Rank-Matching for Multihop Multiflow

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***Abstract*—Seeking fundamental insights into multi-hop multi-flow networks we study the simplest non-trivial setting, a $2\times2\times2$ MIMO interference network comprised of two sources, two relays and two destinations, wherein all nodes have M antennas, all first-hop channels are of rank D_1, and all second hop channels are of rank D_2. For this setting, we show that the optimal sum DoF is $\min(4D_1, 4D_2, 2M - |D_1 - D_2|)$. While $4D_1, 4D_2$ are the obvious min-cut bottlenecks that are active when either hop is severely rank-deficient, what is remarkable is that under moderate rank-deficiencies the DoF are limited not by the higher or the lower of the two ranks D_1, D_2, but only by the difference of the two ranks $|D_1 - D_2|$. This suggests an interesting "rank-matching" design principle for multi-hop networks, reminiscent of "impedance matching", wherein the goal is not necessarily to increase or decrease the rank of each hop, but rather to use linear processing at intermediate hops to create effectively a two-hop setting with matching ranks.**

I. Introduction

Following significant advances in our understanding of single-hop multi-flow [1]–[4] and multi-hop single-flow [5], the natural next step is to extend this understanding to multi-hop multi-flow wireless networks. An early attempt in this direction came from translating the multi-hop multi-flow problem into a single hop interference network, in order to take advantage of interference alignment schemes developed for single hop interference networks. This is known as the precoding based network alignment (PBNA) paradigm [6]–[8], and is based on the assumption that the intelligence resides only at the source and destination nodes, whereas all intermediate nodes only do random linear forwarding operations. Going beyond PBNA, by allowing intelligence at *some* (but not necessarily all) of the intermediate layers of nodes creates a multi-hop setting where a key issue is the optimization of the functionalities of these intermediate layers comprised of intelligent relays. This has motivated canonical layered models such as the $2 \times 2 \times 2$ interference channel, possibly with multiple antennas at each node. DoF studies of the $2 \times 2 \times 2$ MIMO interference channel have identified key design principles, such as aligned interference neutralization [9], and have contributed fundamental insights that have turned out to be useful even in generalized settings such as $K\times K\times K$ networks [10], [11], multi-hop layered networks with arbitrary topologies [12], [13], and even non-layered settings [14].

To seek new fundamental insights into multi-hop multi-flow networks, it is important to further enrich the $2\times2\times2$ MIMO interference channel model to capture other aspects of the multi-hop multi-flow problem. This is the motivation for this work.

One important aspect that is yet unexplored is that the structure of the network can vary across hops, in a way that each hop has a distinct character. For instance, one hop may be comprised of fewer paths and experience greater spatial dependency than another which may have multiple paths and less spatial dependency. Or in a more direct translation to two-hop wireless networks, one hop could be line-of-sight/backhaul and the other could be an indoor environment with abundant scattering. A starting point to capture spatial dependencies is to consider rank-deficient channels [15]–[17]. To this end, we enrich the $2 \times 2 \times 2$ MIMO interference channel model by assuming different rank constraints in each hop. Through this simple model we hope to identify fundamental DoF constraints imposed by the variation in spatial dependencies from one hop to another. Specifically, we seek new DoF outer bounds, *beyond the obvious min-cut bounds*, that depend only on the ranks of the channels within each hop. As a measure of the quality of these outer bounds we will also explore if the bounds are tight almost surely if the channels are generated from continuous distributions subject to given rank-constraints.

The main contribution of this work is to identify a key "rank-mismatch" bottleneck on the DoF of a multi-hop multi-flow network. Specifically, in a $2 \times 2 \times 2$ MIMO interference network, where the channels between each source and relay node have rank D_1 in the first hop, and the channels between each relay and destination node have rank D_2 in the second hop, we show that aside from the usual min-cut bounds which are active only when either of the hops is severely rank constrained, the information theoretic DoF are bounded above by $2M - |D_1 - D_2|$. Thus, for moderate rank-deficiencies, the loss of DoF depends only on the mismatch $|D_1 - D_2|$, of the ranks in the two hops. Remarkably this bound is tight almost surely for generic channel realizations subject to the given rank constraints. This finding has interesting potential implications for the design of multi-hop (more than 2 hops) multi-flow settings. Through linear operations at intermediate hops, one could create effectively a two-hop network with matching ranks, thus allowing the full $2M$ DoF, i.e., the ideal scenario where "everyone gets the entire cake" [9]–[11]. The "rank-matching" principle is reminiscent of the "impedance matching" principle in circuit theory for maximum power transfer. Just as the power transfer in a circuit is maximized when the effective load impedance matches the effective source impedance, the DoF of the $2 \times 2 \times 2$ MIMO multi-hop multi-flow network are maximized when the effective first hop rank matches the effective second hop rank.

