Useful identities

\(g^*(t) \leftrightarrow G^*(-f)\)

\[\nabla_c e^H Re = 2Re\quad \nabla_c Re[e^H p] = p\]

\[\sum_{m=0}^{N-1} e^{j\frac{2\pi}{N}(k-l)m} = \sum_{m=0}^{N-1} e^{-j\frac{2\pi}{N}(k-l)m} = N\delta_{k-l}\]

\[\text{DFT}\{x_k\} = X_n = \sum_{k=0}^{N-1} x_k e^{-j\frac{2\pi}{N}nk}\]

\[\mathcal{L}\{t^n u(t)\} = \frac{n!}{s^{n+1}} \quad n = 0, 1, 2, \ldots\]
1. (10 points)

(a) If $X(f)$ is the Fourier transform of $x(t)$, what is the Fourier transform of $\text{Re}[x(t)]$?
(b) Let $x_1(t) \leftrightarrow X_1(f)$ where $X_1(f)$ is as shown above. Does $x_1(t)$ satisfy the Nyquist condition?
(c) Does $\text{Re}[x_1(t)]$ satisfy the Nyquist condition?
(d) Let $x_2(t) \leftrightarrow X_2(f)$ where $X_2(f)$ is as shown above. Does $x_2(t)$ satisfy the Nyquist condition?
(e) Does $\text{Re}[x_2(t)]$ satisfy the Nyquist condition?
2. (12 points) We wish to estimate the random variable \( X \) from its noisy observation \( Y = X + N \). The random variables \( X \) and \( N \) are uncorrelated. The random variable \( X \) has mean \( m \) and variance \( E[(X - m)^2] = \sigma^2_X \), whereas \( N \) has zero mean and variance \( \sigma^2_N \).

(a) We would like to use an estimate \( \hat{X}_1 = a_1 Y \) so that mean squared error \( E[(\hat{X}_1 - X)^2] \) is minimized. Calculate the optimum value of \( a_1 \).

(b) Calculate \( E[\hat{X}_1] \). Is it equal to \( E[X] \)?

(c) We now wish to use an estimate \( \hat{X}_2 = a_2 Y + b \) such that \( E[(\hat{X}_2 - X)^2] \) is minimized as well as \( E[\hat{X}_2 - X] = 0 \). Calculate the values of \( a_2 \) and \( b \) that will provide this estimate.
3. (15 points) In this problem you will derive the finite-length MMSE equalizer using a different approach. Let the channel coefficients be given as \(\{f_k\}_{k=0}^L\) and the equalizer coefficients as \(\{c_k\}_{k=-K}^{K+L}\) where \(K \geq L\). We define their combination as \(x_k = f_k \ast c_k\) where \(K \geq L\). We express the output of the equalizer as

\[
\hat{I}_k = x_0 I_k + \sum_{n=-K}^{K+L} x_n I_{k-n} + \sum_{n=-K}^{K} c_n \eta_{k-n}
\]

where \(\eta_k\) is zero-mean white Gaussian noise with PSD \(N_0\). The data symbols \(I_k\) are independent and identically distributed with zero mean and unit variance. The data symbols \(I_k\) and noise \(\eta_k\) are uncorrelated. We define the vector \(\mathbf{c} = (c_{-K}, \ldots, c_0, \ldots, c_K)^T\). One value of \(x_k, x_0\), is particularly important. It is given as

\[
x_0 = \sum_{k=0}^{L} f_k c_{-k} = \mathbf{c}^T \mathbf{p}
\]

where \(\mathbf{p} = (0, \ldots, 0, f_L, \ldots, f_0, 0, 0, \ldots, 0)^T\).

