FINAL EXAMINATION

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Student ID #: ______________________

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Homework Average
Midterm
Final
Extra Credit
Total
Course Grade

Useful identities

\[
\cos(A) \cos(B) = \frac{1}{2} \cos(A - B) + \frac{1}{2} \cos(A + B)
\]

\[
\sin(A) \sin(B) = \frac{1}{2} \cos(A - B) - \frac{1}{2} \cos(A + B)
\]

\[
\sin(A) \cos(B) = \frac{1}{2} \sin(A - B) + \frac{1}{2} \sin(A + B)
\]

\[
\sin(A) + \sin(B) = 2 \sin \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right)
\]

\[
\int \frac{dx}{p + \frac{q}{2} e^{jx} + \frac{q}{2} e^{-jx}} = \int \frac{dx}{p + q \cos x} = \frac{2}{\sqrt{p^2 - q^2}} \tan^{-1} \left( \frac{\sqrt{p^2 - q^2} \tan \frac{x}{2}}{p + q} \right) \quad |p| > |q|.
\]

\[
\mathcal{L}[t^{n-1} u(t)] = \frac{(n-1)!}{s^n} \quad n = 1, 2, \ldots
\]

\[
\frac{1}{1 + az^{-1}} = 1 - az^{-1} + a^2 z^{-2} - a^3 z^{-3} + \ldots \quad |az^{-1}| < 1
\]
1. (10 points)

Consider the channel frequency response $X(f)$ in the figure above.

(a) Does this channel satisfy the Nyquist condition? Why?

(b) What is the impulse response $x(t)$ for this channel?

(c) Show by direct evaluation of the impulse response that $x(kT) = \delta_k$.

(d) Compare this $x(t)$ with that of raised cosine. Which one would you prefer? Why?
2. (15 points) Let $H(z) = 1 + az^{-1}$ where $a$ is a real number, $|a| < 1$.

(a) What is $F(z)$?
(b) Calculate $C(z)$ for LE-ZF.
(c) For this $C(z)$, what is $c_k$, for $k \geq 0$ and $k < 0$?
(d) What is $J_{\text{min}}(\text{LE-ZF})$?
(e) Calculate $C(z)$ for LE-MSE.
(f) What is $J_{\text{min}}(\text{LE-MSE})$?
(g) What is the relation between $J_{\text{min}}(\text{LE-ZF})$ and $J_{\text{min}}(\text{LE-MSE})$? Why?
3. (10 points) Let $H(z) = 1 + az^{-1}$ where $a$ is a real number, $|a| > 1$.

(a) What is $F(z)$?
(b) Calculate $C(z)$ for LE-ZF.
(c) What is $J_{\text{min}}(\text{LE-ZF})$?
(d) For this $C(z)$, what is $c_k$, for $k \geq 0$ and $k < 0$?
(e) Calculate $C(z)$ for LE-MSE.
(f) What is $J_{\text{min}}(\text{LE-MSE})$?
4. (15 points) Recall that in the true gradient algorithm
\[
c_k - c_{opt} = \prod_{i=1}^{k} (I - \Delta \Gamma)(c_0 - c_{opt})
= (I - \Delta \Gamma)^k(c_0 - c_{opt}) \quad k = 1, 2, 3, \ldots
\]
(a) By multiplying out, show that
\[
\Gamma = U \Lambda U^H = \sum_{j=-K}^{K} \lambda_j u_j u_j^H
\]
where \(\lambda_j\) and \(u_j\) are the \(j\)th eigenvalue and eigenvector of \(\Gamma\), columns of \(U\) are \(u_j\) and \(\Lambda\) is a diagonal matrix whose diagonal entries are the corresponding eigenvalues of \(\Gamma\).
(b) Show that the eigenvectors of \(I - \Delta \Gamma\) are the same as the eigenvectors of \(\Gamma\) and the eigenvalues of \(I - \Delta \Gamma\) are \((1 - \Delta \lambda_j), -K \leq j \leq K\), and establish the following decomposition
\[
(I - \Delta \Gamma)^k = \sum_{j=-K}^{K} (1 - \Delta \lambda_j)^k u_j u_j^H.
\]
(c) Show that
\[
c_k - c_{opt} = \sum_{j=-K}^{K} (1 - \Delta \lambda_j)^k u_j^H(c_0 - c_{opt}) u_j \quad k = 1, 2, \ldots
\]
(d) Note \(u_j^H(c_0 - c_{opt})\) is a scalar. What does this last equation imply as \(k \to \infty\) and under what condition?

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5. (10 points) The order of a PLL is defined as the number of poles \( \Phi(s)/\Psi(s) \) has, which is one more than the number of poles \( G(s) \) has.

(a) Consider a second order filter

\[
G(s) = 1 + \frac{a}{s}
\]

and assume that there is an input

\[
\phi(t) = \left( \frac{1}{2} R \dot{v}^2 + 2\pi \Delta f t + \theta_0 \right) a(t)
\]

which is an abrupt change in the phase, frequency, and a ramp input to the frequency at time \( t = 0 \). Calculate \( \lim_{t \to \infty} e(t) \) where \( e(t) = \phi(t) - \dot{\phi}(t) \).