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II. System Model

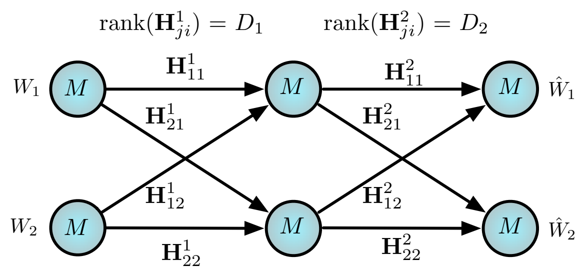

Fig. 1: $2 \times 2 \times 2$ MIMO rank deficient interference channel

The $2 \times 2 \times 2$ MIMO interference channel with M antennas at each node, where the 2 transmitters send 2 independent messages W_1, W_2 to the 2 receivers, is shown in Fig. 1. Over the t^{th} channel use, let $H^l_{ji}(t) \in \mathbb{C}^{M\times M}, i,j,l \in \{1,2\}$ denote the channel in the l-th hop between node i and node j. The signal transmitted by the i^{th} transmitter ($l = 1$) or the i^{th} relay ($l = 2$) is the $M \times 1$ vector denoted as $X^l_i(t), i,l \in \{1,2\}$. The received signal at each hop is given as

$$Y^{l+1}_j(t) = H^l_{j1}(t)X^l_1(t) + H^l_{j2}(t)X^l_2(t) + Z^{l+1}_j(t) \tag{1}$$

wherein $j,l \in \{1,2\}$ and $Z^{l+1}_j(t)$ is the $M\times 1$ i.i.d. zero mean unit variance circularly symmetric complex Gaussian noise. At each transmitting node, we have the average power constraint P. The time index, t, will be suppressed for concise notation, when no ambiguity would be caused. Define

$$H^1 = \begin{bmatrix} H^1_{11} & H^1_{12} \\ H^1_{21} & H^1_{22} \end{bmatrix} \qquad H^2 = \begin{bmatrix} H^2_{11} & H^2_{12} \\ H^2_{21} & H^2_{22} \end{bmatrix}$$

The crucial assumption of rank-deficiency, is that the channels in the l^{th} hop have rank D_l,

$$\text{rank}(H^l_{ji}) = D_l. \tag{2}$$

That there are no other critical rank-deficiencies, is enforced by the following natural assumptions.

$$\text{rank}\left([H^l_{j1} \quad H^l_{j2}]\right) = \min(M, 2D_l) \ \ j,l \in \{1,2\} \tag{3}$$

$$\text{rank}\left(\begin{bmatrix} H^l_{1i} \\ H^l_{2i} \end{bmatrix}\right) = \min(M, 2D_l) \ \ i,l \in \{1,2\} \tag{4}$$

$$\text{rank}(H^l) = \min(2M, 4D_l) \tag{5}$$

Aside from the rank constraints, the channels can take arbitrary values, bounded away from infinity to avoid degenerate scenarios. Perfect channel knowledge is assumed everywhere.

The definitions of codebooks, achievable rates, capacity, and degrees of freedom are all used here in the standard sense.

III. Main Results

Theorem 1 (Outer Bound): An outer bound on the sum DoF of the $2\times2\times2$ MIMO rank deficient interference channel described above, is

$$d_\Sigma \leq \min\{4D_1, 4D_2, 2M - |D_1 - D_2|\} \tag{6}$$

regardless of whether the channel coefficients are time-varying or constant.

Theorem 2 (Achievability): The outer bound of Theorem 1 is achievable if the channel coefficients are time-varying, and generic.

Remark 1: By generic channels we mean that the channels are drawn according to a continuous distribution over the algebraic variety defined by the rank-constraints. For instance, one may assume that each $M \times M$ channel over the l^{th} hop is a product of an $M \times D_l$ channel matrix and a $D_l \times M$ channel matrix, each of which is generated randomly and independently of the others, both across space and time.

IV. Proof of Theorem 1

Proof: We begin with a change of basis operation (an invertible linear transformation that does not affect the DoF) along the lines of [18]. The subsequent genie-aided dimension counting arguments used for information theoretic outer bounds are consistent with the frameworks developed in [18].

A. *Change of basis operation*

The outcome of the change of basis operation is illustrated in Fig. 2 for the case where $D_l > \frac{M}{2}$. The change of basis for the case where $D_l \leq \frac{M}{2}$ is trivial because there is no overlap between the signal spaces accessed by channels from different nodes, so a complete orthogonalization of all 4 channels is possible. Here we describe the change of basis operation for the first hop, where $D_1 > \frac{M}{2}$. The change of basis for the second hop is very similar, with D_2 replacing D_1, relays replacing transmitters, and destinations replacing relays.

Step 1: At each relay, the received signal is rotated such that the first $M - D_1$ antennas of relay k (denoted by ka) do not hear Transmitter $j, j \neq k$ and the last $M-D_1$ antennas of relay k (denoted by kc) do not hear Transmitter k. This operation is guaranteed because of the rank-deficiency assumptions. The remaining $2D_1 - M$ antennas are denoted as kb.

Step 2: At transmitter k, $k \in \{1,2\}$, there is a D_1-dimensional transmit subspace orthogonal to $M - D_1$ relay antennas ka and another D_1-dimensional subspace orthogonal to $M - D_1$ relay antennas $jc, j \neq k$. These two D_1-dimensional subspaces have $2D_1-M$ dimensional intersection within the M-dimensional space seen from the transmitter. The change of basis at transmitter k maps these $2D_1 - M$ dimensions to the $2D_1 - M$ antennas denoted as kb. Then, the first $M - D_1$ antennas of transmitter k are mapped to the space that is not heard by Relay $j, j \neq k$ and the last $M - D_1$ antennas of Transmitter k are mapped to the space not heard by Relay k. This operation is guaranteed again because of the rank deficiency assumptions.

