Balanced Truncation with Spectral Shaping for RLC Interconnects

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Abstract- ***This paper presents a numerically stable and efficient algorithm for model reduction of large RLC networks using frequency-weighted balanced truncation technique. The salient features of this algorithm include guaranteed stability of the reduced transfer function as well as availability of provable frequency-weighted error bounds. Such frequency weighting is essential to provide better control over time-domain error of the reduced system. The first k largest singular values of the system are obtained using the Lanczos algorithm, and the Lyapunov equations are solved by an iterative Lyapunov equation solver. Experimental results demonstrate the higher accuracy of our technique compared to Krylov-subspace-based model reduction techniques and other truncated balanced realizations that do not use spectral shaping. Based on MATLAB simulations, the run-time of our method is only 5% more than that of PRIMA.***

I. INTRODUCTION

The problem of model reduction for VLSI interconnects has gained considerable attention in the EDA community in recent years. The reasons are that the interconnect parasitic effects have a major effect on the overall circuit delay and that detailed simulation of large interconnect structures is very computation-intensive. The model reduction techniques enable us to model the interconnect effects with a far less computational time than that required for simulation of the full model. Among various classes of model reduction techniques, the explicit moment matching algorithms (e.g. AWE [1], RICE [2]) and Krylov subspace based methods (Pact [3], PVL [4], PRIMA [5]) have received more attention because of their lower computational complexity. These methods, however, do not provide a provable error bound for the reduced system.

An alternative to these model reduction techniques that has received renewed interest and consideration is the balanced realization technique [6], [7], [8], [9]. The reduced system using the balanced realization is guaranteed to be stable and no further processing is required to make the system passive. Moreover, there is a provable error bound on the transfer function of the reduced-order model compared to that of the original system. However, compared to Krylov-subspace-based and pade-based model reduction techniques, these methods tend to have higher computational complexity [8]. Furthermore, the balancing transformation may be poorly conditioned when the system is nearly uncontrollable or unobservable.

This research was funded in part by SRC under contract no. 98-DJ-606.

In the balanced realization-based model reduction methods, each state is equally controllable and observable and the reduced order model of the original transfer function is derived by minimizing the Hankel-norm of the error between the transfer functions of the original and the approximated system. Paper [7] uses the truncated balanced realization and uses the Schur decomposition method to develop an efficient numerical method for solving the required equations. Papers [8], [9] propose efficient algorithms to solve the two Lyapunov equations to obtain the grammians. The algorithms are based on the Alternated Direction Implicit method (ADI) [10]. By using a modification to the ADI method, the authors are able to reduce the computational complexity.

A shortcoming of the proposed model reduction techniques is that they do not reshape the frequency spectrum to emphasize error minimization in some frequency range of interest. Furthermore, they also do not address the numerical difficulties when the system is nearly uncontrollable or unobservable. In analyzing the interconnect, it is often very useful to make the energy of the error to be very small in a certain frequency band. Enns [11] first proposed frequency-weighted model reduction based on the balanced truncation method. Although the method works well for many numerical examples, it suffers from a number of shortcomings that become important when this method is applied to interconnect analysis. Most importantly, the proposed method does not guarantee the stability in general and gives no *a priori* error bounds. Paper [12] gives a complete solution to the frequency-weighted Hankel norm approximation with unstable weighting functions. The method proposed in [13] may cause the L^{∞}-norm error to become very large when the weighting functions have minimum phase response. Furthermore, no *a priori* error bound exists for this technique.

To overcome the numerical difficulties of a nearly uncontrollable or unobservable system, a number of techniques have been proposed to compute the reduced-order models of the system transfer function without actual computation of a balanced realization. Among those techniques the Schur method [14] has gained a lot of attention. The Schur method is based on the Schur decomposition of the product of the controllability and observability grammians. The method involves only orthogonal transformation matrices and is therefore numerically robust [14]. In the model reduction technique based on balanced truncation since we are only interested in the first few largest eigenvalues of the system matrix, we use a Krylov-subspace-based method to compute a few of the largest eigenvalues and eigenvectors of the matrices.

