On the Optimality of "Half the Cake" for K-user Rank-Deficient $(M_k \times M_k)$ Interference Channel

Bofeng Yuan, Hua Sun and Syed A. Jafar

Center for Pervasive Communications and Computing (CPCC)
University of California, Irvine
Email: {bofengy, huas2, syed}@uci.edu

***Abstract*—By introducing a novel outer bound, we find a sufficient condition for optimality of $\sum_{k=1}^{K} M_k/2$ degrees of freedom (half the cake per user) for a K-user multiple-input-multiple-output (MIMO) interference channel (IC) where the cross-channels have arbitrary rank constraints, and the k^{th} transmitter and receiver are equipped with M_k antennas each. The result consolidates and significantly generalizes results from prior studies by Krishnamurthy et al., of rank-deficient interference channels where all users have M antennas; and by Tang et al., of full rank interference channels where the k^{th} user pair has M_k antennas.**

I. Introduction

Degrees of freedom (DoF) studies of wireless interference networks have produced a diverse array of new insights into the accessibility of signal dimensions under a variety of channel models. In order to consolidate these insights and to build upon them, it is important to make progress on unifying the underlying channel models. The motivation for this work, summarized in Fig. 1, is to pursue such a generalization of the results from [1], [2], [3]. Specifically, in this work our goal is to consolidate the key insights regarding the optimality of half-the-cake (the "cake" refers to each user's interference-free DoF, cf. [1]) for the K-user MIMO interference channel settings where the number of antennas at each receiver is equal to the number of antennas at the corresponding transmitter, i.e., all the *desired* channels are *square* matrices.[1]

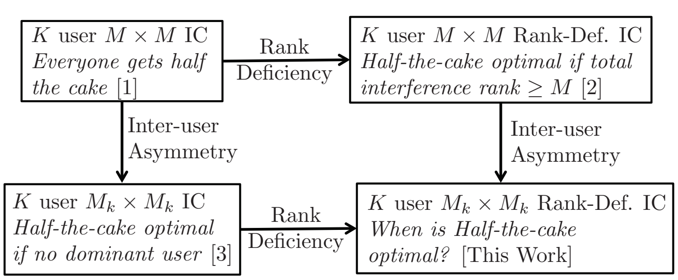

Fig. 1: The motivation of this paper.

A. Everyone gets half the cake

It was shown by Cadambe and Jafar in [1] that in a K-user $M \times M$ MIMO interference channel where each node is equipped with M antennas, the optimal DoF value is $KM/2$. Since each user achieves half of his interference-free DoF, the result is often paraphrased as "everyone gets half the cake". Generalizations of this result have been explored in various directions, in particular to find out when the optimal solution may allow even more than half-the-cake. Indeed rectangular interference channels (cf. [4], [5], [6], [7], [8]), and multi-hop settings (cf. [9]) have shown that more than half-the-cake is possible. Of particular interest to us in this work are the generalizations in [2], [3].

B. Optimality of half-the-cake: Key insight from [2], [3]

The generalization in [2] concerns rank deficient channels. Rank deficient interference channels (cf. [10], [11], [12]) are frequently encountered due to poor scattering, keyhole effects, as well as underlying topological and structural concerns in single-hop abstractions of multihop networks with linear forwarding at intermediate nodes. Cross-channel rank-deficiencies have the potential to be helpful as the scope of zero forcing schemes is enhanced (although the scope of interference alignment schemes is limited by rank-deficiencies), opening the possibility that more than half-the-cake may be achievable. Exploring this possibility in [2], Krishnamurthy and Jafar establish that for the K-user $M \times M$ MIMO interference channel where all the cross channels are rank-deficient with the same rank $D \leq M$ and direct channels are full rank, $KM/2$ DoF (half-the-cake) are optimal if the sum of all interference ranks at each user, is greater than or equal to the number of antennas at the user, $(K-1)D \geq M$. In other words, *every signal dimension is accessible by at least one interfering user*. For $K = 3$ users, [2] considers a bit more general setting, so that at each receiver the interfering channel from the preceding transmitter is of rank D_1 and the interfering channel from the next transmitter (with wrap around) is D_2. For $K = 2$ users the setting is fully general with all interfering channel ranks allowed to take arbitrary values. Remarkably, in all cases, the key insight remains the same:

Original Insight: "*Half-the-cake is optimal if at every transmitter and receiver, the sum of interfering channel ranks is greater than or equal to the number of antennas at that transmitter and receiver, respectively.*"

Finally, Liu, Tuninetti and Jafar in [3] consider a different generalization, to the K-user $M_k \times M_k$ MIMO interference

[1]Rectangular channels present a significantly different set of challenges, and generally allow more than half-the-cake per user, so they remain outside the scope of this paper.

channel with full rank generic channels, where the k^{th} user has M_k transmit and M_k receive antennas. For this setting [3] showed that half-the-cake is optimal provided there is no dominant user (a user with more antennas than all the rest of the users combined). Interestingly, this condition is also identical to the insight from [2] — once again, half-the-cake is optimal if the sum of interfering channel ranks is greater than or equal to the number of antennas at each user.

C. Overview of the contribution of this work

In order to further refine the key insight from [2], [3] and to identify its limitations, it is important to continue to test its validity under generalized settings. To this end, in this work we unify the channel models of [2] and [3] into the rank-deficient K-user $M_k \times M_k$ MIMO interference channel, and study the optimality of half-the-cake under *arbitrary* (no assumptions of symmetry) rank constraints on the cross-channels.

Surprisingly, we discover that the original insight fails in this generalized setting. Indeed, as a *counterexample* consider the 3-user MIMO interference channel with $M_1 = 10, M_2 = 8, M_3 = 6$, where the channel from Transmitter 1 to Receiver 2 has rank 5 and the channel from Transmitter 2 to Receiver 1 has rank 6. All other channels have full rank. Even though in this channel, the sum of interfering channel ranks at every user is greater than or equal to the number of antennas at that user, it is possible to achieve more than half-the-cake (half-the-cake is 12, but 12.5 DoF are achievable, as explained in Section VIII). Therefore, a new outer bound is necessary for the K-user $M_k \times M_k$ MIMO interference channel.

Define $M_\Sigma = \Sigma_{k \in \mathcal{I}_K} M_k$. A key contribution of this work is a novel outer bound argument that shows that the DoF cannot exceed half-the-cake if the overall $M_\Sigma \times M_\Sigma$ channel matrix $\bar{\mathbf{H}}$ where all desired channels have been set to zero, has full rank. In light of our outer bound, the counterexample mentioned above implies that the 24×24 matrix

$$\bar{\mathbf{H}} = \begin{array}{cc} & \begin{matrix} 10 & 8 & 6 \end{matrix} \\ \begin{matrix} 10 \\ 8 \\ 6 \end{matrix} & \begin{pmatrix} \mathbf{0} & \mathbf{H}_{12} & \mathbf{H}_{13} \\ \mathbf{H}_{21} & \mathbf{0} & \mathbf{H}_{23} \\ \mathbf{H}_{31} & \mathbf{H}_{32} & \mathbf{0} \end{pmatrix} \end{array}, \text{ with ranks } \begin{array}{cc} & \begin{matrix} 10 & 8 & 6 \end{matrix} \\ \begin{matrix} 10 \\ 8 \\ 6 \end{matrix} & \begin{pmatrix} 0 & 6 & 6 \\ 5 & 0 & 6 \\ 6 & 6 & 0 \end{pmatrix} \end{array}$$

cannot have full rank for any possible realization. Indeed, this is the case because the 24×18 sub-matrix formed by its first 18 columns is rank-deficient (sum of row ranks cannot be more than $6 + 5 + 6 = 17$).

The new outer bound leads us to a more precise understanding of the original insight, so that we are able to refine it to the following form for generic rank-deficient channels.