B. *Outer Bound*

*1) **Region 1:** $\mathbf{D_1} > \frac{\mathbf{M}}{\mathbf{2}}, \mathbf{D_2} > \frac{\mathbf{M}}{\mathbf{2}}$:*

(1.1) When $D_1 \leq D_2$: Let a genie provide $\mathcal{G}_1 = \{X^n_{2b}, X^n_{2c}, R^n_{2a}\}$ to Receiver 1, which has M antennas. The total number of dimensions available to Receiver 1 (including genie) is:

$$M + |\mathcal{G}_1| = M + |X^n_{2b}| + |X^n_{2c}| + |R^n_{2a}| = 2M - (D_2 - D_1)$$

$M-D_1$	X_{1a}	$\circ$	$\circ$	$S_{1a}(X_{1a})$	$M-D_1$
$2D_1-M$	X_{1b}	$\circ$	$\circ$	$S_{1b}(X_{1a},X_{1b},X_{2b},X_{2c})$	$2D_1-M$
$M-D_1$	X_{1c}	$\circ$	$\circ$	$S_{1c}(X_{2c})$	$M-D_1$

$M-D_1$	X_{2a}	$\circ$	$\circ$	$S_{2a}(X_{2a})$	$M-D_1$
$2D_1-M$	X_{2b}	$\circ$	$\circ$	$S_{2b}(X_{2a},X_{2b},X_{1b},X_{1c})$	$2D_1-M$
$M-D_1$	X_{2c}	$\circ$	$\circ$	$S_{2c}(X_{1c})$	$M-D_1$

$M-D_2$	R_{1a}	$\circ$	$\circ$	$Y_{1a}(R_{1a})$	$M-D_2$
$2D_2-M$	R_{1b}	$\circ$	$\circ$	$Y_{1b}(R_{1a},R_{1b},R_{2b},R_{2c})$	$2D_2-M$
$M-D_2$	R_{1c}	$\circ$	$\circ$	$Y_{1c}(R_{2c})$	$M-D_2$

$M-D_2$	R_{2a}	$\circ$	$\circ$	$Y_{2a}(R_{2a})$	$M-D_2$
$2D_2-M$	R_{2b}	$\circ$	$\circ$	$Y_{2b}(R_{2a},R_{2b},R_{1b},R_{1c})$	$2D_2-M$
$M-D_2$	R_{2c}	$\circ$	$\circ$	$Y_{2c}(R_{1c})$	$M-D_2$

Fig. 2: Change of Basis for Region 1. $D_1 > \frac{M}{2}, D_2 > \frac{M}{2}$

D_1	X_{1a}	$\circ$	$\circ$	$S_{1a}(X_{1a})$	D_1
$M-2D_1$	X_{1b}	$\circ$	$\circ$	$S_{1b}()$	$M-2D_1$
D_1	X_{1c}	$\circ$	$\circ$	$S_{1c}(X_{2c})$	D_1

D_1	X_{2a}	$\circ$	$\circ$	$S_{2a}(X_{2a})$	D_1
$M-2D_1$	X_{2b}	$\circ$	$\circ$	$S_{2b}()$	$M-2D_1$
D_1	X_{2c}	$\circ$	$\circ$	$S_{2c}(X_{1c})$	D_1

$M-D_2$	R_{1a}	$\circ$	$\circ$	$Y_{1a}(R_{1a})$	$M-D_2$
$2D_2-M$	R_{1b}	$\circ$	$\circ$	$Y_{1b}(R_{1a},R_{1b},R_{2b},R_{2c})$	$2D_2-M$
$M-D_2$	R_{1c}	$\circ$	$\circ$	$Y_{1c}(R_{2c})$	$M-D_2$

$M-D_2$	R_{2a}	$\circ$	$\circ$	$Y_{2a}(R_{2a})$	$M-D_2$
$2D_2-M$	R_{2b}	$\circ$	$\circ$	$Y_{2b}(R_{2a},R_{2b},R_{1b},R_{1c})$	$2D_2-M$
$M-D_2$	R_{2c}	$\circ$	$\circ$	$Y_{2c}(R_{1c})$	$M-D_2$

Fig. 3: Change of Basis for Region 2. $D_1 \le \frac{M}{2}, D_2 > \frac{M}{2}$

Receiver 1 can decode its desired message W_1 and can obtain $X_{1a}^n, X_{1b}^n, X_{1c}^n$. Using genie information X_{2b}^n, X_{2c}^n, Receiver 1 can reconstruct the received signal at Relay 1 and obtain $R_{1a}^n, R_{1b}^n, R_{1c}^n$. This enables receiver 1 to remove $R_{1a}^n, R_{1b}^n, R_{1c}^n$ from the received signal and decode R_{2b}^n, R_{2c}^n. With additional genie information R_{2a}^n, Receiver 1 would be able to decode X_{2a}^n and as a result, decodes message W_2 (subject to noise distortion) sent from Transmitter 2. Hence, the sum DoF is bounded as $d_\Sigma \le M+|\mathcal{G}_1| = 2M-(D_2-D_1)$.

