HOMEWORK 1  
(Due 1/18/2018)

1. The conventional discrete-time Fourier transform (DTFT) of a sequence \( x(n) \) is given as

\[
X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}
\]

and the inverse transform is

\[
x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\omega})e^{j\omega n} d\omega.
\]

In this course, we will be using a form of the discrete-time Fourier transform which explicitly specifies the sampling period \( T \) as

\[
X_2(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega nT}.
\]

Note \( T \) is a positive real number.

(a) What is the relation between \( X_1(e^{j\omega}) \) and \( X_2(e^{j\omega T}) \)?

(b) Similarly to \( X_1(e^{j\omega}) \), \( X_2(e^{j\omega T}) \) is periodic. What is the period of \( X_2(e^{j\omega T}) \)?

(c) How can you obtain \( x(n) \) from \( X_2(e^{j\omega T}) \)?

(d) Let \( h(n) \) be the impulse response of a discrete-time linear and time-invariant system. Let \( x(n) = e^{j\omega_0 nT} \) be the input of this system. What is the output \( y(n) \)? What can you conclude about \( e^{j\omega_0 nT} \) as an input function to discrete-time linear and time-invariant systems?

(e) Let \( x(n) \) be the input sequence to a discrete-time linear time-invariant system with impulse response \( h(n) \). Consider the DTFTs \( X(e^{j\omega_0 T}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega_0 nT} \) and \( H(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega nT} \). Assume the output sequence of \( h(n) \) is \( y(n) \) and let \( Y(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} y(n)e^{-j\omega nT} \). Show that

\[
Y(e^{j\omega T}) = H(e^{j\omega T})X(e^{j\omega T}).
\]

2. Let \( X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \) be the Z transform of a sequence \( x(n) \).

(a) Calculate the Z transform of \( x^*(-n) \).

(b) Assume \( x(n) \) has Hermitian symmetry, i.e., \( x(n) = x^*(-n) \). What is the implication of your result in part (a) above for \( X(z) \)?

3. Let \( h(n) \) be the impulse response of a discrete-time linear and time-invariant system. Let the input to this system be a discrete-time random process \( X(n) \) with the autocorrelation sequence \( R_X(k) = E[X(n)X^*(n-k)] \).
(a) Show that the autocorrelation sequence of the output random process $Y(n)$ can be obtained by

$$R_Y(k) = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h(l)h^*(m)R_X(k + m - l).$$

(b) Consider the $Z$ transforms $S_X(z) = \sum_{k=-\infty}^{\infty} R_X(k)z^{-k}$, $H(z) = \sum_{k=-\infty}^{\infty} h(k)z^{-k}$, and $S_Y(z) = \sum_{k=-\infty}^{\infty} R_Y(k)z^{-k}$. Show that

$$S_Y(z) = H(z)H^*(1/z^*)S_X(z).$$

(c) Due to the Wiener-Khinchin theorem, the power spectral density $S_Y(e^{j\omega T})$ is the DTFT of the autocorrelation function $R_Y(k)$. Show that the power in $Y(k)$ can be calculated as

$$E[|Y(n)|^2] = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} |H(e^{j\omega T})|^2 S_X(e^{j\omega T}) d\omega.$$

4. Let

$$X(z) = \frac{1}{1 + az}$$

where $a$ is a complex number.

(a) Expand $X(z)$ into a causal sequence

$$X(z) = \sum_{n=0}^{\infty} b_n z^{-n}.$$  

Specify $b_n$ and the region of convergence. What is the resulting condition on $a$ so that the system is stable?

(b) Expand $X(z)$ into an anticausal sequence

$$X(z) = \sum_{n=0}^{\infty} c_n z^n.$$  

Specify $c_n$ and the region of convergence. What is the resulting condition on $a$ so that the system is stable?