In this paper the high-order interconnect circuit is approximated by a lower-order reduced system using a new numerically attractive frequency-weighted balanced truncation technique. The method is an extension of the techniques presented in [7] and [11]. It provides a definite *a priori* error bound and is guaranteed to be stable even when both input and output weightings are utilized.

Section II provides an overview of the balanced realization method. The new contribution in this section is the adoption of the Schur algorithm based on the Lanczos method to get around the numerical difficulties associated with the balancing transformation. Section III reviews the frequency-weighted model reduction technique. Section IV contains the major contribution of this paper including the new formulation of the frequency-weighted model reduction and the methods described in section II to obtain the reduced order system using balanced truncation without numerical instability. In section V this new model reduction technique is compared with the Vector ADI introduced in [7] and [8] and PRIMA [5]. Section VI concludes our paper.

II. Background

Consider an arbitrary network consisting of inductances, capacitances, and resistances. Modified Nodal Analysis (MNA) can be used to obtain the system of equations:

$$\mathcal{L}\dot{x} = -\mathcal{G}x + B_N u \tag{1}$$

$$y = Cx \tag{2}$$

where the state vector x represents the vector of N node voltages across the circuit capacitances and voltage sources and the vector of M currents flowing through the inductors and current sources. In addition:

$$\mathcal{L} = \begin{bmatrix} C_{cap} & 0 \\ 0 & L \end{bmatrix},\ \mathcal{G} = \begin{bmatrix} G & E \\ -E^T & 0 \end{bmatrix},\ x = \begin{bmatrix} v \\ i \end{bmatrix},\ E = [e_{ij}]$$

$$\text{where } e_{ij} = \begin{cases} 1 & \text{If branch } j \text{ is incident at node } i \text{ and oriented away from it.} \\ 0 & \text{If branch } j \text{ is not incident at node } i. \\ -1 & \text{If branch } j \text{ is incident at node } i \text{ and oriented towards it.} \end{cases}$$

The MNA equations can be rewritten in the standard state-space representation by introducing the following matrices:

$$A = -\mathcal{L}^{-1}\mathcal{G} \qquad , \qquad B = \mathcal{L}^{-1}B_N$$

Hence:

$$\dot{x} = Ax + Bu \tag{3}$$

$$y = Cx \tag{4}$$

The system of Eqs (3) and (4) characterizing the RLC interconnect is a special case of an asymptotically stable system for which the eigenvalues of A are in the open left half plane. For such a class of linear time-invariant (LTI) systems, the controllability and observability grammians are defined as follows:

$$P = \int_0^\infty e^{At}BB^Te^{A^Tt}dt \qquad ; \qquad Q = \int_0^\infty e^{At}C^TCe^{At}dt \tag{5}$$

The P and Q matrices satisfy the Lyapunov equations[6]:

$$AP + PA^T + BB^T = 0 \qquad ; \qquad A^TQ + QA + C^TC = 0 \tag{6}$$

The controllability and observability grammians provide useful insights about the system characteristics. A particularly interesting property is that the Hankel singular values of the system transfer function, $H(s)$, are the square-roots of the eigenvalues of PQ:

$$\sigma_i(H(s)) = \{\lambda_i(PQ)\}^{1/2} \tag{7}$$

where σ_i, $i = 1, 2, \ldots, n$ are the Hankel singular values of the system transfer function $H(s)$ and represent the energy exerted by the *i-th* state variable in the controllability and observability map of the balanced system. It can be proved that there exists a similarity transformation matrix T such that the controllability and observability grammians of the new system $(\hat{A}, \hat{B}, \hat{C})$ are equal and diagonal:

$$\hat{P} = \hat{Q} = \Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{8}$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$. The balanced transformation of an LTI system allows one to choose the state variable set that gives a significant amount of information in the external representation of the system. In fact, on the basis of the computed energy for each state variable x_k, we settle on a criterion for evaluating the possibility of eliminating x_k in the reduced model. If Σ is partitioned into two submatrices:

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \tag{9}$$

where $\Sigma_1 \in R^{k \times k}$, $\Sigma_2 \in R^{(n-k)\times(n-k)}$ and (A, B, C) are also partitioned conformally with Σ as:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad , \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad , \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

The reduced order model based on (A_{11}, B_1, C_1) is stable and the L^∞-error is bounded by [6]:

$$\left\| H(s) - H_r^k(s) \right\|_\infty \leq 2\left(\sum_{i=k+1}^{n} \sigma_i \right) \tag{10}$$

To obtain a state-space realization of the balanced system, we first solve the Lyapunov equations to obtain the grammians. Since the (A, B, C) matrices in Eqs (3) and (4) give a minimal realization of the RLC interconnect, the controllability and observability grammians are positive definite. Consequently, we can apply the Cholesky factorization on matrix Q:

$$Q = R^TR \tag{11}$$

Notice that RPR^T will be a positive-definite matrix, which can be diagonalized as:

$$RPR^T = U\Sigma^2U^T \qquad \text{with} \qquad U^TU = I$$

Now a balancing transformation, T, is given by:

$$T = \Sigma^{-1/2}U^TR \tag{12}$$

The new coordinate-transformed grammians are equal and diagonal [6]. The calculations required to construct the balancing transformation are complicated and sensitive to numerical errors. In particular the balancing transformation T may be poorly conditioned when matrix PQ has a high condition number.[1] Paper [14] proposes an algorithm where the balancing is avoided altogether and as a result numerical difficulties are never encountered. The algorithm uses the Schur decomposition to generate orthogonal bases for eigenspaces. Briefly, instead of finding the balancing transformation, the algorithm uses a Schur decomposition of the product PQ where P and Q

1. Recall that the condition number $cond(M) = \sigma_{max}(M)/\sigma_{min}(M)$ provides a measure of the distance of M to the set of singular matrices.

are the controllability and observability grammians of the original linear system. The resulting similarity transformations which convert the PQ matrices to the upper triangular forms are then partitioned into two submatrices. The submatrices whose columns form the respective right and left eigenvectors of the product PQ associated with the largest singular values of the system matrix A are chosen. The singular value decomposition of these selected submatrices are computed and directly used to compute the reduced order model of the original system. For the complete explanation of the algorithm, please refer to [14]. The algorithm is proven to yield a stable reduced order model without regard to nearness to unobservability or uncontrollability. By using similarity transformations, it is possible to reduce a given matrix to simpler forms (e.g. diagonal, upper/lower triangular, etc). This algorithm, however, involves an eigenvalue problem whose dimension is as large as the order of the original transfer function. Furthermore we still need to solve Lyapunov equations to obtain the controllability and observability grammians, a task which is computationally expensive.

We can avoid the problem of solving a large eigenvalue problem by using the Krylov subspace-based methods. In fact studying the algorithm proposed by [14] shows that it is only necessary to find the first k largest eigenvalues of the product and their corresponding left and right eigenvectors. Based on this observation, a modified version of the Safanov's algorithm was used in paper [7]. In paper [7], the Arnoldi algorithm is utilized to compute the largest eigenvalues and the corresponding left and right eigenvectors. Recall that PQ is a large symmetric matrix, which will also be a positive definite matrix if the system is both controllable and observable.

The problem is thus to efficiently obtain the largest and/or the smallest eigenvalues of a symmetric matrix. This problem is solved by using the Lanczos method. The Lanczos algorithm reduces the original large symmetric matrix, $M = PQ$, to a smaller tridiagonal matrix T_q where $T_q \in \Re^{q \times q}$. The algorithm involves successfully filling in the columns V_L and V_R such that $V_L{}^T V_R = D_q = diag(\delta_1, \delta_2, \ldots, \delta_q)$, where $V_L = [v_{L1}\ v_{L2} \ldots v_{Lq}]$ and $V_R = [v_{R1}\ v_{R2} \ldots v_{Rq}]$ and the vectors $\{v_{Ri}\}_{i=1}^{q}$ and $\{v_{Li}\}_{i=1}^{q}$ span the Krylov subspaces $K_q(v_1, PQ)$ and $K_q(w_1, PQ)$, respectively:

$$colsp(V_L) = K_q(v_{L1}, M) = span\{v_{L1}, Mv_{L1}, \ldots, M^{q-1}v_{L1}\}$$

$$colsp(V_L) = K_q(v_{L1}, M) = span\{v_{L1}, Mv_{L1}, \ldots, M^{q-1}v_{L1}\}$$

$$V_L^T(PQ)V_R = T_q = \begin{bmatrix} \alpha_1 & \beta_2 & 0 & \ldots & 0 \\ \gamma_2 & \alpha_2 & \beta_3 & \ddots & \vdots \\ 0 & \gamma_3 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \beta_q \\ 0 & \ldots & 0 & \gamma_q & \alpha_q \end{bmatrix}$$

$$V_L^T V_R = D_q = diag(\delta_1, \ldots, \delta_q)$$

The columns of the two projection matrices $V_L, V_R \in \Re^{n \times q}$, form bases for the respective right and left eigenspaces of the product PQ associated with their "big" eigenvalues $\sigma_1^2, \ldots, \sigma_q^2$. The matrices V_L and V_R are used as bases for the relevant eigenspaces of the matrix PQ in the derivation of the reduced order model.

The problem of efficiently solving the Lyapunov equations will be addressed in section V when we present our new frequency-weighted balanced truncation method.

III. Review of the Frequency-Weighted Balanced Realization

So far we have seen that the internally-balanced realization is an attractive model reduction technique. An extension to the balancing technique to include weightings is now explained.

Consider the state-space representation of the RLC interconnect given in Eqs (3) and (4). The frequency-weighted balanced realization problem is to calculate $H_r^k(s)$ of degree k ($k < n$) so as to make

$$\left\| W_o(s)\ (H(s) - H_r^k(s)) W_i(s) \right\|_\infty \tag{13}$$

as small as possible. To obtain such a reduced system, we first calculate the grammians of the augmented system and then repeat the same steps that are used for a unity-weighted system. We can write the stable state-space representations of the input and output weightings as:

$$W_i(s) = C_i(sI - A_i)^{-1}B_i + D_i \tag{14}$$

$$W_o(s) = C_o(sI - A_o)^{-1}B_o + D_o \tag{15}$$

If only the input is weighted, the minimal realization of the new system, $H(s)W_i(s)$, is easily realized by the controller-form realization method [13]:

$$\bar{A}_i = \begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix}, \quad \bar{B}_i = \begin{bmatrix} BD_i \\ B_i \end{bmatrix}, \quad \bar{C}_i = \begin{bmatrix} C & 0 \end{bmatrix} \tag{16}$$

Similarly the minimal realization of the augmented system $W_o(s)H(s)$ is obtained by the observer-form realization method [13]:

$$\bar{A}_o = \begin{bmatrix} A & 0 \\ B_iC & A_o \end{bmatrix}, \quad \bar{B}_o = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{C}_o = \begin{bmatrix} D_oC & C_o \end{bmatrix} \tag{17}$$

The grammians corresponding to these equations are given by the upper left corner $n \times n$ submatrices of $\bar{P}$ and $\bar{Q}$:

$$\bar{P} = \begin{bmatrix} P & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} Q & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \tag{18}$$

where $\bar{P}$ and $\bar{Q}$ satisfy the following Lyapunov equations:

$$\bar{A}_i\bar{P} + \bar{P}\bar{A}_i{}^T + \bar{B}_i\bar{B}_i{}^T = 0 \tag{19}$$

$$\bar{A}_o{}^T\bar{Q} + \bar{Q}\bar{A}_o + \bar{C}_o{}^T\bar{C}_o = 0 \tag{20}$$