(1.2) When $D_1 > D_2$: Let a genie provide $\mathcal{G}_1 = \{X_{2c}^n, R_{2a}^n, R_{2b}^n\}$ to Receiver 1, which has M antennas. The total number of dimensions at Receiver 1 (including genie) is:

$$M + |\mathcal{G}_1| = M + |X_{2c}^n| + |R_{2a}^n| + |R_{2b}^n| = 2M - (D_1 - D_2)$$

Receiver 1 can decode its desired message W_1 and can obtain $X_{1a}^n, X_{1b}^n, X_{1c}^n$. Receiver 1 can decode R_{2c}^n using $M - D_2$ antennas. Using genie information R_{2a}^n, R_{2b}^n and decoded R_{2c}^n, Receiver 1 can reconstruct the received signal at Relay 2 and obtain X_{2a}^n, X_{2b}^n. With additional genie information X_{2c}^n, Receiver 1 would be able to decode message W_2 (subject to noise distortion) sent from Transmitter 2. Hence, the sum DoF is bounded as $d_\Sigma \le M + |\mathcal{G}_1| = 2M - (D_1 - D_2)$.

Combining bounds of (1.1) and (1.2), we get the bound:

$$d_\Sigma \le 2M - |D_1 - D_2| \tag{7}$$

2) ***Region 2:*** $\mathbf{D_1 \le \frac{M}{2}, D_2 > \frac{M}{2}}$*:* Let a genie provide $\mathcal{G}_1 = \{X_{2c}^n, R_{2a}^n\}$ to Receiver 1, which has M antennas. The total number of dimensions at Receiver 1 (including genie) is:

$$M + |\mathcal{G}_1| = M + |X_{2c}^n| + |R_{2a}^n| = 2M - (D_2 - D_1)$$

Receiver 1 can decode its desired message W_1 and can obtain $X_{1a}^n, X_{1b}^n, X_{1c}^n$. Using genie information X_{2c}^n, Receiver 1 can reconstruct the received signal at Relay 1 and obtain $R_{1a}^n, R_{1b}^n, R_{1c}^n$. This enables receiver 1 to remove $R_{1a}^n, R_{1b}^n, R_{1c}^n$ from the received signal and decode R_{2b}^n, R_{2c}^n. With additional genie information R_{2a}^n, Receiver 1 would be able to decode X_{2a}^n and as a result, decodes message W_2 (subject to noise distortion) sent from Transmitter 2. Hence, the sum DoF is bounded as $d_\Sigma \le M+|\mathcal{G}_1| = 2M-(D_2-D_1)$.

When $M > D_1 + D_2$, outer bound on the sum DoF is the same as the cutset bound, $d_\Sigma \le 4D_1$. Hence, outer bound on the sum DoF for Region 2, is:

$$d_\Sigma \le \min\{4D_1, 2M - (D_2 - D_1)\} \tag{8}$$

3) ***Region 3:*** $\mathbf{D_1 > \frac{M}{2}, D_2 \le \frac{M}{2}}$*:* Let a genie provide $\mathcal{G}_1 = \{X_{2c}^n, R_{2a}^n, R_{2b}^n\}$ to Receiver 1, which uses only $2D_2$ antennas. The total number of dimensions (including genie):

$$2D_2 + |\mathcal{G}_1| = 2D_2 + |X_{2c}^n| + |R_{2a}^n| + |R_{2b}^n| = 2M - (D_1 - D_2)$$

Receiver 1 can decode its desired message W_1 and can obtain $X_{1a}^n, X_{1b}^n, X_{1c}^n$. Receiver 1 can decode R_{2c}^n using D_2 antennas. Using genie information R_{2a}^n, R_{2b}^n and known R_{2c}^n, Receiver 1 can decode the received signal at Relay 2 and obtain X_{2a}^n, X_{2b}^n. With additional genie information X_{2c}^n, Receiver 1 can decode the message W_2 (subject to noise distortion) sent from Transmitter 2. Hence, the sum DoF is bounded as $d_\Sigma \le M + |\mathcal{G}_1| = 2M - (D_1 - D_2)$.

When $M > D_1 + D_2$, outer bound on the sum DoF is the same as the cutset bound, $d_\Sigma \le 4D_2$. Hence, outer bound on the sum DoF for Region 3, is :

$$d_\Sigma \le \min\{4D_2, 2M - (D_1 - D_2)\} \tag{9}$$

4) ***Region 4:*** $\mathbf{D_1 \le \frac{M}{2}, D_2 \le \frac{M}{2}}$*:* In this region, DoF outer bound is the same as the min-cut, $d_\Sigma \le \min(4D_1, 4D_2)$.

This completed the converse proof. Alternate converse proof for a more asymmetric setting is discussed in [19]. ■

V. Proof of Theorem 2

Proof: Achievability is shown for time-varying generic channels through an allocation of signal dimensions over space or time, for Zero-Forcing (ZF), Interference Alignment in X channel (X), Zero-Forcing over Broadcast channel (BC), and Aligned Interference Neutralization (AIN), with the fraction of signal dimensions used for each denoted as f_Z, f_X, f_B, f_A, respectively. N_Z, N_X, N_B, N_A denote the corresponding number of symbols (listed in Table I). Symbol extensions are used if necessary. Since all achievable schemes are linear and duality applies in the reciprocal direction, we assume that the first hop channel rank is smaller, i.e., $D_1 \le D_2$, and show achievability of $\min\{4D_1, 2M - (D_2 - D_1)\}$ DoF.

