c_0}(X_{3a_0}, X_{3c_0}, X_{1a_0}, X_{1c_0}, X_{1c_1})$

X_{3a_1}	$\circ$
X_{3a_0}	$\circ$
X_{3c_0}	$\circ$
X_{3c_1}	$\circ$

$\circ$	$S_{3a_0}(X_{1a_1}, X_{1a_0}, X_{1c_0}, X_{2a_0}, X_{2c_0})$
$\circ$	$S_{3b_0}(X_{1a_0}, X_{2c_0})$
$\circ$	$S_{3c_0}(X_{1a_0}, X_{1c_0}, X_{2a_0}, X_{2c_0}, X_{2c_1})$

First, a genie provides $\mathcal{G}_1 = (X_{3a_1}^n, X_{3a_0}^n, X_{2a_0}^n, X_{2c_0}^n, X_{2c_1}^n)$ to receiver 1, then we have:

$$
\begin{aligned}
nR_\Sigma &\leq Mn\log\rho + h(X_{2a_0}^n, X_{2c_0}^n) + h(X_{3a_0}^n|X_{3c_0}^n) + n\ o(\log\rho) + o(n) \\
&\quad + h(X_{2c_1}^n|X_{2a_1}^n, X_{2a_0}^n, X_{2c_0}^n) + h(X_{3a_1}^n|X_{3a_0}^n, X_{3c_0}^n, X_{3c_1}^n) \qquad (289) \\
\hookrightarrow 3nR &\leq Mn\log\rho + h(X_{a_0}^n, X_{c_0}^n) + h(X_{a_0}^n|X_{c_0}^n) + n\ o(\log\rho) + o(n) \\
&\quad + h(X_{c_1}^n|X_{a_1}^n, X_{a_0}^n, X_{c_0}^n) + h(X_{a_1}^n|X_{a_0}^n, X_{c_0}^n, X_{c_1}^n) \qquad (290) \\
&\leq Mn\log\rho + h(X_{a_0}^n, X_{c_0}^n) + h(X_{a_0}^n|X_{c_0}^n) + h(X_{a_1}^n, X_{c_1}^n|X_{a_0}^n, X_{c_0}^n) + n\ o(\log\rho) + o(n) \qquad (291) \\
&\leq Mn\log\rho + nR + h(X_{a_0}^n|X_{c_0}^n) + n\ o(\log\rho) + o(n) \qquad (292)
\end{aligned}
$$

Similarly if a genie provides $\mathcal{G}_2 = (X_{2c_1}^n, X_{2c_0}^n, X_{3c_0}^n, X_{3a_0}^n, X_{3a_1}^n)$ to receiver 1, we have:

$$
3nR \leq Mn\log\rho + nR + h(X_{c_0}^n|X_{a_0}^n) + n\ o(\log\rho) + o(n) \qquad (293)
$$

Adding up (292) and (293) we have:

$$
6nR \leq 2Mn\log\rho + 2nR + h(X_{a_0}^n, X_{c_0}^n) + n\ o(\log\rho) + o(n) \qquad (294)
$$

Second, if a genie provides $\mathcal{G}_3 = (X_{3a_1}^n, \underbrace{X_{2a_1}^n, X_{2a_0}^n, X_{2c_0}^n, X_{2c_1}^n}_{W_2})$ to receiver 1, then we have:

$$
\begin{aligned}
nR_\Sigma &\leq Mn\log\rho + nR_2 + h(X_{3a_1}^n|X_{3a_0}^n, X_{3c_0}^n, X_{3c_1}^n) + n\ o(\log\rho) + o(n) \qquad (295) \\
\hookrightarrow 3nR &\leq Mn\log\rho + nR + h(X_{a_1}^n|X_{a_0}^n, X_{c_0}^n, X_{c_1}^n) + n\ o(\log\rho) + o(n) \qquad (296)
\end{aligned}
$$

and similarly if a genie provides $\mathcal{G}_4 = (X_{2c_1}^n, \underbrace{X_{3c_1}^n, X_{3c_0}^n, X_{3a_0}^n, X_{3a_1}^n}_{W_3})$ to receiver 1, then we have:

$$
3nR \leq Mn\log\rho + nR + h(X_{c_1}^n|X_{a_1}^n, X_{a_0}^n, X_{c_0}^n) + n\ o(\log\rho) + o(n) \qquad (297)
$$