Expanding the $n \times n$ upper left corner block of the Lyapunov equations yields [11]:

$$AP + PA^T + BC_iP_{12} + P_{12}^TC_i^TB^T + BD_iD_i^TB^T = 0 \tag{21}$$

$$A^TQ + QA + Q_{12}B_oC + C^TB_o^TP_{12}^T + C^TD_o^TD_oC = 0 \tag{22}$$

Solutions to Eqs (21) and (22) give the weighted controllability and observability grammians of the system. The balancing transformation T that accomplishes the objective is an eigenvector matrix obtained from the eigenvalue decomposition. After coordinate transformation of the original system, we obtain:

$$TPT^T = (T^{-1})^T Q T^{-1} = diag(\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_n) \quad (23)$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$.

The reduced order model is then calculated by truncating the states that are weakly controllable/observable with the weighted input/output [11]. In mathematical terms, it again involves partitioning the frequency-weighted balanced realization [11]. It has been be proved that the error bound of the above model order reduction method is [14]:

$$\left\| W_o(s)(H(s) - H_r^k(s)) W_i(s) \right\|_\infty$$

$$\leq 2 \sum_{k=r+1}^{n} \sqrt{\sigma_k^2 + (\alpha_k + \beta_k)\sigma_k^{3/2} + \alpha_k \beta_k \sigma_k} \quad (24)$$

In the above formula σ_k denotes the Hankel singular values and α_k, β_k denote infinity norms of the transfer functions, which depend on *H(s)*, *$W_o(s)$*, *$W_i(s)$* and system matrices A_i, A_o, *A*. It is clear from the above formula that the error bounds are calculated iteratively, each iteration involving evaluation of a number of infinity norms. Therefore, these bounds have limited practical applications.

IV. A NEW NUMERICALLY STABLE FREQUENCY-WEIGHTED BALANCED REALIZATION WITH AN ERROR BOUND

In this section we propose a frequency-weighted balanced truncation method that yields stable models for both single-sided and double-sided weightings. The error bound formula is a simple equation without a need to be updated iteratively.

From equations (21) and (22) we define the new variables as follows:

$$X = BC_i P_{12} + P_{12}^T C_i^T B^T + BD_i D_i^T B^T \quad (25)$$

$$Y = Q_{12} B_o C + C^T B_o^T P_{12}^T + C^T D_o^T D_o C \quad (26)$$

It can be easily seen that *X* and *Y* are symmetric matrices. As a consequence, there are orthogonal matrices *U* and *V* and diagonal matrices *S* and *Z* such that:

$$X = USU^T \quad (27)$$

$$Y = VZV^T \quad (28)$$

where $S=diag(s_1, s_2, \ldots, s_n)$, and $Z=diag(z_1, z_2, \ldots, z_n)$, and denoting rank(*X*) = *i* and rank(*Y*) = *j* we can write:

$$\bar{B} = U diag(|s_1|^{1/2}, \ldots, |s_i|^{1/2}, 0, \ldots, 0) \quad (29)$$

$$\bar{C} = diag(|z_1|^{1/2}, \ldots, |z_j|^{1/2}, 0, \ldots, 0) V^T \quad (30)$$

Let $\hat{P}$ and $\hat{Q}$ denote the solutions of the following Lyapunov equations:

$$A\hat{P} + \hat{P}A^T + \bar{B}\bar{B}^T = 0 \quad (31)$$

$$A^T\hat{Q} + \hat{Q}A + \bar{C}^T\bar{C} = 0 \quad (32)$$

We find the transformation matrix *T* that simultaneously diagonalizes $\hat{P}$ and $\hat{Q}$ as follows:

$$T\hat{P}T^T = (T^{-1})^T \hat{Q} T^{-1} = diag(\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_n) \quad (33)$$

The new realization $(A, \bar{B}, \bar{C})$ is minimal because $\hat{P}$ and $\hat{Q}$ are positive definite [13]. Since *A* is stable, an immediate conclusion is that the pair $(A, \bar{B})$ is controllable and $(A, \bar{C})$ is observable. As a result, the realization $(A, \bar{B}, \bar{C})$ is minimal. The calculations required to construct the balancing transformation are complicated and sensitive to numerical errors. To avoid any numerical difficulties, we use the definitions and techniques described in section II and obtain the reduced order model of the RLC interconnect without balancing the system.