$M-D_1$	X_{1a}	$\circ$	$\circ$	$S_{1a}(X_{1a})$	$M-D_1$
$2D_1-M$	X_{1b}	$\circ$	$\circ$	$S_{1b}(X_{1a},X_{1b},X_{2b},X_{2c})$	$2D_1-M$
$M-D_1$	X_{1c}	$\circ$	$\circ$	$S_{1c}(X_{2c})$	$M-D_1$

$M-D_1$	X_{2a}	$\circ$	$\circ$	$S_{2a}(X_{2a})$	$M-D_1$
$2D_1-M$	X_{2b}	$\circ$	$\circ$	$S_{2b}(X_{2a},X_{2b},X_{1b},X_{1c})$	$2D_1-M$
$M-D_1$	X_{2c}	$\circ$	$\circ$	$S_{2c}(X_{1c})$	$M-D_1$

D_2	R_{1a}	$\circ$	$\circ$	$Y_{1a}(R_{1a})$	D_2
$M-2D_2$	R_{1b}	$\circ$	$\circ$	$Y_{1b}()$	$M-2D_2$
D_2	R_{1c}	$\circ$	$\circ$	$Y_{1c}(R_{2c})$	D_2

D_2	R_{2a}	$\circ$	$\circ$	$Y_{2a}(R_{2a})$	D_2
$M-2D_2$	R_{2b}	$\circ$	$\circ$	$Y_{2b}()$	$M-2D_2$
D_2	R_{2c}	$\circ$	$\circ$	$Y_{2c}(R_{1c})$	D_2

Fig. 4: Change of Basis for Region 3. $D_1 > \frac{M}{2}, D_2 \leq \frac{M}{2}$

Let us denote the precoding matrix used by Transmitter or Relay k of hop l as V_k^l which is of size $M \times 2D_1$ or $M \times (M - \frac{D_2-D_1}{2})$, depending on the channel ranks.

$$S_1^l = [H_{11}^l V_1^l \quad H_{12}^l V_2^l] \qquad S_2^l = [H_{21}^l V_1^l \quad H_{22}^l V_2^l] \quad (10)$$

The signal space matrices $S_1^l, l \in \{1,2\}$, can be shown to be full rank, since H_{11}^l and H_{12}^l are time-varying, generic channels, and they rotate the vectors in different directions. Hence, apart from the precoding vectors that are chosen to align at Relay 1 or Receiver 1, the remaining rotated vectors would lie in different directions. Similarly, the signal space matrices $S_2^l, l \in \{1,2\}$ can be shown to be full rank. We now discuss the construction of the precoding matrices.

A. $\mathbf{D_1 + D_2 \leq M \rightarrow d_\Sigma = 4D_1}$

In this region, zero-forcing (ZF) is sufficient to achieve $4D_1$ DoF. In both hops, $4D_1$ vectors are chosen by the transmitters and the relays from the nullspace of the 4 channels.

B. $\mathbf{\frac{3}{2}D_1 + \frac{1}{2}D_2 \leq M < D_1 + D_2 \rightarrow d_\Sigma = 4D_1}$

In the first hop, $4D_1$ zero-forcing vectors are chosen similar to Region A. In the second hop, each relay constructs a $M \times 2D_1$ precoding matrix with vectors for ZF and X-channel alignment.

$$V_i^2 = [V_{Z1i}^2 \quad V_{Z2i}^2 \quad V_{X1i}^2 \quad V_{X2i}^2] \qquad i \in \{1,2\}$$

$$\dim(V_{Zji}^2) = M - D_2 \quad \dim(V_{Xji}^2) = D_1 + D_2 - M$$

wherein V_{Zji}^2 denotes the vectors from the nullspace of channel $H_{ji}^2, \;\; i,j \in \{1,2\}$.

Alignment vectors $V_{Xji}^2, \;\; i,j \in \{1,2\}$ are chosen to satisfy the following X-channel interference alignment conditions.

$$H_{11}^2 V_{X21}^2 = -H_{12}^2 V_{X22}^2 \subseteq H_{11}^2 \cap H_{12}^2 \quad (11)$$
$$H_{21}^2 V_{X11}^2 = -H_{22}^2 V_{X12}^2 \subseteq H_{21}^2 \cap H_{22}^2 \quad (12)$$

Note that

$$\begin{aligned} \dim(H_{11}^2 \cap H_{12}^2) &= \dim(H_{21}^2 \cap H_{22}^2) = 2D_2 - M \\ &\geq D_1 + D_2 - M = \dim(V_{Xji}^2) \end{aligned}$$

Hence $2D_1$ linearly independent vectors can be chosen at each relay. Signal spaces at the receivers are as follows

$$S_1^2 = \left[H_{11}^2[V_{Z21}^2 \;\; V_{X11}^2 \;\; V_{X21}^2] \quad H_{12}^2[V_{Z22}^2 \;\; V_{X12}^2]\right] \quad (13)$$
$$S_2^2 = \left[H_{22}^2[V_{Z12}^2 \;\; V_{X12}^2 \;\; V_{X22}^2] \quad H_{21}^2[V_{Z11}^2 \;\; V_{X21}^2]\right] \quad (14)$$

The receiver signal spaces S_1^2, S_2^2 are full rank since the channels are generic and non-aligned vectors would be rotated in different directions.

