HOMEWORK 3
(Due 2/1/2018)

1. Let a channel impulse response be given as

\[ h(t) = \sqrt{2a}e^{-at}u(t) \]

where

\[ u(t) = \begin{cases} 
1 & t \geq 0, \\
0 & t < 0.
\end{cases} \]

(a) Calculate \( x_n \) for \( n \geq 0 \) as well as \( n < 0 \) where

\[ x_n = \int_{-\infty}^{\infty} h^*(-t)h(nT - t)dt = \int_{-\infty}^{\infty} h^*(t - nT)h(t)dt. \]

(b) Based on \( x_n \) found above, calculate

\[ X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n}. \]

2. For the channel given in question 1 above

(a) What is the expression for the z-transform of the zero-forcing linear equalizer, where we express the channel as \( X(z) \) and the equalizer as \( C'(z) \)?

(b) How many taps are there in \( C'(z) \)?

(c) What is \( F(z) \) in the decomposition \( X(z) = F(z)F^*(1/z^*) \)?

(d) What is the zero-forcing linear equalizer \( C(z) \) for this channel?

(e) How many taps are there in \( C(z) \)?

(f) What is the minimum MSE realizable by the zero-forcing linear equalizer \( J_{\text{min}}(\text{LE-ZF}) \)?

3. For the same channel

(a) What is the expression for the z-transform of the MMSE linear equalizer, where we express the channel as \( X(z) \) and the equalizer as \( C'(z) \)?

(b) How many taps are there in \( C'(z) \)?

(c) What is the MMSE linear equalizer \( C(z) \) for this channel?

(d) Calculate an expression for the minimum MSE realizable by the MMSE linear equalizer \( J_{\text{min}}(\text{LE-MSE}) \). Hint:

\[ \int \frac{dx}{p + q \cos ax} = \frac{2}{a\sqrt{p^2 - q^2}} \tan^{-1} \left[ \frac{\sqrt{(p - q)/(p + q)} \tan \frac{1}{2} ax}{a} \right] \quad p > q. \]
4. Let $R_R$ and $R_I$ be defined as the real and imaginary parts, respectively, of a complex-valued Hermitian symmetric matrix $R$. Similarly, let $c_R$ and $c_I$ ($p_R$ and $p_I$) be defined as the real and imaginary parts, respectively, of a complex-valued vector $c$ ($p$).

(a) Show that as a consequence of the Hermitian symmetry

$$R_R = R_R^T, \quad R_I = -R_I^T.$$ 

(b) Show that

$$\nabla_{c_R} c^H R c = 2R_R c_R - 2R_I c_I, \quad \nabla_{c_I} c^H R c = 2R_R c_I + 2R_I c_R,$$

$$\nabla_{c_R} \text{Re} \left[ c^H p \right] = p_R, \quad \nabla_{c_I} \text{Re} \left[ c^H p \right] = p_I.$$

(c) Show that if we define the gradient of a real-valued function with respect to a complex vector $c$ as

$$\nabla c = \nabla_{c_R} + j \nabla_{c_I},$$

then

$$\nabla c^H R c = 2R c, \quad \nabla c \text{Re} \left[ c^H p \right] = p.$$ 

Note that these vector differential calculus results are similar to those in scalar differential calculus.

5. Recall that the mean square error for a Wiener filter can be expressed as

$$J = \sigma_d^2 - w^H p - p^H w + w^H R w.$$ 

Use the result from the previous problem to calculate the optimum weight vector $w^o$ by direct vector differentiation of $J$.

6. Binary PAM is used to transmit information over an unequalized linear filter channel. When $a = 1$ is transmitted, the noise-free output of the demodulator is

$$f_m = \begin{cases} 
0.3 & m = 1, \\
0.9 & m = 0, \\
0.3 & m = -1, \\
0 & \text{otherwise}.
\end{cases}$$

(a) Design a three-tap zero-forcing linear equalizer so that the output is

$$q_m = \begin{cases} 
1 & m = 0, \\
0 & m = \pm 1.
\end{cases}$$

(b) Determine $q_m$ for $m = \pm 2, \pm 3$, by convolving the impulse response of the equalizer with the channel response.