Refined Insight: "*Half-the-cake is optimal if at every transmitter and receiver, the sum of* **reduced** *interfering channel ranks* **equals** *the number of antennas at that transmitter and receiver, respectively.*"

So according to the refined condition, we are allowed to reduce the ranks of the cross-channels, but the reduced interference channel ranks must then add up at each transmitter and receiver to precisely equal the number of antennas at that transmitter and receiver, respectively. The counterexample presented earlier does not satisfy the refined condition. Indeed, it is not possible to assign any (possibly reduced) rank values that add up to the row and column index for every row and every column.

On the other hand, consider a different $\bar{\mathbf{H}}$ with ranks

$$\begin{array}{cc} & \begin{matrix} 10 & 8 & 6 \end{matrix} \\ \begin{matrix} 10 \\ 8 \\ 6 \end{matrix} & \begin{pmatrix} 0 & 8 & 3 \\ 5 & 0 & 4 \\ 6 & 2 & 0 \end{pmatrix} \end{array} \text{ which can be reduced to } \begin{array}{cc} & \begin{matrix} 10 & 8 & 6 \end{matrix} \\ \begin{matrix} 10 \\ 8 \\ 6 \end{matrix} & \begin{pmatrix} 0 & 8 & 2 \\ 4 & 0 & 4 \\ 6 & 0 & 0 \end{pmatrix} \end{array}$$

so that the reduced ranks add up to the row and column index for every row and column. Therefore, any realization of $\bar{\mathbf{H}}$ channels with these (unreduced or reduced) ranks cannot achieve more than half-the-cake. Also, as we show, for generic channels half-the-cake is always achievable, so it is optimal.

As a "sufficient" condition for optimality of half-the-cake, the additional requirements in the refined condition may appear to weaken its impact. This is not the case, however, as we note that the refined condition still recovers all prior results on the optimality of half-the-cake from [1], [2], [3] as special cases of the K-user $M_k \times M_k$ rank-deficient MIMO channel model.

Notation: We denote the set $\{1, ..., K\}$ by $\mathcal{I}_K$ for a positive integer K. For $l \in \mathcal{I}_K$, we have $\mathcal{I}_K \backslash l = \{1, ..., l-1, l+1, ..., K\}$. Indexing is interpreted in a circular wrap-around manner, modulo the number of users, e.g., the K^{th} user is same as the 0^{th} user.

II. System Model

We consider a K-user MIMO interference channel where there are M_i antennas at the i-th transmitter and receiver. Each transmitter sends one independent message to its corresponding receiver. At time slot $t \in \mathbb{Z}^+$, the received signal vector at receiver j is given by

$$Y_j(t) = \sum_{i=1}^{K} \mathbf{H}_{ji}(t) X_i(t) + Z_j(t) \tag{1}$$

where $X_i(t) \in \mathbb{C}^{M_i \times 1}$ is the signal vector sent from transmitter i which satisfies an average power constraint $\mathbb{E}(\|X_i(t)\|^2) \leq \rho$, $\mathbf{H}_{ji}(t) \in \mathbb{C}^{M_j \times M_i}$ is the channel matrix from transmitter i to receiver j, $Z_j(t) \in \mathbb{C}^{M_j \times 1}$ is the i.i.d. circularly symmetric complex additive white Gaussian noise (AWGN) at receiver j, each entry of which is an i.i.d. Gaussian random variable with zero-mean and unit-variance. We assume that perfect global channel knowledge is available at all nodes.

The achievable rates, capacity region and DoF of this network are defined in the standard sense (see [1]). We define the sum-DoF value as $d_\Sigma = \lim_{\rho \to \infty} R_\Sigma(\rho) / \log(\rho)$, where $R_\Sigma(\rho)$ is the maximum sum rate at Signal-to-noise ratio, ρ.

Without loss of generality, let $M_1 \geq M_2 \geq \cdots \geq M_K$ throughout this paper. Unless stated otherwise, by default it is assumed that channel coefficients are generic subject to rank constraints and are ergodically fading. The desired channel matrices $\mathbf{H}_{ii}(t)$ are assumed to be full rank[2] while the cross

[2]Similar to [2], the extension to rank-deficient desired channels is straightforward.

channels $\mathbf{H}_{ij}$ are subject to rank constraint D_{ij}. A rank-constrained generic $M_i \times M_j$ channel matrix of rank D_{ij} is representable as a product of a $M_i \times D_{ij}$ matrix with a $D_{ij} \times M_j$ matrix, all of whose entries are drawn from continuous distribution.

