Detection of Gauss-Markov Random Field on Nearest-Neighbor Graph

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Introduction: Distributed Detection

Setup

- **Sensors**: transmit local decisions
- **Fusion center**: Global Decision
- **Classical data model**: Conditionally IID

Sensor signal field

- Correlated sensor readings
- Large coverage area
- Large number of sensors
- Arbitrary sensor placement

Influence of correlation structure on detection performance
Detection of Correlation

Binary hypothesis testing

\( \mathcal{H}_1: \) Correlated data vs. \( \mathcal{H}_0: \) Independent observations

Questions

- How to model correlation?
- Is there an analytically tractable performance metric?
- How does correlation affect performance?
- How does node density affect performance?

New tradeoffs not encountered in IID scenario
Detection of Correlation

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\( \mathcal{H}_1 \): Correlated data vs. \( \mathcal{H}_0 \): Independent observations

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Summary of Results

Questions Answered

- How to model correlation?
  - Gauss-Markov random field

- Is there an analytically tractable performance metric?
  - Closed-form detection error exponent for Neyman-Pearson

- How does correlation affect performance?
  - Depends on variance ratio
    - If signal under $\mathcal{H}_1$ is weak (low variance), correlation helps
    - If signal under $\mathcal{H}_1$ is strong (high variance), correlation hurts

- How does node density affect performance?
  - More node density more correlation as edge length is reduced
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Previous Results on Detection Error Exponent

I.I.D case
- Closed-form for optimal detector and threshold
- Error exponent - Stein’s lemma

Correlated case
- Stationary Gaussian process (Donsker & Varadhan, 85)
- General formulas for Neyman-Pearson exponent (Chen, 96)
- Closed-form for Gauss-Markov random process (Sung & etal, 06)

Limitations of the closed form
- Requires causality, valid in 1-D case
- Cannot handle random placement of nodes
Outline

1. Introduction
2. Gauss-Markov Random Field
3. Statistical Inference
4. Results on Error Exponent
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**Model for Correlated Data: Graphical Model**

\[ X(i-1)X(i)X(i+1) \]

\[ X_{i-1} \perp X_{i+1} | X_i \]

Linear graph corresponding to autoregressive process of order 1

**Temporal signals**
- Conditional independence based on ordering
- Fixed number of neighbors
- Causal (random processes)

**Spatial signals**
- Conditional independence based on (undirected) **Dependency Graph**
- Variable set of neighbors
- Maybe acausal

**Remark**

Dependency graph is **NOT** related to communication capabilities, but to the correlation structure of data!
**Markov Random Field**

**Definition**: MRF with Dependency Graph $G_d(\mathcal{V}, \mathcal{E})$

$\mathbf{Y}(\mathcal{V}) = \{Y_i : i \in \mathcal{V}\}$ is MRF with $G_d(\mathcal{V}, \mathcal{E})$ if $\mathbf{Y}$ is Gaussian random field, PDF satisfies positivity condition and Markov property.

**Markov Property**

- $A, B, C$ are disjoint
- $A, B$ non-empty
- $C$ separates $A, B$

$$\mathbf{Y}_A \perp \mathbf{Y}_B | \mathbf{Y}_C$$
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Likelihood Function of MRF

Hammersley-Clifford Theorem (1971)

For a MRF $\mathbf{Y}$ with dependency graph $\mathcal{G}_d(\mathcal{V}, \mathcal{E}_d)$,

$$\log \mathbb{P}(\mathbf{Y}; \mathcal{G}_d) = Z + \sum_{c \in \mathcal{C}} \Psi_c(\mathbf{Y}_c), \quad Z \triangleq e^{-\int \prod_{c \in \mathcal{C}} \Psi_c(\mathbf{Y}_c)},$$

where $\mathcal{C}$ is the set of all cliques in $\mathcal{G}_d$ and $\Psi_C$ the clique potential.
Potential Matrix of GMRF

- Inverse of covariance matrix of a GMRF
- Non-zero elements of Potential matrix correspond to graph edges

```
\begin{bmatrix}
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times \\
\end{bmatrix}
```

\( \times : \) Non-zero element of Potential Matrix

Form of Log-Likelihood of zero-mean GMRF with potential matrix \( A \)

\[
- \log P(Y_n; G_d, A) = \frac{1}{2} (-n \log 2\pi + \log |A| + \sum_{(i,j) \in E_d} A(i,j)Y_iY_j + \sum_{i \in V} A(i,i)Y_i^2)
\]

Acyclic Dependency Graph

Given Covariance matrix, closed-form expression of likelihood
Outline

1 Introduction

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4 Results on Error Exponent
Hypothesis Testing for Independence

$H_1$: GMRF with dependency graph $G_d$

$H_0$: Independent observations

Model for Dependency Graph $G_d$ under $H_1$

- Dependency graph is a proximity graph (edges between nearby points)
- Simplest proximity graph: nearest-neighbor graph

Definition of Nearest-Neighbor Graph

In NNG, $(i, j)$ is an edge if $i$ is nearest neighbor of $j$ or vice versa

Additional assumptions

- Random placement of nodes (Uniform or Poisson distribution)
- Correlation function $g$: function of spatial distance
Optimal Detection

Log Likelihood Ratio (LLR) Detector

\[
\log \frac{P[Y_n, \mathcal{V}; \mathcal{H}_1]}{P[Y_n, \mathcal{V}; \mathcal{H}_0]} \leq \tau_n
\]

Neyman-Pearson Detection

Minimize Miss Probability

\[
P_M^\Delta = P[\text{Decision} = \mathcal{H}_0 | \mathcal{H}_1]
\]

with false alarm constraint

\[
P_F = P[\text{Decision} = \mathcal{H}_1 | \mathcal{H}_0] \leq \alpha
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Error Exponent $D$

