MIDTERM EXAMINATION

Name: 
Student ID #: 

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Useful identities

\[
\int \frac{dx}{p + q \cos ax} = \frac{2}{a \sqrt{p^2 - q^2}} \tan^{-1} \sqrt{\frac{p - q}{p + q}} \tan \frac{1}{2} ax \quad p > q.
\]

Specifically,

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dx}{p + \frac{q}{2} e^{jx} + \frac{q}{2} e^{-jx}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dx}{p + q \cos x} = \frac{1}{\sqrt{p^2 - q^2}} \quad p > q.
\]
1. (10 points) A voiceband telephone channel has bandpass frequency response spanning the range 300-3500 Hz. We wish to design a QAM-based digital modulation system for data transmission through this channel.

(a) What operating frequency should one use? Why?

(b) We wish to transmit data at a bit rate of 19.2 kb/s using 256-QAM. What is the roll-off factor $\alpha$ that should be employed?

(c) What is the largest bit rate that can be achieved in this channel employing 256-QAM?
2. (10 points)

The figure above depicts the spectra of four different channels which are candidates for data transmission at a symbol rate of \( T \) symbols/sec. Only the positive part of the spectrum is shown, the negative part of the spectrum is symmetric. Considering both positive and negative parts, for each spectrum \( H_1(f) \) through \( H_4(f) \) draw the folded spectrum, showing contributions of each segment clearly. For \( H_1(f) \) through \( H_4(f) \) state whether the spectrum satisfies the Nyquist condition, explaining clearly why or why not.
3. (25 points) Let $W_n$ be a white discrete-time random process with zero mean and unit variance

$$E[W_n] = 0, \quad E[W_n W_{n-k}] = \delta_k \quad k = 0, \pm 1 \pm 2, \ldots$$

Let $X_n$ be generated from $W_n$ via an autoregressive process

$$X_n = a X_{n-1} + W_n \quad n = 0, \pm 1 \pm 2, \ldots$$

where $|a| < 1$. We observe $Y_n$, a noisy version of $X_n$,

$$Y_n = X_n + Z_n$$

where $Z_n$ is a white discrete-time random process with zero mean and unit variance

$$E[Z_n] = 0, \quad E[Z_n Z_{n-k}] = \delta_k \quad k = 0, \pm 1 \pm 2, \ldots$$

which is uncorrelated with $W_n$.

(a) Calculate $E[X_n X_{n-k}]$ for $k = 0, \pm 1, \pm 2, \ldots$

(b) Calculate $E[Y_n Y_{n-k}]$ for $k = 0, \pm 1, \pm 2, \ldots$

(c) Let $d_n = X_{n+1}$. Calculate $E[d_n Y_{n-k}]$ for $k = 0, 1, 2, \ldots$

(d) We will employ a Wiener filter with $M$ coefficients $w_M = [w_1, w_2, \ldots, w_M]^T$ to estimate $d_n = X_{n+1}$. Write the matrix equation to calculate $w_M$.

(e) Calculate $w_M$ for $M = 1, 2$. 
4. (20 points)

Let the combination of the transmit filter and the channel impulse response for a modem, $h(t)$, be as in the figure above. The symbol period is $T$.

(a) Calculate $x_n = \int_{-\infty}^{\infty} h^*(t) h(t + nT) dt$ for $n = 0, \pm 1, \pm 2, \ldots$.

(b) Calculate and plot $X(e^{j\omega T})$ for $\omega \leq \pi/T$.

(c) What is $\sum_{m=-\infty}^{\infty} |H(\omega + 2\pi m/T)|^2$?

(d) Calculate $F(z)$ and $F^*(1/z^*)$.

(e) Calculate $C(z)$ for the zero-forcing linear equalizer (LE-ZF).

(f) Calculate $J_{\text{min}}(LE - ZF)$. 
5. (20 points) In this problem we will develop a form of the LMS algorithm intended for fast convergence. We will do this by making the scalar coefficient in the algorithm time-varying. For simplicity, we will develop the algorithm for real-valued data \( \mathbf{v}_k \). With a change of notation from \( \Delta \) to \( \mu \), and incorporating time variance as an index \( k \), we write the LMS algorithm as

\[
\mathbf{c}_{k+1} = \mathbf{c}_k + \mu_k \epsilon_k \mathbf{v}_k
\]

where \( \mu_k \) will be allowed to change with time. Let’s write the LMS algorithm as

\[
\mathbf{c}_{k+1} = \mathbf{c}_k + \delta_{c_k}.
\]

(a) Calculate \( \epsilon_k^2 = (d_k - \mathbf{c}_k^T \mathbf{v}_k)^2 \).

(b) Assume we replace \( \mathbf{c}_k \) with \( \mathbf{c}_k + \delta_{c_k} \). Calculate

\[
\epsilon_k^2 = (d_k - (\mathbf{c}_k + \delta_{c_k})^T \mathbf{v}_k)^2.
\]

(c) Calculate \( \delta_{\epsilon_k^2} = \epsilon_k^2 - \tilde{\epsilon}_k^2 \).

(d) You can combine two terms in \( \delta_{\epsilon_k^2} \) above by recognizing that they have a common factor which multiplies \( \epsilon_k \). As a result, express \( \delta_{\epsilon_k^2} \) with an equation that has two terms.

(e) Now, make the change \( \delta_{c_k} = \mu_k \epsilon_k \mathbf{v}_k \) in the expression you calculated in (d) above.

You will get an expression in terms of \( \mu_k \) with two terms, one with \( \mu_k \) and another with \( \mu_k^2 \).

(f) Now, calculate the derivative of \( \delta_{\epsilon_k^2} \) with respect to \( \mu_k \) and set it equal to zero. This will give you the optimum value of \( \mu_k \) for fastest convergence.

(g) Express the LMS algorithm with \( \mu_k \) as you found above. This is called the \emph{normalized LMS algorithm}.
6. (15 points) Let $h(t)$ be equal to the convolution of the transmitter filter and the channel impulse response. Let $x_n$ be its time autocorrelation function $x_n = \int_{-\infty}^{\infty} h^*(t)h(t + nT)dt$ and $X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n}$, as in class. Now, let $X(z)$ be given as

$$X(z) = \frac{2z^2 - 5z + 2}{3z^2 - 10z + 3}.$$ 

In this problem, you will design a zero-forcing decision feedback equalizer for this channel.

(a) Calculate the feedback filter $D(z)$.

(b) Calculate the feedforward filter $C(z)$.

(c) The lowpass channel has additive white Gaussian noise with power spectral density $N_0$. Calculate the mean squared error $J_{\text{min}}$ associated with this filter.