We still need to solve the Lyapunov equations to obtain the grammians of the system. To efficiently solve the Lyapunov equations, we make use of the ADI procedure which is an iterative method for solving the Lyapunov equation,

$$AP + PA^T + X = 0 \quad .$$

The system is first reduced to tridiagonal form with a Gaussian similarity transformation, T_{trid}, as follows:

$$S = T_{trid} A T_{trid}^{-1}$$

$$Z = T_{trid} P T_{trid}^T$$

$$X_s = T_{trid} X T_{trid}^T$$

Reducing a matrix to its tridiagonal form requires $O(n^3)$ computations where *n* is the order of the system matrix, *A*. The resulting system is solved with the ADI iteration [10]:

$$Z_0 = 0$$

$$(S + p_j I) Z_{j-\frac{1}{2}} = X_s - [(S - p_j I) Z_{j-1}]^T,$$

$$(S + p_j I) Z_j = X_s - \left[(S - p_j I) Z_{j-\frac{1}{2}}\right]^T \text{ for } j = 1, 2, \ldots, J$$

Iterative solution of the reduced Lyapunov equation is accomplished in $O(12Jn^2)$ flops[10].

The steps to calculate the reduced order model of the matrix transfer function is as follows:

Given *H(s)*, *$W_i(s)$*, *$W_o(s)$* :

1. Using the ADI iteration, compute the controllability and observability grammians *P* and *Q* by solving the Lyapunov equations (21) and (22).
2. Use equations (25) and (26) or to compute *X* and *Y*.
3. Decompose *X* and *Y* using eigenvalue decomposition into USU^T and VZV^T.
4. Use equations (29) and (30) to compute $\bar{B}$ and $\bar{C}$.
5. Using the ADI iteration, solve the mapped Lyapunov equations (31) and (32) to compute $\hat{P}$ and $\hat{Q}$.
6. Using the Lanczos algorithm, obtain the reduced order left and right eigen matrices, $\hat{V}_L$ and $\hat{V}_R$, associated with the *q* largest singular values of the matrix $\hat{P}\hat{Q}$.
7. Let $\hat{E} = \hat{V}_L^T \hat{V}_R$ and compute the singular-value decomposition of matrix $\hat{E}$. $\hat{E} = \hat{U}_E \hat{\Sigma}_E \hat{V}_E^T$
8. Let $\hat{S}_L = \hat{V}_L \hat{U}_E \hat{\Sigma}_E^{-1/2} \in \Re^{n \times q}$

   $\hat{S}_R = \hat{V}_R \hat{V}_E \hat{\Sigma}_E^{-1/2} \in \Re^{n \times q}$
9. Find a transformation matrix *T* that simultaneously diagonalizes $\hat{P}$ and $\hat{Q}$.

10. Compute the reduced order space realization using the matrices $\hat{S}_L$ and $\hat{S}_R$ as follows:

$$\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{c|c} \hat{S}_L A \hat{S}_R & \hat{S}_L^T B \\ \hline C S_R & D \end{array}\right]$$

Theorem 2. The L^∞ error of the frequency-weighted model reduction technique is:

$$\|W_o(s)(H(s) - H_r^k(s))W_i(s)\|_\infty \le k\left(\sum_{i=k+1}^{n} \sigma_i\right) \quad (34)$$

where $k = \|W_o(s)L\|_\infty \|KW_i(s)\|_\infty$ and

$$K = diag(|s_1|^{-1/2}, \ldots, |s_n|^{-1/2}, 0, \ldots, 0)U^T B$$

$$L = CVdiag(|z_1|^{-1/2}, \ldots, |z_n|^{-1/2}, 0, \ldots, 0)$$

Proof is not given due to the lack of space.