III. RESULTS

The goal of this section is to state the main results of this work. The proofs are presented in subsequent sections.

For the main inner bound, although a more sophisticated proof that does not require ergodicity may be possible as shown in [2], we will use the much simpler ergodic interference alignment [13] argument, also used as an alternative proof in [2] to show that half-the-cake is achievable.

Theorem 1: *For generic ergodic fading rank-deficient channels, regardless of interference rank-constraints*

$$\text{DoF} \geq M_\Sigma/2$$

Since achievability of half-the-cake is settled, the main question of interest is, when is half-the-cake optimal? To answer this we introduce a new outer bound argument that is actually surprisingly simple and yet quite broadly applicable. For our present purpose, the bound is presented below.

Theorem 2: *For arbitrary channel realizations, if*

$$\text{rank}(\bar{\mathbf{H}}) = M_\Sigma \quad \text{then} \quad \text{DoF} \leq M_\Sigma/2.$$

Note that the bound applies to any given channel realization.

The next result translates the rank-condition on $\bar{\mathbf{H}}$ to rank-conditions on individual interfering channels.

Lemma 1: For generic channel realizations, rank$(\bar{\mathbf{H}}) = M_\Sigma$ *if and only if there exist reduced ranks $\bar{D}_{ij} \leq D_{ij}$ which satisfy the following condition,*

$$\sum_{j \in \mathcal{I}_K \backslash i} \bar{D}_{ji} = \sum_{j \in \mathcal{I}_K \backslash i} \bar{D}_{ij} = M_i, \forall i \in \mathcal{I}_K. \tag{2}$$

Combined with Theorem 2, Lemma 1 directly proves the following theorem.

Theorem 3: *For a K-user generic rank-deficient MIMO interference channel, if there exist reduced ranks $\bar{D}_{ij} \leq D_{ij}$ for each interference link, which satisfy the following condition,*

$$\sum_{j \in \mathcal{I}_K \backslash i} \bar{D}_{ji} = \sum_{j \in \mathcal{I}_K \backslash i} \bar{D}_{ij} = M_i, \forall i \in \mathcal{I}_K. \tag{3}$$

then almost surely $\text{DoF} = \sum_{k=1}^{K} \frac{M_k}{2}$.

For $K = 3$ users it may be useful to state the condition more explicitly as follows. *The 3-user generic rank-deficient MIMO interference channel has DoF$= M_\Sigma/2$, if*

$$\begin{aligned} &\min\{M_1 + D_{32}, M_2 + D_{13}, M_3 + D_{21}\} + \\ &\min\{M_3 + D_{12}, M_1 + D_{23}, M_2 + D_{31}\} \leq M_1 + M_2 + M_3. \end{aligned} \tag{4}$$

IV. RECOVERING PRIOR RESULTS AS SPECIAL CASES

In this section, we will show that, the prior half-the-cake DoF results in [2], [3] can be recovered as special cases of Theorem 1.

1) Full rank case: In [3], half-the-cake DoF is shown to be optimal in K-user $M_k \times M_k$ MIMO interference channel where there is no dominant user and all channels have full rank. To prove that full rank K-user $M_k \times M_k$ MIMO interference channels satisfy the condition in Theorem 1, it is sufficient to show that for any $M_1 \leq M_2 + \cdots + M_K$, we can always find a set of values for $\bar{D}_{ij} \leq \min(M_i, M_j)$ that satisfy the condition in Lemma 1.

