Useful identities

\[ \cos(2\pi f_c t) \leftrightarrow \frac{1}{2} [\delta(f - f_c) + \delta(f + f_c)] \]

\[ \sin(2\pi f_c t) \leftrightarrow \frac{1}{j2} [\delta(f - f_c) - \delta(f + f_c)] \]

\[ \sin^2(A) = \frac{1}{2} - \frac{1}{2} \cos(2A) \]

\[ \cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B) \]
1. (16 points) Recall that the probability density function for two Gaussian variables $X$ and $Y$ with zero mean can be written as

$$
p_{X,Y}(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \left(\frac{x^2}{\sigma_X^2} - 2\rho \frac{xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right].$$

(a) Calculate the scalar $c$ in the expansion below

$$\left(\frac{x^2}{\sigma_X^2} - 2\rho \frac{xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right) = \frac{1}{\sigma_X^2} \left(x - \rho \frac{\sigma_X}{\sigma_Y} y\right)^2 + c \frac{y^2}{\sigma_Y^2}.$$

(b) Calculate $p_{X|Y}(x|y)$. Do you recognize this density?

(c) Based on this density, specify the conditional mean $m_{X|Y=y}$ defined as

$$m_{X|Y=y} = E[X|Y=y] = \int_{-\infty}^{\infty} x \ p_{X|Y}(x|y) \ dx.$$

(d) Specify the conditional variance

$$\sigma^2_{X|Y=y} = E[(X - m_{X|Y=y})^2|Y=y] = \int_{-\infty}^{\infty} (x - m_{X|Y=y})^2 \ p_{X|Y}(x|y) \ dx.$$
2. (16 points) Recall that we employ the notation that the Hilbert transform of a signal \( g(t) \) is given as \( \hat{g}(t) \). The Fourier transforms of \( g(t) \) and \( \hat{g}(t) \), \( G(f) \) and \( \hat{G}(f) \) respectively, are related as

\[
\hat{G}(f) = -j \operatorname{sgn}(f) G(f)
\]

where the signum function \( \operatorname{sgn}(f) \) is defined as

\[
\operatorname{sgn}(f) = \begin{cases} 
1 & f > 0, \\
0 & f = 0, \\
-1 & f < 0.
\end{cases}
\]

(a) Show that \( |\hat{G}(f)|^2 = |G(f)|^2 \).

(b) Calculate the Hilbert transform of \( \cos(2\pi f_c t) \).

(c) Show that if \( \hat{g}(t) \) is the Hilbert transform of \( g(t) \), then the Hilbert transform of \( \hat{g}(t) \) is \( -g(t) \).

(d) Based on your responses above, can you state what the Hilbert transform of \( \sin(2\pi f_c t) \) will be, without actually calculating it?
3. (20 points) Let a square $M$-QAM signal $s(t)$ for an infinite succession of symbols be given as

$$
s(t) = \sum_{n=-\infty}^{\infty} a_n g(t - nT) \cos(2\pi f_c t) - b_n g(t - nT) \sin(2\pi f_c t)$$

$$= \text{Re} \left[ \sum_{n=-\infty}^{\infty} A_n g(t - nT) e^{j2\pi f_c t} \right] \quad (1)
$$

where $A_n = a_n + j b_n$, $a_n, b_n \in \{\pm 1, \pm 3, \ldots, \pm (\sqrt{M} - 1)\}$, and $M = 2^{2k}$ for a small positive integer $k$.

(a) By inserting $1 = e^{j2\pi f_c nT} e^{-j2\pi f_c nT}$ into the summation in (1) above, and by setting $A_n = A_n e^{j2\pi f_c nT}$, $s(t)$ can be written as

$$s(t) = \text{Re} \left[ \sum_{n=-\infty}^{\infty} \tilde{A}_n g_1(t - nT) \right]. \quad (2)
$$

Specify $g_1(t)$ in terms of $g(t), \cos(2\pi f_c t)$, and $\sin(2\pi f_c t)$.

(b) We assume that the shaping function $g(t)$ is bandlimited to $W < f_c$. In this case, it is known that the Hilbert transform of $p(t) = g(t) \cos(2\pi f_c t)$ is $\hat{p}(t) = g(t) \sin(2\pi f_c t)$. Specify $g_1(t)$ in terms of $p(t)$ and $\hat{p}(t)$. As studied in class, $g_1(t)$ is in a special form. What is $g_1(t)$ called (in terms of $p(t)$)?