In the case of single-sided weightings, the same L^∞-norm error bound is obtained. Corollary 1. gives the error bound for single-sided weightings.

Corollary 1. If the frequency weighting is applied on the input vector only, the error bound is given by:

$$\|(H(s) - H_r^k(s))W_i(s)\|_\infty \le k\left(\sum_{i=k+1}^{n} \sigma_i\right) \quad (35)$$

where $k = 2\|KW_i(s)\|_\infty$. Similarly if the frequency weighting is applied on the output vector only, the error bound is:

$$\|W_o(s)\,(H(s) - H_r^k(s))\|_\infty \le k\left(\sum_{i=k+1}^{n} \sigma_i\right) \quad (36)$$

where $k = 2\|W_o(s)\,L\|_\infty$.

The input and output weightings are determined based on the range of frequencies where we like to have the maximum accuracy. The weighting functions should emphasize the frequency ranges where more accuracy is required. Similarly they must de-emphasize the range of frequencies where the noise arising from the order reduction has small energy or it is out of the desired frequency bound.

For the RLC interconnect the goal is to generate a reduced-order model that has a small magnitude frequency error with respect to that of the original system over a wide range of frequencies starting from the DC frequency.

Recall that in the balanced truncation approach the k first singular values are retained while the remaining singular values are truncated. The truncated singular values represent the higher frequency fluctuations of the impulse response of the system. To have the magnitude and the phase spectrums matched with those of the original system, the weighting functions are normally chosen to have the high-pass frequency behavior, albeit with a high bandwidth.

V. EXPERIMENTAL RESULTS

In this section the results of applying the frequency-weighted balanced truncation model reduction technique are compared with those obtained by utilizing our implementations of PRIMA and the truncated balanced realization using Vector ADI (VADI) [8]. First consider a single lossy transmission line. The values of per-unit capacitances, inductances and resistances are normalized. The transmission line is modeled by 50 RLC lumped sections as depicted in Fig. 1. In Fig. 2., the Bode diagram of the magnitude and phase of the reduced transfer function obtained by our technique is compared with those of the VADI algorithm and PRIMA. The order of the system is reduced to 3. Clearly, our proposed technique yields more accurate results than the other techniques. The input weighting function is a low-pass first-order function:

$$W_i(s) = \frac{1}{s + 0.4}$$

In the second example we consider model reduction of two capacitively coupled transmission lines. The order of the reduced model is 3. The circuit is depicted in Fig. 3. The Bode diagram of the original system along with that of the reduced system are shown in Fig. 4. The output weighting function is:

$$W_o(s) = \frac{10}{s + 0.2}$$

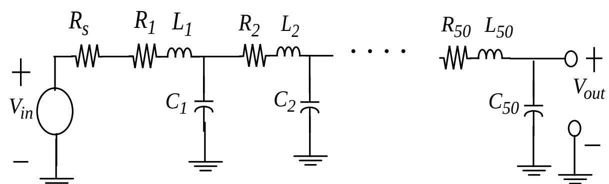

Fig. 1. A two-port lumped RLC network consisting of 50 sections.