C. $\mathbf{2D_1 \leq M < \frac{3}{2}D_1 + \frac{1}{2}D_2 \rightarrow d_\Sigma = 2M - (D_2 - D_1)}$

In the first hop, when common symbols are sent to both relays, relays can cooperate and the second hop could be treated as a Broadcast channel (BC), over which zero-forcing could be performed to send symbols only to the intended receivers. Hence, each transmitter sends one fraction of the symbols privately to the 2 relays, and another fraction as common information to the 2 relays.

In the second hop, ZF vectors are chosen to the extent possible ($4(M-D_2)$) and the remaining signal dimensions are used for treating the second hop as a BC and an X channel.

C.1. Beamforming in the first hop

In the first hop, Transmitter i constructs the precoding matrix V_i^1 with $M - \frac{D_2-D_1}{2}$ vectors, as follows.

$$V_i^1 = [V_{Z1i}^1 \quad V_{Z2i}^1 \quad V_{Bi}^1] \quad i \in \{1,2\} \quad (15)$$

wherein $V_{Bi}^1, \;\; i \in \{1,2\}$ denotes the $\frac{3D_1+D_2-2M}{2}$ vectors chosen so that both relays can receive the symbols, and V_{Zji}^1 denotes the $M - \frac{D_1+D_2}{2}$ vectors from the nullspace of channel $H_{ji}^1, \;\; i,j \in \{1,2\}$. Since $M - \frac{D_1+D_2}{2} \leq M - D_1$ and $M - \frac{D_1+D_2}{2} + \frac{3D_1+D_2-2M}{2} = D_1$, such precoding vectors exist.

$$S_1^1 = \left[H_{11}^1[V_{Z21}^1 \quad V_{B1}^1] \quad H_{12}^1[V_{Z22}^1 \quad V_{B2}^1]\right] \quad (16)$$
$$S_2^1 = \left[H_{22}^1[V_{Z12}^1 \quad V_{B2}^1] \quad H_{21}^1[V_{Z11}^1 \quad V_{B1}^1]\right] \quad (17)$$

The relay signal spaces S_1^1, S_2^1 are full rank ($2D_1$), since the channels are generic, and the vectors are rotated in different directions. Thus, each relay decodes $2M - (D_1 + D_2)$ symbols sent privately through zero-forcing and $3D_1 + D_2 - 2M$ symbols sent common to both relays.

C.2. Beamforming in the second hop

In the second hop, Relay i uses an $M \times 2D_1$ precoding matrix V_i^2 to send $2M - (D_2 - D_1)$ symbols, as follows.

$$V_i^2 = [V_{Z1i}^2 \quad V_{Z2i}^2 \quad V_{X1i}^2 \quad V_{X2i}^2 \quad V_{Bi}^2] \quad (18)$$
$$\dim(V_{Zji}^2) = M - D_2, \; \dim(V_{Xji}^2) = \frac{D_2 - D_1}{2},$$
$$\dim(V_{Bi}^2) = 3D_1 + D_2 - 2M, \qquad i,j \in \{1,2\}$$

wherein V_{Zji}^2 denotes the vectors from the nullspace of the channel $H_{ji}^2, \;\; i,j \in \{1,2\}$. Vectors $V_{Xji}^2, \;\; i,j \in \{1,2\}$ are chosen to satisfy the conditions in (11),(12) as in Region B. Since $\frac{D_2-D_1}{2} < 2D_2 - M$ (signal space overlap), vectors can be chosen to align interference in $\frac{D_2-D_1}{2}$ dimensions.

Broadcast channel vectors are constructed as $V_{B1}^2 = [V_{B11}^2 \quad V_{B21}^2]$ and $V_{B2}^2 = [V_{B12}^2 \quad V_{B22}^2]$. Symbols intended for Receiver 1 are precoded using $\frac{3D_1+D_2-2M}{2}$ vectors (V_{B11}^2 and V_{B12}^2) from the nullspace of $[H_{21}^2 \quad H_{22}^2]$. Symbols intended for Receiver 2 are precoded using $\frac{3D_1+D_2-2M}{2}$ vectors (V_{B21}^2 and

Table I: Dimension of precoding vectors for $2 \times 2 \times 2$ rank deficient interference channel with $D_2 > D_1$

D_1, D_2 Region	N_A	N_B	N_X	N_Z	Total DoF
$D_1 + D_2 \leq M$	0	0	0	$4D_1$	$4D_1$
$\frac{3}{2}D_1 + \frac{1}{2}D_2 \leq M < D_1 + D_2$	0	0	$4(D_1 + D_2 - M)$	$4(M - D_2)$	$4D_1$
$2D_1 \leq M < \frac{3}{2}D_1 + \frac{1}{2}D_2$	0	$3D_1 + D_2 - 2M$	$2(D_2 - D_1)$	$4(M - D_2)$	$2M - (D_2 - D_1)$
$M < 2D_1$	$2(2D_1 - M)$	$D_2 - D_1$	$2(D_2 - D_1)$	$4(M - D_2)$	$2M - (D_2 - D_1)$

V^2_{B22}) from the nullspace of $[H^2_{11} \quad H^2_{12}]$. Thus, $\frac{3D_1+D_2-2M}{2}$ symbols can be decoded at each receiver free of interference.