To start, suppose $\forall k \in \mathcal{I}_K$, each transmitter k has M_k chips and each receiver k has an empty bin that can hold M_k chips. Transmitter 1 starts by dropping as many chips as possible into receiver 2's bin, and then if the bin is full and he still has chips left over, he continues with receiver 3's bin, and so on. After Transmitter 1 is done, Transmitter 2 does the same, starting with receiver 3's bin. Transmitter 2 is followed by Transmitters $3, 4, \cdots, K$, in that order. At the end, the number of chips in receiver bin i from transmitter j is chosen to be the rank $\bar{D}_{ij}$. Since there is no dominant user, the total capacity of all bins is the same as the total number of chips, and users are arranged as $M_1 \geq M_2 \geq \cdots \geq M_K$, it is easy to see that this allocation works.

2) Symmetric case: In [2], it is shown that for a K-user rank deficient MIMO interference channel with M antennas at each node, if all the direct channels have full rank, and all cross channels have rank D, then half-the-cake DoF is optimal when $(K-1)D \geq M$. We now show that this result is also a special case of Theorem 1.

Note that if $\frac{M}{K-1}$ is an integer, then we just need to reduce D to the value $\frac{M}{K-1}$. When $\frac{M}{K-1}$ is not an integer, we can write $M = \left\lfloor \frac{M}{K-1} \right\rfloor (K-1) + \Delta$ for some positive integer $\Delta < K-1$. Now, assign reduced interference ranks as follows.

$$\begin{aligned} \bar{D}_{ij} &= \left\lfloor \frac{M}{K-1} \right\rfloor + 1 \leq D, \text{ if } i \in \{j+1, j+2, \cdots, j+\Delta\}, \\ \bar{D}_{ij} &= \left\lfloor \frac{M}{K-1} \right\rfloor \leq D, \text{ otherwise.} \end{aligned}$$

With these reduced ranks, the condition in Lemma 1 is always satisfied. Thus, Theorem 3 applies and half-the-cake is DoF optimal.

V. PROOF OF THEOREM 1

Each transmission takes 2 channel uses t_1 and t_2, where all channel matrices of interference links remain the same $\mathbf{H}_{ij}(t_1) = \mathbf{H}_{ij}(t_2)$, $i \neq j$, and all channel matrices of direct links change $\mathbf{H}_{ii}(t_1) \neq \mathbf{H}_{ii}(t_2)$ in a generic sense, i.e., their difference is also full rank. Thus by letting each transmitter repeat its symbols over the 2 channel uses, each receiver can eliminate interference by subtracting the output at t_2 from the output at t_1, and obtain an $M_k \times M_k$ interference free channel, over which M_k DoF are obtained. Since, this requires two channel uses, effectively $\frac{M_k}{2}$ DoF per user are achieved.

VI. PROOF OF THEOREM 2

Given the K-user interference channel, we use $\hat{\mathbf{H}}_{ij}$, $\forall i, j \in \mathcal{I}_K$, to denote each channel matrix. Now create a $2K$-user interference channel by adding an auxiliary user k' for each original user k, and choosing the new channels so that $\forall i,j,i',j' \in \mathcal{I}_K$, 1) $\mathbf{H}_{i'j} = \mathbf{H}_{ij'} = \hat{\mathbf{H}}_{ij}$ whenever $i \neq j$, 2) $\mathbf{H}_{i'i'} = \mathbf{H}_{ii} = \hat{\mathbf{H}}_{ii}$, 3) $\mathbf{H}_{i'j'} = \mathbf{H}_{ij}$ is the matrix of zeros whenever $i \neq j$, and 4) $\mathbf{H}_{i'i} = \mathbf{H}_{ii'}$ is the matrix of zeros. Any coding scheme for the original channel still works if each auxiliary user i' uses the same codebook as user i. Since users i and i' in the new network achieve the same rates as user i in the original network, the DoF value for the new network is at least twice that of the original network. Now in the new network, allow all original transmitters to cooperate, all original receivers to cooperate, all auxiliary transmitters to cooperate and all auxiliary receivers to cooperate, which can only help. This creates a 2-user interference channel where everyone has M_Σ antennas, and where the interference matrix is $\bar{\mathbf{H}}$. If this interference matrix is full rank, then each user, after decoding its desired signal, can subtract it out and then proceed to decode the interfering signal as well (subject to noise distortion, inconsequential for DoF). Thus, the sum-DoF of the interference channel cannot be more than M_Σ, and therefore the DoF of the original network cannot be more than $\frac{1}{2}M_\Sigma$.

