FINAL EXAMINATION

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Useful identities

\[
\cos(A) \cos(B) = \frac{1}{2} \cos(A - B) + \frac{1}{2} \cos(A + B)
\]

\[
\sin(A) \sin(B) = \frac{1}{2} \cos(A - B) - \frac{1}{2} \cos(A + B)
\]

\[
\sin(A) \cos(B) = \frac{1}{2} \sin(A - B) + \frac{1}{2} \sin(A + B)
\]

\[
\text{rect}(t) \triangleq \begin{cases} 
1 & |t| \leq 1/2 \\
0 & |t| > 1/2 
\end{cases}
\]

\[
\mathcal{F}\{\text{rect}(t/2T)\} = 2T \sin\left(\frac{\pi T}{2}\right)
\]

\[
\text{tri}(t) \triangleq \begin{cases} 
1 - |t| & |t| \leq 1 \\
0 & |t| > 1 
\end{cases}
\]

\[
\mathcal{F}\{\text{tri}(t/T)\} = T \sin^2(Tf) = T \frac{\sin^2(\pi Tf)}{(\pi Tf)^2}
\]
1. (15 points)

(a) Let $\mathbf{X}_i$ be an $N$-dimensional Gaussian random vector with mean $\mathbf{m}_i$ and covariance matrix $\mathbf{M}_i$. Let

$$\mathbf{Y} = \sum_{i=1}^{n} \mathbf{X}_i + \mathbf{b}_i$$

where $\mathbf{b}_i$ is an $N$-dimensional vector. Assume that Gaussian vectors $\mathbf{X}_i$ are uncorrelated. Calculate the characteristic function of $\mathbf{Y}$.

(b) Can you conclude that $\mathbf{Y}$ is a Gaussian vector? If so, what are its mean vector and covariance matrix?
2. (15 points) Three messages $m_1, m_2,$ and $m_3$ are to be transmitted over a zero-mean additive white Gaussian noise channel with power spectral density $N_0/2$ W/Hz. The signals are

$$s_1(t) = \frac{1}{2} \sin(2\pi f_c t) + \frac{\sqrt{3}}{2} \cos(2\pi f_c t)$$

$$s_2(t) = \frac{1}{2} \sin(2\pi f_c t) - \frac{\sqrt{3}}{2} \cos(2\pi f_c t)$$

$$s_3(t) = -\sin(2\pi f_c t)$$

where $0 \leq t \leq T$, and $T$ is an integer multiple of $1/f_c$.

(a) What is the dimensionality of the signal space?

(b) Find the appropriate basis for the signal space. (Hint: You can find the basis without using the Gram-Schmidt Orthogonalization Procedure.)

(c) Draw the signal constellation for this problem.

(d) Derive and sketch the optimal decision regions $R_1$, $R_2$, and $R_3$. 
3. (20 points) The signal component of a coherent (synchronous) system is defined by
\[
S_m(t) = A_c k \sin(2\pi f_c t) + a_m A_c \sqrt{1 - k^2} \cos(2\pi f_c t)
\]
where \(0 \leq t \leq T\) and \(a_m \in \{\pm \sqrt{1/5}, \pm \sqrt{9/5}\}\). The first term represents a carrier component included for the purpose of synchronizing the receiver to the transmitter. Assuming all signals are equally likely

(a) Draw a signal-space diagram for the scheme described here; what observations can you make about this diagram?

(b) Calculate the average symbol energy.

(c) In the presence of additive white Gaussian noise of zero mean and power spectral density \(N_0/2\ \)W/Hz, what is the average probability of symbol error in terms of the average symbol energy you found in (b) above?
Consider the two signals \( f_1'(t) \) and \( f_2'(t) \) in the figure above. Calculate the exact probability of symbol error when a signal
\[
s(t) = \sum_k a_k f_1'(t - kT) + b_k f_2'(t - kT)
\]
where \( a_k, b_k \in \{\pm 1, \pm 3, \pm 5, \pm 7, \pm 9\} \) is transmitted on a zero-mean additive white Gaussian noise channel with power spectral density \( N_0/2 \) W/Hz. All symbols have independent and identical probability. Justify all of the steps in your calculation.
5. (15 points)

The functions $g_1(t)$, $g_2(t)$, and $g_3(t)$ depicted above are given as follows:

$$g_1(t) = \sqrt{\frac{E}{2T}} \text{rect} \left( \frac{t}{2T} \right)$$

$$g_2(t) = \sqrt{\frac{3E}{2T}} \text{tri} \left( \frac{t}{T} \right)$$

$$g_3(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos \left( \frac{\pi t}{2T} \right) & -T \leq t \leq T \\ 0 & \text{otherwise.} \end{cases}$$

(a) Calculate the power spectra $|G_1(f)|^2$, $|G_2(f)|^2$, and $|G_3(f)|^2$ (you can use the Fourier transforms given on the cover page and previously derived results).

(b) Taking power spectral considerations into account, which one of $g_1(t)$, $g_2(t)$, or $g_3(t)$ would you use as the pulse shaping function $g(t)$ in a digital modulation system? Why?
6. (20 points)

Consider the finite-state machine whose state transition diagram is shown above as a graph. The edges of the graph are labeled as input/output where the input is a bit (0 or 1) and the output is a voltage value transmitted to a zero-mean additive white Gaussian noise channel. The finite-state machine is originally at state 0.

(a) Draw the trellis associated with this finite-state machine.

(b) Assume that after the signal goes through the channel, the following sequence of scalars (voltages) is received at the receiver: 0, 2, -2, -2, 0, 1. Using the Viterbi algorithm, determine the input bit sequence optimum in the sense of maximum likelihood sequence detection.