$$S^2_1 = \left[H^2_{11}[V^2_{Z21} \;\; V^2_{X11} \;\; V^2_{X21}] \quad H^2_{12}[V^2_{X12} \;\; V^2_{Z22}] \quad \sum_{i=1}^{2} H^2_{1i} V^2_{B1i} \right]$$

$$S^2_2 = \left[H^2_{22}[V^2_{Z12} \;\; V^2_{X12} \;\; V^2_{X22}] \quad H^2_{21}[V^2_{X21} \;\; V^2_{Z11}] \quad \sum_{i=1}^{2} H^2_{2i} V^2_{B2i} \right]$$

The receiver signal spaces S^2_1, S^2_2 are full rank (M), since the channels are generic with $D_2 \geq \frac{D_1+D_2}{2}, D_2 \geq M - \frac{D_1+D_2}{2}$, and non-aligned vectors would lie in different directions. Each receiver decodes $2(M - D_2)$ symbols through zero-forcing, $D_2 - D_1$ symbols through X-channel alignment and $\frac{3D_1+D_2-2M}{2}$ symbols through ZF on Broadcast channel.

D. $\mathbf{M < 2D_1 \to d_\Sigma = 2M - (D_2 - D_1)}$

In this region, interference can be aligned in the first hop enabling us to perform Aligned Interference Neutralization over $2D_1 - M$ dimensions. We solve a linear programming problem to determine the fraction of signal dimensions for the 4 schemes in the second hop, and the fractions are:

$$f_A = \frac{2(2D_1 - M)}{2M} \quad f_B = \frac{D_2 - D_1}{2M} \quad f_X = \frac{3(D_2 - D_1)}{2M} \quad f_Z = \frac{4(M - D_2)}{2M}$$

D.1. Beamforming in the first hop

In the first hop, Transmitter i uses a precoding matrix V^1_i with $M - \frac{D_2-D_1}{2}$ vectors, as follows

$$V^1_i = [V^1_{Z1i} \;\; V^1_{Z2i} \;\; V^1_{Bi} \;\; V^1_{Ai}] \qquad i \in \{1, 2\} \tag{19}$$

wherein V^1_{B2} denotes the $\frac{D_2-D_1}{2}$ vectors chosen so that both relays can receive the symbols, V^1_{Zji} denotes the $M - \frac{D_1+D_2}{2}$ vectors from the nullspace of channel $H^1_{ji}, \;\; i, j \in \{1, 2\}$ and $V^1_{Ai}, i \in \{1, 2\}$ denotes the $2D_1 - M$ vectors used for Aligned Interference Neutralization. Since $M - \frac{D_1+D_2}{2} \leq M - D_1$ and $M - \frac{D_1+D_2}{2} + \frac{D_2-D_1}{2} + (2D_1 - M) = D_1$, zero-forcing and broadcast vectors exist.

In order to perform Aligned Interference Neutralization in the first hop, we identify matrices U^1_1, U^1_2 each of size $M \times (2D_1 - M)$, such that

$$H^1_{11} U^1_1 = H^l_{12} U^1_2 \tag{20}$$

$$H^1_{21} U^1_1 = H^l_{22} U^1_2 \tag{21}$$

For the solution of (20), the basis of U^1_1 has rank D_1, $M - D_1$ of which will have $H^1_{11} U^1_1 = 0$ and the remaining $2D_1 - M$ will produce $H^1_{11} U^1_1 = H^1_{11} \cap H^1_{12}$. Similarly, for the solution of (21), U^1_1 has rank D_1. These two D_1 dimensional spaces will intersect in a $2D_1 - M$ dimensional space, which is the solution that we seek since it satisfies both equations. Similar solution can be found for U^1_2 as well. Thus, we have found two $2D_1 - M$ dimensional spaces, one at each relay, that are accessible by the same space at each transmitter. Using these $2D_1 - M$ dimensional spaces, we perform Aligned Interference Neutralization as in [9], where Transmitter 1 sends $p \triangleq 2D_1 - M$ symbols with p precoding vectors $V^1_{A1,1}, \cdots, V^1_{A1,p}$ and Transmitter 2 sends $p - 1 = 2D_1 - M - 1$ symbols with $p - 1$ precoding vectors $V^1_{A2,1}, \cdots, V^1_{A2,p-1}$. Each precoding vector has size $M \times 1$. The alignment relationship is same as that in Table I of [9].

Vectors are chosen to align at the two relays, as follows

$$H^1_{11} V^1_{A1,q+1} = H^1_{12} V^1_{A2,q}, \quad q = 1, \cdots, p - 1 \tag{22}$$

$$H^1_{21} V^1_{A1,q} = H^1_{22} V^1_{A2,q}, \quad q = 1, \cdots, p - 1 \tag{23}$$

Here to find a solution, we will start from a random one dimensional subspace of U^1_1 and set it as $V^1_{A1,1}$, then go through (22),(23) to find all other vectors. Note that as $p = 2D_1 - M$, we are guaranteed to find such independent vectors.

Signal spaces at the relays are given as

$$S^1_1 = \left[H^1_{11}[V^1_{Z21} \quad V^1_{B1} \quad V^1_{A1}] \quad H^1_{12}[V^1_{Z22} \quad V^1_{B2}] \right] \tag{24}$$

$$S^1_2 = \left[H^1_{21}[V^1_{Z11} \quad V^1_{B1} \quad V^1_{A1}] \quad H^1_{22}[V^1_{Z12} \quad V^1_{B2}] \right] \tag{25}$$

The relay signal spaces S^1_1, S^1_2 are full rank (M), since the channels are generic with $D_1 > M - D_1$ and based on the precoding vectors construction. Hence, the non-aligned vectors are rotated in different directions. Thus, each relay receives $2M - (D_1 + D_2)$ symbols sent privately through zero-forcing, $2D_1 - M$ symbols through Interference Alignment and $D_2 - D_1$ symbols sent common to both relays.