(b) What can you conclude from this result?

(c) Now consider a third order filter

\[
G(s) = 1 + \frac{a}{s} + \frac{b}{s^2}
\]

and calculate \( \lim_{t \to \infty} e(t) \) for this filter.

(d) What can you conclude from this result?
6. (15 points)

The circuit above is a Costas loop for an 8-PSK signal which is given as

\[ s(t) = \sum_{k} \sin(2\pi f_c t + \theta_k + \phi) \]
\[ = \sum_{k} a_k \cos(2\pi f_c t + \phi) + b_k \sin(2\pi f_c t + \phi). \]

In this equation \( a_k = \sin(\theta_k) \), \( b_k = \cos(\theta_k) \), and \( \theta_k = (2l_k - 1)\pi/8 \) where \( l_k \in \{1, 2, \ldots, 8\} \) is the message at time \( k \).

(a) Calculate \( a_k^2 + b_k^2 \).

(b) Fill in the table below

<table>
<thead>
<tr>
<th>( F(a_k) )</th>
<th>( \sin \pi/8 )</th>
<th>( \sin 3\pi/8 )</th>
<th>( \sin 5\pi/8 )</th>
<th>( \sin 7\pi/8 )</th>
<th>( \sin 9\pi/8 )</th>
<th>( \sin 11\pi/8 )</th>
<th>( \sin 13\pi/8 )</th>
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Can you conclude from this table that \( F(a_k) = a_k \)? Why?

(c) Ignoring the effect of noise, during one symbol duration, the received signal is

\[ r(t) = a_k \cos(2\pi f_c t + \phi) + b_k \sin(2\pi f_c t + \phi). \]

Calculate \( c(t) \) and \( d(t) \) based on \( r(t) \) and local oscillator signals as shown.

(d) Calculate \( c(t) \) and \( d(t) \).

(e) Assuming \( \Delta\phi = \phi - \hat{\phi} \) is small, and considering your response to (a) above, calculate \( F(c(t)) \) and \( F(d(t)) \).

(f) Now calculate \( e(t) \).

(g) Can \( e(t) \) be used to drive the VCO? Why?
7. (15 points) We represent the impulse response of a discrete-time channel as $h_0, h_1, \ldots, h_{\nu-1}$. We have a block $s_0, s_1, \ldots, s_{N-1}$ of data to transmit ($N > \nu$). We add a cyclic prefix of size $\nu$ so that $s_{-i} = s_{N-i}$ for $i = 1, 2, \ldots, \nu$. We transmit the data sequentially from $s_{-\nu}$ to $s_{N-1}$. The received data are in the form of

$$r_k = \sum_{l=0}^{\nu-1} h_l s_{k-l} + v_k \quad k = 0, 1, \ldots, N - 1$$

where $v_k$ is a noise term. Let $r = (r_0, r_1, \ldots, r_{N-1})^T$, $s = (s_0, s_1, \ldots, s_{N-1})^T$, and $v = (v_0, v_1, \ldots, v_{N-1})^T$. Because $s_{-i} = s_{N-i}$ for $i = 1, 2, \ldots, \nu - 1$, the following holds

$$r = Hs + v$$

where $H$ is a matrix consisting of some zeros and values of $h_i$ placed in a structured manner.

(a) As an example, let $N = 4$ and $\nu = 2$, and write down the $H$ matrix in terms of $h_0$, $h_1$, and 0s.

(b) Matrices with this structure are known as circulant matrices. For a circulant matrix $H$, there is a spectral decomposition which states

$$H = W^H \Lambda W$$

where $W = \frac{1}{\sqrt{N}} [e^{-\frac{2\pi i j}{N}}]_{j=0}^{N-1}$, $\Lambda = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{N-1})$ and $(\lambda_0, \lambda_1, \ldots, \lambda_{N-1})^T$ is the discrete Fourier transform of $(h_0, h_1, \ldots, h_{\nu-1})^T$. Note $W^H W = I$ because $W$ is the discrete Fourier transform matrix. Let $s = W^H d$ for some $d = (d_0, d_1, \ldots, d_{N-1})^T$. Also, assume at the receiver we calculate

$$y = W r.$$

Find $y$ in terms of $\Lambda$, $d$, and $v$.

(c) What is the importance of your result in part (b) above?
8. (10 points) A CDMA system consists of 15 equal power users that transmit information at a rate of 10 kb/s, each using a DS-SS signal operating at a chip frequency of 1 MHz. The modulation is binary PSK.

(a) Determine $\frac{E_s}{I_0}$, where $I_0$ is the spectral density of the combined interference.
(b) What is the processing gain?
(c) How much should the processing gain be increased to allow for doubling the number of users without affecting the output SNR?