By defining the matrix

\[
\mathbf{H} = \begin{bmatrix}
 f_0 & f_1 & \cdots & f_L \\
 f_0 & f_1 & \cdots & f_L \\
 \vdots & \vdots & \ddots & \vdots \\
 f_0 & f_1 & \cdots & f_L \\
\end{bmatrix}_{(2K+1) \times (2K+L+1)}
\]

and the vectors

\[
\mathbf{I}_k = (I_{k+K}, \ldots, I_k, \ldots, I_{k-K-L})^T, \quad \mathbf{n}_k = (\eta_{k+K}, \ldots, \eta_k, \ldots, \eta_{k-K})^T,
\]

we can represent the output of the equalizer as \(\hat{I}_k = \mathbf{c}^T \mathbf{H} \mathbf{I}_k + \mathbf{c}^T \mathbf{n}_k\), or the error as

\[
e_k = I_k - \mathbf{c}^T \mathbf{H} \mathbf{I}_k - \mathbf{c}^T \mathbf{n}_k.
\]

(a) Calculate the mean square error \(E[|e_k|^2]\) in a compact form. Note that \(I_k\) is statistically independent of all members of \(\mathbf{I}_k\), except one.

(b) By using vector calculus, calculate the optimum value of \(\mathbf{c}\) that minimizes \(E[|e_k|^2]\).

(c) Verify your response in part (b) above by using the principle of orthogonality.
4. (15 points)

In this problem, you will show that the LMS algorithm can be employed to analyze a signal in the frequency domain. Consider the LMS complex algorithm which employs the recursion

\[ \mathbf{c}_{k+1} = \mathbf{c}_k + \Delta \epsilon_k \mathbf{v}_k^* \]

where \( \epsilon_k = d_k - \mathbf{v}_k^T \mathbf{c}_k \) (this is slightly different than the development in class, but it is a valid form of the LMS algorithm; the one we developed employed \( \epsilon_k = d_k - \mathbf{v}_k^T \mathbf{c}_k^* \)). This implementation of the LMS algorithm uses the “desired signal” \( d_k \) and the “input” \( \mathbf{v}_k \) in an unconventional way, but if you follow the steps below, you should be able to reach at what the problem tries to show. We will employ the “input” vector \( \mathbf{v}_k = (1, e^{j \frac{2\pi}{N} k}, e^{j \frac{2\pi}{N} 2k}, \ldots, e^{j \frac{2\pi}{N} (N-1)k})^T \).

(a) First, show that \( \mathbf{v}_k^T \mathbf{v}_k^* = N \delta_{k-l} \).

(b) Show that the LMS recursion above with \( \epsilon_k = d_k - \mathbf{v}_k^T \mathbf{c}_k \) can be written as

\[ \mathbf{c}_{k+1} = (\mathbf{I} - \Delta \mathbf{v}_k^* \mathbf{v}_k^T) \mathbf{c}_k + \Delta d_k \mathbf{v}_k^* . \]

(c) Let \( \mathbf{c}_0 = 0 \), show that \( \mathbf{c}_1 = \Delta d_0 \mathbf{v}_0^* \).

(d) Calculate the next step with the same assumption for \( \mathbf{c}_0 \), and express \( \mathbf{c}_2 \) in terms of \( \Delta, d_1, \mathbf{v}_1^*, d_2, \mathbf{v}_2^* \). Use the result from part (a) above to simplify. You should get an expression with two terms.

(e) By induction, generalize your response in parts (c) and (d) above to \( \mathbf{c}_k \) for general \( k \).

(f) Specifically, what is \( \mathbf{c}_k \) for \( k = N \)? Do you recognize this expression as an “analysis” of the signal \( d_k, k = 1, \ldots, N \)?
5. (16 points) Let the transfer function of a channel be given as $H(z) = 1 + az^{-1}$ where $a$ is a real number such that $|a| < 1$. We would like to design a decision feedback equalizer for this channel, employing the zero-forcing criterion.

(a) What is the transfer function of the feedback filter $D(z)$?

(b) What is the transfer function of the feedforward filter $(1 + D(z))/X(z)$?

(c) What is the mean square error $J_{\min}(DFE - ZF)$?

(d) Compare your result in part (c) above with that of a linear equalizer designed using the zero-forcing criterion, $J_{\min}(LE - ZF) = N_0/(1 - a^2)$.

Now let the transfer function of the channel be given as $H(z) = 1 + az^{-1}$ where $a$ is a real number such that $|a| > 1$. We would like to design a decision feedback equalizer for this channel, employing the zero-forcing criterion.

(e) What is the transfer function of the feedback filter $D(z)$?

(f) What is the transfer function of the feedforward filter $(1 + D(z))/X(z)$?