(c) Assume that $f_c$ is an integer multiple of $1/T$. With that assumption, and by inserting $g_1(t)$ in terms of $p(t)$ and $\hat{p}(t)$ into (2) and calculating the real part of the resulting complex expression, calculate the simplest form of $s(t)$ possible.

(d) This system can be implemented as in the figure below. Note that, although the QAM signal is at carrier frequency $f_c$, there is no explicit generation of the carriers $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$. Based on your response to part (c) above, specify $c_n, d_n, p_1(t)$, and $p_2(t)$.

(e) It can be shown that $p_1(t)$ and $p_2(t)$ in the figure above are orthogonal. Draw an optimum matched filter receiver for the transmitter in the figure above.
4. (16 points) Let

\[ s_m(t) = \begin{cases} 
A_m \cos(2\pi f_1 t) + B_m \sin(2\pi f_1 t) + C_m \cos(2\pi f_2 t) + D_m \sin(2\pi f_2 t) & 0 \leq t \leq T \\
0 & \text{otherwise.}
\end{cases} \]

where \( A_m \in \{ \pm 1, \pm 3, \pm 5, \pm 7 \} \), \( B_m \in \{ \pm 1, \pm 3, \pm 5 \} \), \( C_m \in \{ \pm 1, \pm 3 \} \) and \( D_m \in \{ \pm 1 \} \) with equal probability, \( f_1 = n_1/T \), and \( f_2 = n_2/T \) for two integers \( n_1 \neq n_2 \). The channel has zero mean additive white Gaussian noise with power spectral density \( N_0/2 \) W/Hz.

(a) What is the set of orthonormal basis functions to represent the signal set \( \{ s_m(t) \}_{m=1}^{M} \)? What are \( N \), the dimensionality of the signal space, and \( M \), the number of messages?

(b) Draw the optimum receiver.

(c) Calculate the exact probability of symbol error.
5. (16 points) The message signal for a quaternary digital modulation system is given as

\[ s_m(t) = \begin{cases} 
\cos \left( \frac{\pi t}{T} \right) \cos(2\pi f_c t) + I_m g_1(t) \sin(2\pi f_c t) & 0 \leq t \leq T \\
0 & \text{otherwise}
\end{cases} \]

where \( I_1 = 1, I_2 = 3, I_3 = 5, \) and \( I_4 = 7, \) and where the messages \( m = 1\text{-}4 \) are equally likely. The symbol shaping function is given as

\[ g_1(t) = \begin{cases} 
\sin^2 \left( \frac{\pi t}{T} \right) & 0 \leq t \leq T \\
0 & \text{otherwise.}
\end{cases} \]

(a) Draw a rough sketch of \( g_1(t) \), labeling all pertinent points.

(b) Find the relation between \( g_1(t) \) and the raised cosine function \( g(t) \) discussed in class given as

\[ g(t) = \begin{cases} 
\frac{A}{2} \left[ 1 + \cos \left( \frac{2\pi}{T} \left( t - \frac{T}{2} \right) \right) \right] & 0 \leq t \leq T \\
0 & \text{otherwise.}
\end{cases} \]

(c) Calculate the power spectral density of the infinite duration signal obtained by time shifting and superposing \( s_m(t) \), as discussed in class.

(d) Draw a sketch of the power spectral density you calculated in part (c) above.
6. (16 points) Alternate Mark Inversion (AMI) is a line encoding technique where a 0 at the input is represented by a 0 at the output. For the first 1 at the input, the transmitter generates a 1 at the output, for the next 1 at the input, the transmitter generates a -1 at the output. Thereafter, for each consecutive 1 at the input, the output alternates between 1 and -1, respectively.

(a) Draw the state transition diagram for the finite state machine associated with AMI.

(b) When AMI is used for transmission, assume that the following sequence is received: 0, 2, -2, -1, 1. Using the Viterbi algorithm and with the goal of minimizing \( \sum_{k=1}^{K} (r_k - z_k)^2 \), where \( r_k \) is the received symbol and \( z_k \) is the transmitted one, calculate the optimum decisions on the transmitted sequence and the input sequence in the maximum likelihood sense.

(c) Compare your result in part (b) above with symbol-by-symbol detection of the received sequence.