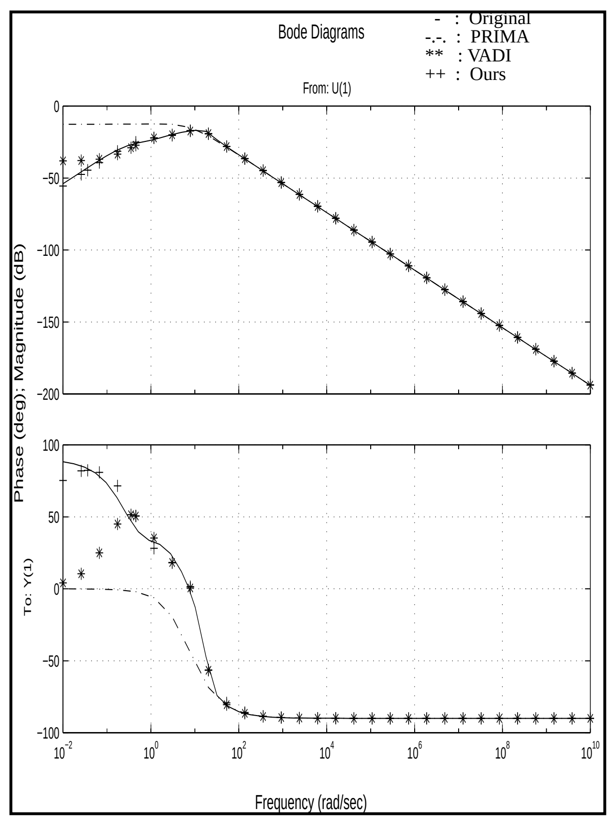

Fig. 2. The Bode diagram of a single interconnect having 50 lumped RLC sections and the corresponding reduced-order systems.

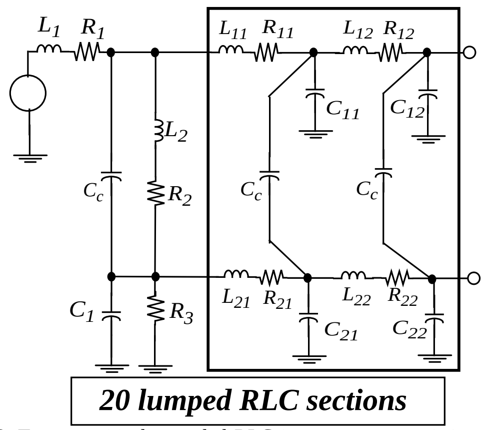

Fig. 3. Two capacitively coupled RLC interconnects consisting of 20 RLC lumped sections.

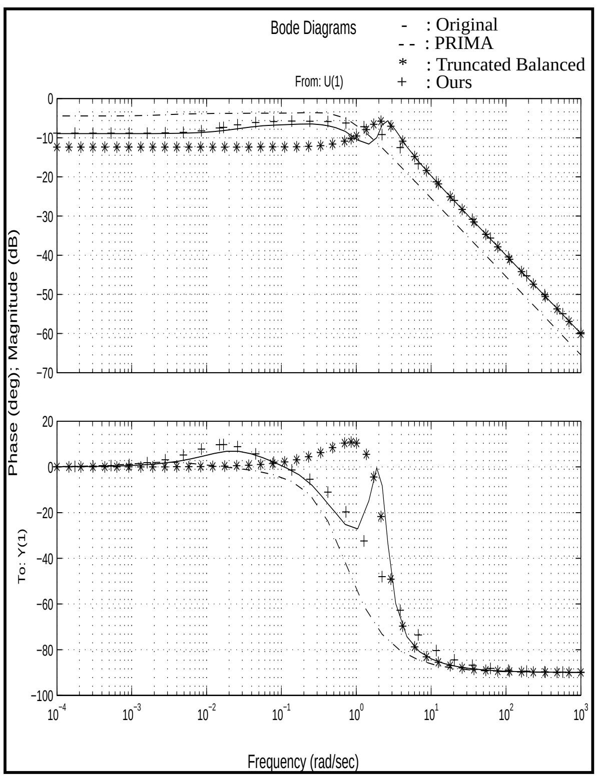

Fig. 4. The Bode diagram of the system in Fig. 2.

By comparing the number of flops resulting from MATLAB simulation, the CPU-time of our algorithm is only 5% more than that of PRIMA.

VI. Conclusion

In this paper a frequency-weighted balanced truncation technique for model-order reduction of multiport RLC interconnect was proposed. The method yields stable reduced order models even when both input and output weighting functions are applied. The reduced order model is computed directly without the need to calculate a balanced realization of the original system. The Lyapunov equations are efficiently solved by Krylov-subspace based methods and an iterative Lyapunov equation solver is presented. The proposed algorithm also provides *a priori* error bound. Experimental results and comparison with truncated balanced realization and PRIMA shows the effectiveness of our approach.

VII. References

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