VII. Proof of Lemma 1

In this section, we prove Lemma 1 by first showing that condition (2) is sufficient for $\bar{\mathbf{H}}$ to have full rank, and then showing that this condition is also necessary.

A. Sufficiency

To prove that $\bar{\mathbf{H}}$ is full-rank almost surely for generic rank-deficient channels with given ranks, it suffices to show that its determinant polynomial is not identically zero. To show this, it suffices to find one realization of $\bar{\mathbf{H}}$ for which the determinant is not zero. Such a realization is constructed as follows. At Receiver i, starting from the first antenna, label the first set of $\bar{D}_{i,i+1}$ antennas as $S_R(i,i+1)$, the next $\bar{D}_{i,i+2}$ as $S_R(i,i+2)$, and so on, until the final set of $\bar{D}_{i,i+K-1}$ antennas is labeled as $S_R(i,i+K-1)$. Similarly, at Transmitter j, starting from the first antenna, label the first set of $\bar{D}_{j+1,j}$ antennas as $S_T(j+1,j)$, the next set of $\bar{D}_{j+2,j}$ antennas as the set $S_T(j+2,j)$, and so on until the last set of $\bar{D}_{j+K-1,j}$ antennas is labeled as $S_T(j+K-1,j)$. Now connect transmit antennas in $S_T(i,j)$ with the receive antennas in $S_R(i,j)$ through identity matrices. With this channel realization, each transmit antenna is connected to exactly one undesired receive antenna, so that $\bar{\mathbf{H}}$ has exactly one 1 in each row and each column, and is therefore full rank. Increasing any of the ranks only introduces additional variables into the polynomial which can be set to zero to return to the same realization described above, thus proving that the polynomial is not identically zero.

B. Necessity

If the following partitioned matrix $\bar{\mathbf{H}}$ has full rank,

$$\bar{\mathbf{H}} = \begin{array}{c} \\ M_1 \\ M_2 \\ \vdots \\ M_K \end{array} \begin{array}{c} \begin{array}{ccccc} M_1 & M_2 & \cdots & M_{K-1} & M_K \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{H}_{12} & \cdots & \mathbf{H}_{1(K-1)} & \mathbf{H}_{1K} \\ \mathbf{H}_{21} & \mathbf{0} & \cdots & \mathbf{H}_{2(K-1)} & \mathbf{H}_{2K} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{H}_{K1} & \mathbf{H}_{K2} & \cdots & \mathbf{H}_{K(K-1)} & \mathbf{0} \end{pmatrix} \end{array} \quad (5)$$

then the first observation is that each column of sub-matrix and each row of sub-matrix must have full rank, i.e., the rank of each sub-matrix must satisfy the following conditions.

$$\sum_{j\in\mathcal{I}_K\backslash i} D_{ji} \geq M_i, \ \sum_{j\in\mathcal{I}_K\backslash i} D_{ij} \geq M_i, \ \forall i \in \mathcal{I}_K. \quad (6)$$

With the help of this observation, the necessity of condition (2) can be proved as follows. Any sub-matrix $\mathbf{H}_{ij}$ of rank D_{ji} can be represented as a sum of D_{ji} matrices, each of which has rank 1, i.e.,

$$\mathbf{H}_{ij} = a_{ij}^{[1]}\mathbf{v}_{ij}^{[1]}\mathbf{u}_{ij}^{[1]} + a_{ij}^{[2]}\mathbf{v}_{ij}^{[2]}\mathbf{u}_{ij}^{[2]} + \cdots + a_{ij}^{[D_{ji}]}\mathbf{v}_{ij}^{[D_{ji}]}\mathbf{u}_{ij}^{[D_{ji}]} \quad (7)$$

where $\mathbf{v}_{ij}^{[k]}$ and $\mathbf{u}_{ij}^{[k]}$ are $M_i \times 1$ and $1 \times M_j$ unit vectors, respectively. Now let us consider the $a_{ij}^{[k]}$ as variables while $\mathbf{v}_{ij}^{[k]}$ and $\mathbf{u}_{ij}^{[k]}$ are treated as constants. After all the $\mathbf{H}_{ij}$ are represented in the form as (7), we use $\mathcal{A}$ to denote the set of all the $a_{ij}^{[k]}$ in $\bar{\mathbf{H}}$. Then we go through the following steps.