Thus, we have:

$$6nR \quad \leq \quad 2Mn\log\rho + 2nR + h(X_{a_1}^n, X_{c_1}^n | X_{a_0}^n, X_{c_0}^n) + n\ o(\log\rho) + o(n) \tag{298}$$

Adding (294), (298), we obtain

$$12nR \quad \leq \quad 4Mn\log\rho + 4nR + nR + n\ o(\log\rho) + o(n) \tag{299}$$

$$\Rightarrow d \quad \leq \quad \frac{4M}{7} = \frac{12}{7} \tag{300}$$

B.2.2 Case $(N, M) = (6, 5) \Rightarrow$ DoF $\leq \frac{30}{11}$

X_{1a_2}	$\circ$
X_{1a_1}	$\circ$
X_{1a_0}	$\circ$
X_{1c_0}	$\circ$
X_{1c_1}	$\circ$
X_{1c_2}	$\circ$

$\circ$	$S_{1a_1}(X_{2a_2}, X_{2a_1}, X_{2a_0}, X_{2c_0}, X_{2c_1}, X_{3a_1}, X_{3a_0}, X_{3c_0}, X_{3c_1})$
$\circ$	$S_{1a_0}(X_{2a_1}, X_{2a_0}, X_{2c_0}, X_{3a_0}, X_{3c_0})$
$\circ$	$S_{1b_0}(X_{2a_0}, X_{3c_0})$
$\circ$	$S_{1c_0}(X_{2a_0}, X_{2c_0}, X_{3a_0}, X_{3c_0}, X_{3c_1})$
$\circ$	$S_{1c_1}(X_{2a_1}, X_{2a_0}, X_{2c_0}, X_{2c_1}, X_{3a_1}, X_{3a_0}, X_{3c_0}, X_{3c_1}, X_{3c_2})$

X_{2a_2}	$\circ$
X_{2a_1}	$\circ$
X_{2a_0}	$\circ$
X_{2c_0}	$\circ$
X_{2c_1}	$\circ$
X_{2c_2}	$\circ$

$\circ$	$S_{2a_1}(X_{3a_2}, X_{3a_1}, X_{3a_0}, X_{3c_0}, X_{3c_1}, X_{1a_1}, X_{1a_0}, X_{1c_0}, X_{1c_1})$
$\circ$	$S_{2a_0}(X_{3a_1}, X_{3a_0}, X_{3c_0}, X_{1a_0}, X_{1c_0})$
$\circ$	$S_{2b_0}(X_{3a_0}, X_{1c_0})$
$\circ$	$S_{2c_0}(X_{3a_0}, X_{3c_0}, X_{1a_0}, X_{1c_0}, X_{1c_1})$
$\circ$	$S_{2c_1}(X_{3a_1}, X_{3a_0}, X_{3c_0}, X_{3c_1}, X_{1a_1}, X_{1a_0}, X_{1c_0}, X_{1c_1}, X_{1c_2})$

X_{3a_2}	$\circ$
X_{3a_1}	$\circ$
X_{3a_0}	$\circ$
X_{3c_0}	$\circ$
X_{3c_1}	$\circ$
X_{3c_2}	$\circ$

$\circ$	$S_{3a_1}(X_{1a_2}, X_{1a_1}, X_{1a_0}, X_{1c_0}, X_{1c_1}, X_{2a_1}, X_{2a_0}, X_{2c_0}, X_{2c_1})$
$\circ$	$S_{3a_0}(X_{1a_1}, X_{1a_0}, X_{1c_0}, X_{2a_0}, X_{2c_0})$
$\circ$	$S_{3b_0}(X_{1a_0}, X_{2c_0})$
$\circ$	$S_{3c_0}(X_{1a_0}, X_{1c_0}, X_{2a_0}, X_{2c_0}, X_{2c_1})$
$\circ$	$S_{3c_1}(X_{1a_1}, X_{1a_0}, X_{1c_0}, X_{1c_1}, X_{2a_1}, X_{2a_0}, X_{2c_0}, X_{2c_1}, X_{2c_2})$