Closed-form of error probability not tractable

\[ P_M \approx e^{-nD} \quad \text{Number of samples} \]

\[ \log P_M \approx -nD \quad \text{Number of samples} \]

Sensors Placed in region with constant node density $\lambda$
**Error Exponent** \(D\)

Closed-form of error probability not tractable

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P_M \approx e^{-nD}
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Sensors Placed in region with constant node density \(\lambda\)
Error Exponent $D$

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Sensors Placed in region with constant node density $\lambda$
Our Methodology

Approaches

- LLR as sum of node and edge functionals of dependency graph
- Error exponent through limit of LLR
- Evaluate limit using Law of Large Numbers for graph functionals
- Error exponent for performance analysis
Detailed Methodology

LLR as sum of node and edge functionals of dependency graph

\[
\text{LLR}(Y_n, G_d) = n \log \frac{\sigma_1}{\sigma_0} + \frac{1}{2} \left[ \sum_{i \in V} \left( \frac{1}{\sigma_1} - \frac{1}{\sigma_0} \right) Y_i^2 
+ \sum_{(i,j) \in E_{d}} \left\{ \log[1 - \frac{g^2(R_{ij})}{1 - g^2(R_{ij})}] + \frac{g^2(R_{ij})}{1 - g^2(R_{ij})} \frac{Y_i^2 + Y_j^2}{\sigma_1^2} - \frac{2g(R_{ij})}{1 - g^2(R_{ij})} \frac{Y_i Y_j}{\sigma_1^2} \right\} \right]
\]

Error exponent through limit of LLR

\[
D = \lim_{n \to \infty} \frac{1}{n} \text{LLR}(Y_n; G_d), \quad \mathcal{H}_0
\]

LLR is sum of graph functionals of a Marked process

\(Y_i\) are independent under \(\mathcal{H}_0\)
Detailed Methodology

LLR as sum of node and edge functionals of dependency graph

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+ \left. \sum_{(i,j) \in E_d} \left\{ \log [1 - g^2(R_{ij})] + \frac{g^2(R_{ij})}{1 - g^2(R_{ij})} \frac{Y_i^2 + Y_j^2}{\sigma_1^2} - \frac{2g(R_{ij})}{1 - g^2(R_{ij})} \frac{Y_i Y_j}{\sigma_1^2} \right\} \right]
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+ \sum_{\substack{(i,j) \in E \atop i < j}} \left\{ \log \left[ 1 - g^2(R_{ij}) \right] + \frac{g^2(R_{ij})}{1 - g^2(R_{ij})} \frac{Y_i^2 + Y_j^2}{\sigma_1^2} - \frac{2g(R_{ij})}{1 - g^2(R_{ij})} \frac{Y_i Y_j}{\sigma_1^2} \right\} \right]
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+ \sum_{(i,j) \in \mathcal{E}_d, i < j} \left\{ \log[1 - g^2(R_{ij})] + \frac{g^2(R_{ij})}{1 - g^2(R_{ij})} \frac{Y_i^2 + Y_j^2}{\sigma_1^2} - \frac{2g(R_{ij})}{1 - g^2(R_{ij})} \frac{Y_i Y_j}{\sigma_1^2} \right\}
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Error exponent through limit of LLR

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\]

LLR is sum of graph functionals of a Marked process

\( Y_i \) are independent under \( \mathcal{H}_0 \)
LLN for graph functionals (Penrose & Yukich, 02)

Pictorial Representation of result

Normalized sum of edge weights

\[ \frac{\sum_{e \in E} \Phi(R_e)}{n} \]

Expectation of edges of origin of Poisson process

\[ \mathbb{E} \sum_{X \in \mathcal{P}_\lambda} \phi(R_0, X) \]

Remarks

LLN states that limit is a localized effect around origin
Result on Error Exponent $D$

Applying LLN (Penrose & Yukich, 02)

$$D = \frac{1}{2} \left[ \mathbb{E} \sum_{x \in \mathcal{P}_\lambda} f(g(R_0, x)) + \log K + \frac{1}{K} - 1 \right],$$

$$f(x) \overset{\Delta}{=} \log[1 - x^2] + \frac{2x^2}{K[1 - x^2]}, \quad K \overset{\Delta}{=} \frac{\sigma_1^2}{\sigma_0^2},$$

- $R_0, x$: edge-lengths in a NNG of origin of a homogeneous Poisson process of intensity $\lambda$

Closed-form Expression for $D$

$$D = \frac{1}{2} \left[ \mathbb{E} f(g(Z_1)) - \frac{\pi}{2\omega} \mathbb{E} f(g(Z_2)) + \log K + \frac{1}{K} - 1 \right]$$

- $Z_1, Z_2$: Rayleigh distributed with Variances $(2\pi \lambda)^{-1}, (2\omega \lambda)^{-1}$
- $\omega \approx 5.06$: area of union of two unit radii circles, with centers unit distant apart
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Exponential Correlation Function

\[ g(r) = Me^{-ar}, \quad a > 0, 0 < M < 1 \]
Minimum Energy Routing for Optimal Inference

Minimize total energy of routing such that LLR is delivered to fusion center

Summary of Results

- Concept of dependency graph based routing
  - Exploit correlation to fuse data
- Proposed 2-approximation algorithm

Transmission scheme delivering LLR

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Conclusion

Summary

- Derived a closed-form expression for error exponent of detection a GMRF with nearest-neighbor dependency
- Studied effect of correlation and node density on performance

Outlook

- Relax assumptions
- Extend to other dependency models
- Study Performance-Routing Energy tradeoff
Thank You!