1) Step 1: Choose any one of the variables $a_{ij}^{[k]}$ from $\mathcal{A}$. We set this variable $a_{ij}^{[k]}$ to zero, then we have reduced the rank of corresponding sub-matrix $\mathbf{H}_{ij}$ by 1.

2) Step 2: We check the determinant polynomial of $\bar{\mathbf{H}}$ with the rank-reduced sub-matrix $\mathbf{H}_{ij}$. If det($\bar{\mathbf{H}}$) is not the zero polynomial, then $\bar{\mathbf{H}}$ is full rank almost surely, fix $a_{ij}^{[k]} = 0$. And if det($\bar{\mathbf{H}}$) is the zero polynomial, leave $a_{ij}^{[k]}$ as a generic variable. Remove this $a_{ij}^{[k]}$ from the set $\mathcal{A}$.

3) Step 3: If the set $\mathcal{A}$ is not an empty set, go back to step 1. And if $\mathcal{A}$ is empty, i.e., all $a_{ij}^{[k]}$ have been tested, we now have a situation that the remaining $a_{ij}^{[k]}$ must all be non-zero for $\bar{\mathbf{H}}$ to have full rank M_Σ. At this stage, the number of remaining $a_{ij}^{[k]}$ variables for each sub-matrix define the reduced rank value $\bar{D}_{ij}$ for that matrix.

Now, based on the following two facts, it can be claimed that the number of remaining $a_{ij}^{[k]}$ variables cannot be more than M_Σ.

Fact 1: "*Since setting any $a_{ij}^{[k]}$ to zero will make* $\det(\bar{\mathbf{H}})$ *the zero polynomial, then it must be true that every remaining $a_{ij}^{[k]}$ variable appears in every term of the polynomial.*"

Fact 2: "*Since each term in $\bar{\mathbf{H}}$ is linear in each $a_{ij}^{[k]}$ variable, each term of the determinant polynomial cannot involve more than M_Σ of $a_{ij}^{[k]}$ variables.*"

Fact 1 says that every remaining $a_{ij}^{[k]}$ must be present in every term of $\det(\bar{\mathbf{H}})$. Fact 2 says that there cannot be more than M_Σ remaining $a_{ij}^{[k]}$ that are presented in any given term of $\det(\bar{\mathbf{H}})$. Thus the two facts imply that the number of

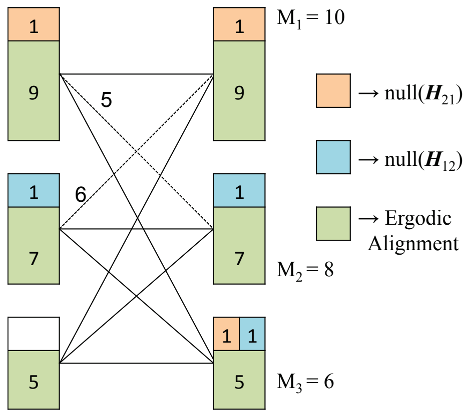

Fig. 2: Example for achieving more than half-the-cake DoF.

remaining $a_{ij}^{[k]}$ cannot be more than M_Σ. i.e., $\sum(\bar{D}_{ij}) = M_\Sigma$. Since all the $\bar{D}_{ij}$ must also satisfy condition (6) in order for $\bar{\mathbf{H}}$ to have full rank, all the inequalities in (6) must take equality. In other words, for any full rank matrix $\bar{\mathbf{H}}$, there always exist reduced ranks $\bar{D}_{ij} \leq D_{ij}$ which satisfy the condition (2). This completes the proof.

