## FINAL EXAMINATION

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### Useful identities

\[
\cos(A) \cos(B) = \frac{1}{2} \cos(A - B) + \frac{1}{2} \cos(A + B) \\
\sin(A) \sin(B) = \frac{1}{2} \cos(A - B) - \frac{1}{2} \cos(A + B) \\
\sin(A) \cos(B) = \frac{1}{2} \sin(A - B) + \frac{1}{2} \sin(A + B) \\
\]

\[
\text{rect}(t) \triangleq \begin{cases} 
1 & |t| \leq 1/2 \\
0 & |t| > 1/2
\end{cases} \quad \mathcal{F}\{\text{rect}(t/T)\} = \text{sinc}(fT) \triangleq \frac{\sin(\pi ft)}{\pi ft} \\
\mathcal{F}\{\cos(2\pi f_0 t)\} = \frac{\delta(f - f_0) + \delta(f + f_0)}{2} \\
\mathcal{F}\{g_1(t)g_2(t)\} = G_1(f) * G_2(f) \\
\delta(f - f_0) * G(f) = G(f - f_0) \\
\mathcal{F}\{g(t - t_0)\} = G(f)e^{-j2\pi ft_0}
\]
1. (15 points) Let the random variable $X$ be distributed according to the Laplacian distribution

$$p_X(x) = \frac{a}{2} e^{-a|x|} \quad -\infty < x < \infty$$

where $a > 0$.

(a) Calculate the characteristic function $\psi_X(j\nu)$.

(b) Using $\psi_X(j\nu)$, calculate $E[X]$.

(c) Using $\psi_X(j\nu)$, calculate $E[X^2]$. 

2. (15 points) In a detection problem there are two hypotheses, \(H_0\) and \(H_1\). There are \(N\) observations made: \(r_1, r_2, \ldots, r_N\). Under both hypotheses, the observations are independent, identically distributed zero-mean Gaussian random variables. Under \(H_0\), each \(r_k\) has a variance \(\sigma_0^2\) and under \(H_1\), each \(r_k\) has a variance \(\sigma_1^2\). Assuming \(\sigma_1^2 > \sigma_0^2\), calculate the simplest form of the detection rule that will minimize the probability of error.
3. (20 points) Let six signals be given as

\[ s_1(t) = \sqrt{\frac{2P}{T}} \cos \left( \frac{2\pi n_c t}{T} \right) \]
\[ s_2(t) = \sqrt{\frac{2P}{T}} \cos \left( \frac{2\pi (n_c + 1) t}{T} \right) \]
\[ s_3(t) = \sqrt{\frac{2P}{T}} \cos \left( \frac{2\pi (n_c + 2) t}{T} \right) \]
\[ s_4(t) = -\sqrt{\frac{2P}{T}} \cos \left( \frac{2\pi n_c t}{T} \right) \]
\[ s_5(t) = -\sqrt{\frac{2P}{T}} \cos \left( \frac{2\pi (n_c + 1) t}{T} \right) \]
\[ s_6(t) = -\sqrt{\frac{2P}{T}} \cos \left( \frac{2\pi (n_c + 2) t}{T} \right) \]

over a period of \([0, T]\) for \(n_c = n_c + i\) where \(n_c\) is an integer and \(i = 1, 2, 3\) as in Sunde’s FSK. All six signals are equal to 0 outside this interval. The noise in the channel is AWGN with PSD \(N_0/2\) W/Hz.

(a) What is the dimensionality \(N\) of \(\{s_k(t)\}_{k=1}^{6}\)?

(b) Draw the signal constellation.

(c) What is the modulation technique classification for this signal set?

(d) Draw the optimum receiver with the minimum number \(K\) of correlators or matched filters.

(e) Let the outputs of correlation receivers or matched filters in (d) be \(r_1, r_2, \ldots, r_K\). What is the optimum ML decision rule that maps \(r_1, r_2, \ldots, r_K\) to the transmitted message \(m\)?

(f) What is the exact probability of symbol error for this system?
4. (20 points)

(a) Derive a set of basis functions \( \{ f_i(t) \}_{i=1}^{N} \) based on \( \{ s_k(t) \}_{k=1}^{3} \) above. What is \( N \)?

(b) Draw the vector space representation of \( \{ s_k(t) \}_{k=1}^{3} \) using the set of basis functions you found in part (a) above. What is \( d_{\text{min}}^{(e)} \)?

(c) Draw the optimum receiver for this signal set, assuming the transmitted signal goes through an AWGN channel with PSD \( N_0/2 \) W/Hz.

(d) What is the probability of error incurred by this system?

(e) Consider the following signal set.

What is the dimensionality \( N' \) of \( \{ s'_k(t) \}_{k=1}^{3} \)? What is \( d_{\text{min}}^{(e)} \)?

(f) Draw the vector space representation of \( \{ s'_k(t) \}_{k=1}^{3} \) above.

(g) What is the probability of error incurred by this system?
5. (15 points)

The functions $g_1(t)$ and $g_2(t)$ are given as follows

$$g_1(t) = \begin{cases} \sqrt{\frac{2\pi}{T}} \cos \left( \frac{\pi t}{2T} \right) & -T \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

$$g_2(t) = \begin{cases} \sqrt{\frac{2\pi}{T}} \sin \left( \frac{\pi t}{2T} \right) & 0 \leq t \leq 2T \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that

$$|G_1(f)|^2 = \frac{32\pi T}{\pi^2} \left[ \frac{\cos(2\pi T f)}{16T^2 f^2 - 1} \right]^2$$

(b) Calculate $|G_2(f)|^2$ and state why.
6. (15 points)

The finite-state machine whose state transition diagram is shown above is known to be at state \( a \) initially. State transitions are shown with notation \( x/y \) to denote input and output, respectively. At the receiver, during time \( k \), the output of the matched filter or correlation receiver is

\[
 r_k = y_k + n_k
\]

where \( y_k \) is the output of the finite-state machine and \( n_k \) is additive noise. Assume that the additive noise is distributed as

\[
 p_{N_k}(n_k) = Ce^{-|n_k|^\beta} \quad -\infty < n_k < \infty.
\]

where \( C \) is a constant. Let the received sequence be 0, 3, -2, 2, and -1. Determine the input sequence optimum in the maximum likelihood sequence detection sense for this noise distribution.