(g) What is the mean square error $J_{\min}(DFE - ZF)$?

(h) Compare your result in part (g) above with that of a linear equalizer designed using the zero-forcing criterion, $J_{\min}(LE - ZF) = N_0/(a^2 - 1)$. 

6. (12 points) Consider the linearized model of the PLL as discussed in class with loop filter $G(s)$. Let $e(t) = \phi(t) - \dot{\phi}(t)$ be the error in phase. We define the steady-state of the error as $\lim_{t \to \infty} e(t)$.

(a) Consider the phase transfer function defined as $H(s) = \frac{\Phi(s)}{\Psi(s)}$. The linear loop filter $G(s)$ is in the form of the ratio of two polynomials $N(s)$ and $D(s)$, $G(s) = \frac{N(s)}{D(s)}$ where the degree of $N(s)$ is less than or equal to the degree of $D(s)$. The order of a PLL is defined as the largest power of $s$ in the denominator of $H(s)$. Show that this number is equal to one plus the highest power of $s$ in $D(s)$.

(b) Assume the input is $\phi(t) = \phi_0 u(t)$ (phase step). What is the condition on $G(0)$ that will ensure that the steady-state error is zero?

(c) Assume the input is $\phi(t) = 2\pi \Delta f t u(t)$ (frequency step). What is the condition on $G(0)$ that will ensure that the steady-state error is zero?

(d) Fill in the following table.

<table>
<thead>
<tr>
<th>$G(s)$</th>
<th>$G(0)$</th>
<th>PLL Order</th>
<th>Steady-State Error for Phase Step</th>
<th>Steady-State Error for Frequency Step</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_0s$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{K_1}{s + K_1}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K_1 + \frac{K_2}{s}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K_1 + s$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{K_2}{K_2 + s}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
7. (10 points) OFDM is employed in the IEEE 802.11a wireless local area network standard. In this standard, the maximum delay spread to be tolerated is set as 800 nanoseconds and the cyclic prefix overhead is set as 20%. The bandwidth is chosen as 20 MHz. It is desired to transmit 6 Mb/s using a code rate of 1/2 and BPSK. (A nanosecond is $10^{-9}$ seconds.)

(a) What is the duration of the useful part of the OFDM symbol?
(b) What is the total duration of the OFDM symbol?
(c) What is the FFT size?
(d) How many carriers are used for data transmission?
(e) Assuming that the same number of carriers are used in all modes, fill in the following table.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Modulation</th>
<th># of Bits per QAM Symbol</th>
<th>Code Rate</th>
<th># of Bits per OFDM Symbol</th>
<th>Data Rate [Mb/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>BPSK</td>
<td>1</td>
<td>1/2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>BPSK</td>
<td>1</td>
<td>3/4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>QPSK</td>
<td>2</td>
<td>1/2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>QPSK</td>
<td>2</td>
<td>3/4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>16-QAM</td>
<td>4</td>
<td>1/2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>16-QAM</td>
<td>4</td>
<td>3/4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>64-QAM</td>
<td>6</td>
<td>2/3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>64-QAM</td>
<td>6</td>
<td>3/4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
8. (10 points) The Hadamard matrix $H_1$ is defined as 1. The Hadamard matrix $H_{2^k+1}$ is defined as

$$H_{2^k+1} = \begin{bmatrix} H_{2^k} & H_{2^k} \\ H_{2^k} & -H_{2^k} \end{bmatrix}.$$ 

For example,

$$H_1 = 1 \quad H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$ 

Due to this construction, a Hadamard matrix has orthogonal rows and columns.

In a CDMA system employing BPSK, the chip rate is 1,228,800 chips per second. The channel encoder has a code rate of 1/2. The value of $d_{\text{min}}$ for this code is 8. The spreading is carried out by the rows of a Hadamard matrix with a processing gain of 64. A different row is used for a different user so that the users are orthogonal and the sum of their transmissions appear as Gaussian interference to each other.

(a) What is the user data rate prior to channel encoding in this system?

(b) Based on the description of the Hadamard matrix above, how many users can be supported by the Hadamard matrix that provides orthogonal codes with processing gains of 64?

(c) What is the value of $E_b/J_0$ achieved by a BPSK system with equal probability of bit error?