VIII. Counterexample to original insight

Here we briefly summarize how more than half-the-cake DoF can be achieved in the 3-user setting shown in Fig. 2 where $D_{12} = 6$, $D_{21} = 5$ and all other links have full rank.

The transmission takes place over 2 channel uses, where all cross channels remain the same, and all direct channels change to different generic values. We use $\mathbf{V}_1^z$ and $\mathbf{V}_2^z$ to denote the beamforming vectors at transmitters 1 and 2 that need to be aligned at receiver 3 after being chosen from the null space they see at each other. The symbols carried by $\mathbf{V}_1^z$ and $\mathbf{V}_2^z$ are different over two channel uses. Mathematically, we have

$$\begin{array}{r} \mathbf{H}_{21}\mathbf{V}_1^z = \mathbf{0}, \\ \mathbf{H}_{12}\mathbf{V}_2^z = \mathbf{0}, \\ \mathbf{H}_{31}\mathbf{V}_1^z = \mathbf{H}_{32}\mathbf{V}_2^z. \end{array} \Rightarrow \underbrace{\begin{bmatrix} \mathbf{H}_{21} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{12} \\ \mathbf{H}_{31} & -\mathbf{H}_{32} \end{bmatrix}_{24\times 18}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{V}_1^z \\ \mathbf{V}_2^z \end{bmatrix}}_{\mathbf{v}} = \mathbf{0}.$$

Note that matrix $\mathbf{A}$ has rank 17, thus $\mathbf{v}$ can be chosen from the right null space of $\mathbf{A}$. In the same manner, we choose the receive combining vectors $\mathbf{U}_1^z$ and $\mathbf{U}_2^z$ at receivers 1 and 2 satisfying the following equations

$$\mathbf{U}_1^z\mathbf{H}_{12} = \mathbf{0}, \quad \mathbf{U}_2^z\mathbf{H}_{21} = \mathbf{0}, \quad \mathbf{U}_1^z\mathbf{H}_{13} = \mathbf{U}_2^z\mathbf{H}_{23}.$$

Next, we use $\mathbf{V}_k^e$ and $\mathbf{U}_k^e$ to denote the $M_k \times (M_k - 1)$ and $(M_k - 1) \times M_k$ matrices at each transmitter and receiver, respectively. These matrices carry the signals for ergodic alignment (green area in Fig. 2), i.e. signals repeated over the two channel uses. User 3 needs to choose its beamforming/combining matrices to satisfy $\mathbf{V}_3^e = \mathrm{span}(\mathrm{null}(\mathbf{U}_2^z\mathbf{H}_{23}))$ and $\mathbf{U}_3^e = \mathrm{span}(\mathrm{null}(\mathbf{H}_{32}\mathbf{V}_2^z))$. And as s a result, each receiver can eliminate interference by only subtracting the part of received signals correspond to $\mathbf{U}_k^e$ of two time slots. Thus, a total of 25 DoF is achieved over the two channel uses, or equivalently, 12.5 DoF per channel use (half-the-cake is 12 DoF per channel use).

IX. Conclusion

We considered a K-user MIMO interference channel with rank-deficient cross-channels, where there are M_i antennas at i-th user pair, and all direct links have full rank. A novel outer bound argument was proposed and it was shown that the DoF cannot exceed half-the-cake if the overall $M_\Sigma \times M_\Sigma$ channel matrix $\bar{\mathbf{H}}$ where all desired channels have been set to zero, has full rank. While it is easy to see that the new bounds presented here are not necessary for optimality of half-the-cake, for $K = 3$ we conjecture that combining them with existing bounds based on cooperation and 2-user settings produces a necessary and sufficient set of conditions. In particular, for $K = 3$, $D_{ij} = D_{ji}$ and generic channels, the condition presented here is necessary and sufficient for optimality of half-the-cake (almost surely). In the full paper, currently under preparation, we are able to show that the novel outer bound that we introduce here has broad applications, e.g., to DoF of K-user $M \times N$ MIMO settings as well.

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