D.2. Beamforming in the second hop

In the second hop, Relay i uses a precoding matrix V^2_i with M vectors to send $2M - (D_2 - D_1)$ symbols.

$$V^2_i = [V^2_{Z1i} \quad V^2_{Z2i} \quad V^2_{X1i} \quad V^2_{X2i} \quad V^2_{Bi} \quad V^2_{Ai}] \tag{26}$$

$$\dim(V^2_{Zji}) = M - D_2, \;\; \dim(V^2_{Ai}) = 2D_1 - M$$

$$\dim(V^2_{Xji}) = \dim(V^2_{Bi}) = \frac{D_2 - D_1}{2} \qquad i, j \in \{1, 2\}$$

wherein V^2_{Zji} denotes the vectors from the nullspace of channel $H^2_{ji}, \;\; i, j \in \{1, 2\}$. Vectors $V^2_{Xji}, \;\; i, j \in \{1, 2\}$ are chosen such that the conditions in (11),(12) are satisfied, as in Region B. Since $\frac{D_2-D_1}{2} < 2D_2 - M$ (signal space overlap), vectors can be chosen to align interference in $\frac{D_2-D_1}{2}$ dimensions. In order to perform Aligned Interference Neutralization, precoding matrices V^2_{A1}, V^2_{A2} of size $M \times (2D_1 - M)$

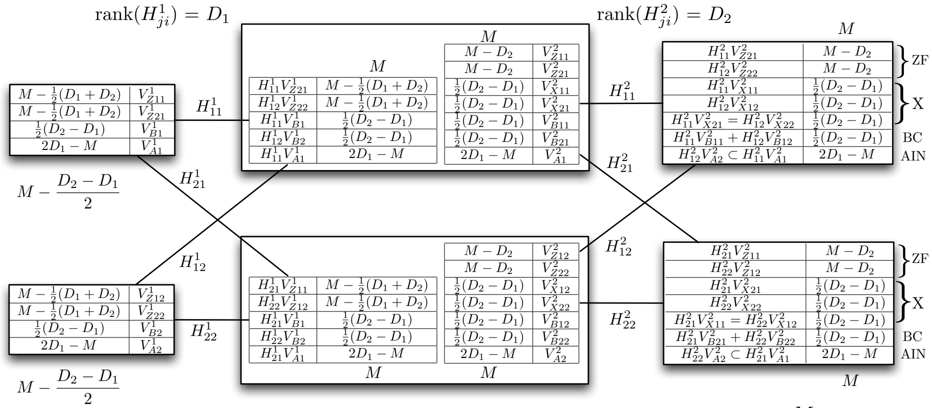

Fig. 5: Achieving $2M - (D_2 - D_1)$ DoF in Region D: $D_1 > \frac{M}{2}$

are constructed similar to the alignment vectors construction in the first hop (see Table II of [9]), using which we can achieve $2p - 1 = 2(2D_1 - M) - 1$ DoF. By considering a k-symbol extension, we can send $2k(2D_1 - M) - 1$ symbols over such symbol-extended network by Aligned Interference Neutralization, resulting in $2(2D_1 - M)$ DoF asymptotically.

Vectors V_{B1}^2 and V_{B2}^2 are constructed similar to that in Region C. Thus, $\frac{D_2 - D_1}{2}$ symbols are decoded at each receiver free of interference.

$$S_1^2 = \left[H_{11}^2 [V_{Z21}^2 \ V_{X11}^2 \ V_{X21}^2 \ V_{A1}^2] \quad H_{12}^2 [V_{Z22}^2 \ V_{X12}^2] \quad \sum_{i=1}^{2} H_{1i}^2 V_{B1i}^2 \right]$$

$$S_2^2 = \left[H_{22}^2 [V_{Z12}^2 \ V_{X22}^2 \ V_{X12}^2] \quad H_{21}^2 [V_{Z11}^2 \ V_{X21}^2 \ V_{A1}^2] \quad \sum_{i=1}^{2} H_{2i}^2 V_{B2i}^2 \right]$$

The receiver signal spaces S_1^2, S_2^2 are full rank (M), since the channels are generic with $D_2 \geq \frac{D_1 + D_2}{2}, D_2 \geq M - \frac{D_1 + D_2}{2}$, and the non-aligned vectors lie in different directions. Each receiver decodes $2(M - D_2)$ symbols through ZF, $D_2 - D_1$ through X-channel alignment, $\frac{D_2 - D_1}{2}$ symbols through zero-forcing for BC, and $2D_1 - M$ symbols through Aligned Interference Neutralization, as illustrated in Fig. 5. ■

VI. Conclusions

Genie-based outer bounds are developed for the DoF of the $2 \times 2 \times 2$ MIMO rank deficient interference channel with time-varying or constant channels. These outer bounds are better than the cutset bounds (function of the rank mismatch, $|D_1 - D_2|$) and are shown to be tight when the channel coefficients are time-varying and generic. DoF of multi-hop multi-flow networks are discussed in the full version of this paper [19].

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