First, a genie provides $\mathcal{G}_1 = (X_{3a_2}^n, X_{3a_1}^n, X_{3a_0}^n, X_{2a_0}^n, X_{2c_0}^n, X_{2c_1}^n, X_{2c_2}^n)$ to receiver 1, then we have:

$$\begin{aligned}
nR_\Sigma \leq\ & Mn\log\rho + h(X_{2a_0}^n, X_{2c_0}^n) + h(X_{3a_0}^n | X_{3c_0}^n) + n\ o(\log\rho) + o(n) \\
& + h(X_{2c_1}^n | X_{2a_1}^n, X_{2a_0}^n, X_{2c_0}^n) + h(X_{3a_1}^n | X_{3a_0}^n, X_{3c_0}^n, X_{3c_1}^n) \\
& + h(X_{2c_2}^n | X_{2a_2}^n, X_{2a_1}^n, X_{2a_0}^n, X_{2c_0}^n, X_{2c_1}^n) + h(X_{3a_2}^n | X_{3a_1}^n, X_{3a_0}^n, X_{3c_0}^n, X_{3c_1}^n, X_{3c_2}^n) && (301) \\
\hookrightarrow 3nR \leq\ & Mn\log\rho + h(X_{a_0}^n, X_{c_0}^n) + h(X_{a_0}^n | X_{c_0}^n) + h(X_{c_1}^n | X_{a_1}^n, X_{a_0}^n, X_{c_0}^n) + h(X_{a_1}^n | X_{a_0}^n, X_{c_0}^n, X_{c_1}^n) \\
& + h(X_{c_2}^n | X_{a_2}^n, X_{a_1}^n, X_{a_0}^n, X_{c_0}^n, X_{c_1}^n) + h(X_{a_2}^n | X_{a_1}^n, X_{a_0}^n, X_{c_0}^n, X_{c_1}^n, X_{c_2}^n) + n\ o(\log\rho) + o(n) && (302) \\
\leq\ & Mn\log\rho + h(X_{a_0}^n, X_{c_0}^n) + h(X_{a_0}^n | X_{c_0}^n) + h(X_{a_1}^n, X_{c_1}^n | X_{a_0}^n, X_{c_0}^n) + h(X_{a_2}^n, X_{c_2}^n | X_{a_1}^n, X_{a_0}^n, X_{c_0}^n, X_{c_1}^n) \\
& + n\ o(\log\rho) + o(n) && (303) \\
\leq\ & Mn\log\rho + nR + h(X_{a_0} | X_{c_0}) + n\ o(\log\rho) + o(n) && (304)
\end{aligned}$$

Similarly if a genie provides $\mathcal{G}_2 = (X^n_{2c_2}, X^n_{2c_1}, X^n_{2c_0}, X^n_{3c_0}, X^n_{3a_0}, X^n_{3a_1}, X^n_{3a_2})$ to receiver 1, then we have:

$$3nR \leq Mn \log \rho + nR + h(X^n_{c_0}|X^n_{a_0}) + n\ o(\log \rho) + o(n) \tag{305}$$

and thus we have:

$$6nR \leq 2Mn \log \rho + 2nR + h(X^n_{a_0}, X^n_{c_0}) + n\ o(\log \rho) + o(n) \tag{306}$$

Second, a genie provides $\mathcal{G}_3 = (X^n_{3a_2}, X^n_{3a_1}, X^n_{2a_1}, X^n_{2a_0}, X^n_{2c_0}, X^n_{2c_1}, X^n_{2c_2})$ to receiver 1, then we have:

$$\begin{aligned}
nR_\Sigma &\leq Mn \log \rho + h(X^n_{2a_1}, X^n_{2a_0}, X^n_{2c_0}, X^n_{2c_1}, X^n_{2c_2}) + n\ o(\log \rho) + o(n) \\
&\quad + h(X^n_{3a_1}|X^n_{3a_0}, X^n_{3c_0}, X^n_{3c_1}) + h(X^n_{3a_2}|X^n_{3a_1}, X^n_{3a_0}, X^n_{3c_0}, X^n_{3c_1}, X^n_{3c_2}) \qquad (307) \\
\hookrightarrow 3nR &\leq Mn \log \rho + h(X^n_{a_1}, X^n_{a_0}, X^n_{c_0}, X^n_{c_1}, X^n_{c_2}) + n\ o(\log \rho) + o(n) \\
&\quad + h(X^n_{a_1}|X^n_{a_0}, X^n_{c_0}, X^n_{c_1}) + h(X^n_{a_2}|X^n_{a_1}, X^n_{a_0}, X^n_{c_0}, X^n_{c_1}, X^n_{c_2}) \qquad (308) \\
\Rightarrow 3nR &\leq Mn \log \rho + nR + h(X^n_{a_1}|X^n_{a_0}, X^n_{c_0}, X^n_{c_1}) + n\ o(\log \rho) + o(n) \qquad (309)
\end{aligned}$$

Similarly if a genie provides $\mathcal{G}_4 = (X^n_{2c_2}, X^n_{2c_1}, X^n_{3c_1}, X^n_{3c_0}, X^n_{3a_0}, X^n_{3a_1}, X^n_{3a_2})$ to receiver 1, then we have:

$$3nR \leq Mn \log \rho + nR + h(X^n_{c_1}|X^n_{a_1}, X^n_{a_0}, X^n_{c_0}) + n\ o(\log \rho) + o(n) \tag{310}$$

and thus we obtain:

$$6nR \leq 2Mn \log \rho + 2nR + h(X^n_{a_1}, X^n_{c_1}|X^n_{a_0}, X^n_{c_0}) + n\ o(\log \rho) + o(n) \tag{311}$$

Last, a genie provides $\mathcal{G}_5 = (X^n_{3a_2}, \underbrace{X^n_{2a_2}, X^n_{2a_1}, X^n_{2a_0}, X^n_{2c_0}, X^n_{2c_1}, X^n_{2c_2}}_{W_2})$ to receiver 1, then we have:

$$\begin{aligned}
nR_\Sigma &\leq Mn \log \rho + nR_2 + h(X^n_{3a_2}|X^n_{3a_1}, X^n_{3a_0}, X^n_{3c_0}, X^n_{3c_1}, X^n_{3c_2}) + n\ o(\log \rho) + o(n) \qquad (312) \\
\hookrightarrow 3nR &\leq Mn \log \rho + nR + h(X^n_{a_2}|X^n_{a_1}, X^n_{a_0}, X^n_{c_0}, X^n_{c_1}, X^n_{c_2}) + n\ o(\log \rho) + o(n) \qquad (313)
\end{aligned}$$

Similarly if a genie provides $\mathcal{G}_6 = (X^n_{2c_2}, \underbrace{X^n_{3c_2}, X^n_{3c_1}, X^n_{3c_0}, X^n_{3a_0}, X^n_{3a_1}, X^n_{3a_2}}_{W_3})$ to receiver 1, then we have:

$$3nR \leq Mn \log \rho + nR + h(X^n_{c_2}|X^n_{a_2}, X^n_{a_1}, X^n_{a_0}, X^n_{c_0}, X^n_{c_1}) + n\ o(\log \rho) + o(n) \tag{314}$$

and thus we have:

$$6nR \leq 2Mn \log \rho + 2nR + h(X^n_{a_2}, X^n_{c_2}|X^n_{a_1}, X^n_{a_0}, X^n_{c_0}, X^n_{c_1}) + n\ o(\log \rho) + o(n) \tag{315}$$

Adding the final equations from each of the genies, we obtain:

$$\begin{aligned}
18nR &\leq 6Mn \log \rho + 6nR + nR + n\ o(\log \rho) + o(n) \qquad (316) \\
d &\leq \frac{6M}{11} = \frac{30}{11} \qquad (317)
\end{aligned}$$

B.2.3 Case $(N, M) = (2L+2, 2L+1) \Rightarrow$ DoF $\leq \frac{MN}{M+N}$

X_{1a_L}	$\circ$
$\vdots$	$\circ$
X_{1a_1}	$\circ$
X_{1a_0}	$\circ$
X_{1c_0}	$\circ$
X_{1c_1}	$\circ$
$\vdots$	$\circ$
X_{1c_L}	$\circ$

$\circ$	$S_{1a_{L-1}}(X_{2a_L}, \cdots, X_{2c_{L-1}}, X_{3a_{L-1}}, \cdots, X_{3c_{L-1}})$
$\circ$	$\vdots$
$\circ$	$S_{1a_0}(X_{2a_1}, X_{2a_0}, X_{2c_0}, X_{3a_0}, X_{3c_0})$
$\circ$	$S_{1b_0}(X_{2a_0}, X_{3c_0})$
$\circ$	$S_{1c_0}(X_{2a_0}, X_{2c_0}, X_{3a_0}, X_{3c_0}, X_{3c_1})$
$\circ$	$\vdots$
$\circ$	$S_{1c_{L-1}}(X_{2a_{L-1}}, \cdots, X_{2c_{L-1}}, X_{3a_{L-1}}, \cdots, X_{3c_L})$

X_{2a_L}	$\circ$
$\vdots$	$\circ$
X_{2a_1}	$\circ$
X_{2a_0}	$\circ$
X_{2c_0}	$\circ$
X_{2c_1}	$\circ$
$\vdots$	$\circ$
X_{2c_L}	$\circ$

$\circ$	$S_{2a_{L-1}}(X_{3a_L n}, \cdots, X_{3c_{L-1}}, X_{1a_{L-1}}, \cdots, X_{1c_{L-1}})$
$\circ$	$\vdots$
$\circ$	$S_{2a_0}(X_{3a_1}, X_{3a_0}, X_{3c_0}, X_{1a_0}, X_{1c_0})$
$\circ$	$S_{2b_0}(X_{3a_0}, X_{1c_0})$
$\circ$	$S_{2c_0}(X_{3a_0}, X_{3c_0}, X_{1a_0}, X_{1c_0}, X_{1c_1})$
$\circ$	$\vdots$
$\circ$	$S_{2c_{L-1}}(X_{3a_{L-1}}, \cdots, X_{3c_{L-1}}, X_{1a_{L-1}}, \cdots, X_{1c_L})$

X_{3a_L}	$\circ$
$\vdots$	$\circ$
X_{3a_1}	$\circ$
X_{3a_0}	$\circ$
X_{3c_0}	$\circ$
X_{3c_1}	$\circ$
$\vdots$	$\circ$
X_{3c_L}	$\circ$

$\circ$	$S_{3a_{n-1}}(X_{1a_L}, \cdots, X_{1c_{L-1}}, X_{2a_{L-1}}, \cdots, X_{2c_{L-1}})$
$\circ$	$\vdots$
$\circ$	$S_{3a_0}(X_{1a_1}, X_{1a_0}, X_{1c_0}, X_{2a_0}, X_{2c_0})$
$\circ$	$S_{3b_0}(X_{1a_0}, X_{2c_0})$
$\circ$	$S_{3c_0}(X_{1a_0}, X_{1c_0}, X_{2a_0}, X_{2c_0}, X_{2c_1})$
$\circ$	$\vdots$
$\circ$	$S_{3c_{L-1}}(X_{1a_{L-1}}, \cdots, X_{1c_{L-1}}, X_{2a_{L-1}}, \cdots, X_{2c_L})$

First, a genie provides $\mathcal{G}_1 = (X^n_{3a_L}, \cdots, X^n_{3a_0}, X^n_{2a_0}, \cdots, X^n_{2c_L})$ and $\mathcal{G}_2 = (X^n_{2c_L}, \cdots, X^n_{2c_0}, X^n_{3c_0}, \cdots, X^n_{3a_L})$ to receiver 1, respectively, and we eventually have:

$$6nR \leq 2Mn \log\rho + 2nR + h(X^n_{a_0}, X^n_{c_0}) + n\ o(\log\rho) + o(n). \tag{318}$$

Second, a genie provides $\mathcal{G}_3 = (X^n_{3a_L}, \cdots, X^n_{3a_1}, X^n_{2a_1}, \cdots, X^n_{2c_L})$ and $\mathcal{G}_4 = (X^n_{2c_L}, \cdots, X^n_{2c_1}, X^n_{3c_1}, \cdots, X^n_{3a_L})$ to receiver 1, respectively, and we eventually have:

$$6nR \leq 2Mn \log\rho + 2nR + h(X^n_{a_1}, X^n_{c_1} | X^n_{a_0}, X^n_{c_0}) + n\ o(\log\rho) + o(n). \tag{319}$$

$\vdots$

Last, a genie provides $\mathcal{G}_{2L+1} = (X^n_{3a_L}, X^n_{2a_L}, \cdots, X^n_{2c_L})$ and $\mathcal{G}_{2L+2} = (X^n_{2c_L}, X^n_{3c_L}, \cdots, X^n_{3a_L})$ to receiver 1, respectively, then we eventually obtain:

$$6nR \leq 2Mn \log\rho + 2nR + h(X^n_{a_L}, X^n_{c_L} | X^n_{a_{L-1}}, \cdots, X^n_{c_{L-1}}) + n\ o(\log\rho) + o(n). \tag{320}$$

Adding the final equations from each of the genies, we obtain:

$$6(L+1)nR \leq 2(L+1)Mn\log\rho + 2(L+1)nR + nR + n\ o(\log\rho) + o(n). \quad (321)$$

$$\Rightarrow d \leq \frac{(2L+2)M}{4L+3} = \frac{MN}{M+N} \quad (322)$$

B.2.4 Case $M/N \geq (2L+1)/(2L+2) \Rightarrow$ DoF $\leq \frac{(2L+2)M}{4L+3}$

The proof is identical to the previous section.

B.2.5 Case $M/N \in [2L/(2L+1), (2L+1)/(2L+2)] \Rightarrow$ DoF $\leq \frac{(2L+1)N}{4L+3}$

This setting is also simple, but needs a small manipulation in the size of X_{a_0}, X_{c_0} (the two cannot be equal) if N is not even. As always:

$$|X_{a_i}| = |X_{c_i}| = N - M, \quad \forall i \in \{1, \cdots, L\} \quad (323)$$

In this case, if N is not even, let us choose

$$|X_{a_0}| = \left\lceil \frac{N - 2L(N-M)}{2} \right\rceil \quad (324)$$

$$|X_{c_0}| = \left\lfloor \frac{N - 2L(N-M)}{2} \right\rfloor \quad (325)$$

so that

$$|X_{a_0}| + |X_{c_0}| = N - 2L(N-M)$$

Only Genie 1 needs to provide *extra* dimensions. The Genie usually includes two symmetric cases, one that provides two terms of the form $|X_{a_o}|$ and another that provides two terms of the form $|X_{c_0}|$. Because the two have different sizes, the two genies usually involved within Genie 1, are no longer symmetric.

The number of extra dimensions needed for the Genie that provides two terms of the form $|X_{a_o}|$ is

$$(N-M) - |X_{a_0}| \quad (326)$$

so that the corresponding bound instead of (292) is

$$3nR \leq [M + (N-M)]n\log\rho - |X_{a_0}^n| + h(X_{a_0}^n | X_{c_0}^n) + n\ o(\log\rho) + o(n) \quad (327)$$

Similarly, the corresponding bound instead of (293) is

$$3nR \leq [M + (N-M)]n\log\rho - |X_{c_0}^n| + h(X_{c_0}^n | X_{a_0}^n) + n\ o(\log\rho) + o(n) \quad (328)$$

and the resulting bound instead of (294) is

$$6nR \leq [2M + 2(N-M) - N + 2n(N-M)]n\log\rho + h(X_{a_0}^n, X_{c_0}^n) + n\ o(\log\rho) + o(n) \quad (329)$$

Thus, adding the bounds from all the genies, instead of (321) we get

$$6(L+1)nR \leq 2(L+1)Mn\log\rho + 2(L+1)nR + nR + n\ o(\log\rho) + o(n)$$

$$
\begin{aligned}
& +[2(N-M)-N+2L(N-M)]n\log\rho && (330)\\
6(L+1)nR \;\leq\; & 2(L+1)nR+nR+(2L+1)Nn\log\rho+n\ o(\log\rho)+o(n) && (331)\\
\Rightarrow d \;\leq\; & \frac{(2L+1)N}{4L+3}. && (332)
\end{aligned}
$$

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