A Performance Analysis of Subspace-Based Methods in the Presence of Model Errors, Part I: The MUSIC Algorithm

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Abstract—Application of subspace-based algorithms to narrowband direction-of-arrival (DOA) estimation requires that certain modeling assumptions be made. Most importantly, both the array response in all directions of interest and the spatial covariance of the noise must be known. In practice, however, neither of these quantities is known precisely. Depending on the degree to which they deviate from their nominal values, serious performance degradation may result. In this two-part paper, the performance of subspace-based algorithms is examined for situations in which the noise covariance and array response are perturbed from their assumed values. Part I focuses on the popular MUSIC algorithm. Theoretical expressions for the error in the MUSIC DOA estimates are derived and compared with simulations performed for several representative cases, and with the appropriate Cramér–Rao bound. In addition, an optimally weighted version of MUSIC is proposed for a particular class of array errors. In the companion paper, a similar analysis is performed for various multidimensional algorithms.

I. INTRODUCTION

WITHIN the class of so-called signal-subspace algorithms for direction-of-arrival (DOA) estimation, MUSIC [1], [2] has been the most widely studied. In a detailed performance evaluation based on hundreds of simulations, M.I.T.’s Lincoln Laboratory concluded that among high-resolution algorithms then available, MUSIC was the most promising [3]. The popularity of the MUSIC algorithm is in large part due to its generality; for example, it is applicable to arrays of arbitrary but known configuration and response, and can be used to estimate multiple parameters per source (e.g., azimuth, elevation, range, polarization, etc.). The price paid for this generality is that the array response must be known for all possible combinations of source parameters; i.e., the response must either be measured (calibrated) and stored, or one must be able to characterize it analytically (e.g.,

as in the case of root-MUSIC [1], [4]). In addition, MUSIC requires a priori knowledge of the second-order spatial statistics of the background noise and interference field.

The assumptions of a known array response and noise covariance are never satisfied in practice. Due to changes in weather, the surrounding environment, and antenna location, the response of the array may be significantly different than when it was last calibrated. Furthermore, the calibration measurements themselves are subject to gain and phase errors. For the case of analytically calibrated arrays of nominally identical, identically oriented elements, errors result since the elements are not really identical and their locations are not precisely known. Depending on the degree to which the actual antenna response differs from its nominal value, algorithm performance may be significantly degraded.

Since the surrounding environment and orientation of the array may be time varying, the requirement of known noise statistics is also difficult to satisfy in practice. In addition, one is often unable to account for the effect of unmodeled “noise” phenomena such as distributed sources, reverberation, noise due to the antenna platform, and undesirable channel crosstalk. Measurement of the noise statistics is complicated by the fact that often there are signals of interest observed along with the noise and interference. Consequently, when signal subspace methods are actually applied, it is often assumed that the noise field is isotropic, that it is independent from channel to channel, and that its power in each channel is equal. When the signal-to-noise ratio (SNR) is high, deviations of the noise from these assumptions are not critical since they contribute little to the array covariance. However, at low SNR, the degradation may be severe.

Most of the analysis performed to date for MUSIC and root-MUSIC [5]–[9] has been concerned with the finite sample effects induced by additive noise in the array measurements. While techniques have been proposed to mitigate the effects of the antenna and noise model errors described above [10]–[15], relatively little work has focused on obtaining analytical expressions for the resulting error in the DOA estimates. Several authors have, however, recently considered the performance of MUSIC for some special error models [16]–[20].

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In this paper, we take a more general approach and evaluate the performance of the MUSIC algorithm for a wide class of perturbations. The analysis is applicable to a wide variety of model errors, such as an imprecisely known noise covariance, angle-dependent and angle-independent gain and phase perturbations, sensor position errors, mutual coupling effects, and channel perturbations (i.e., errors that affect both the signal and noise components in a given channel). Our approach is statistical in the sense that the model errors are assumed to be expressed in probabilistic terms. Instead of computing gradients of the MUSIC cost function with respect to the model error parameters, their first-order effects on the noise subspace eigenvectors are determined directly. Once this connection is made, arguments similar to those in [8] are used to link the statistics of the noise eigenvectors with those of the DOA estimates. The resulting error expressions for the DOA estimates also turn out to be valid for root-MUSIC as well.

The principal advantage of this approach is that it yields simple expressions for the root-mean-square (rms) DOA estimation error, facilitating comparison of MUSIC’s performance with that of other algorithms. This advantage will be manifest in two ways. First, it will be shown that for certain perturbation models, the rms error of the DOA estimates may be minimized through use of a weighted version of MUSIC. Second, and somewhat surprisingly, it will be shown in the companion paper [21] that for a particular class of array perturbations, the weighted MUSIC algorithm achieves a lower rms error than all multidimensional subspace-fitting algorithms (e.g., deterministic maximum-likelihood, multidimensional MUSIC, weighted subspace-fitting, etc.). This latter result extends work previously presented in [22], [23].

The organization of this paper is as follows: In the next section the narrow-band DOA estimation problem and data model are briefly described, and the MUSIC and root-MUSIC algorithms are introduced. In Section III, a first-order analysis of MUSIC is conducted for a very general class of model errors. The effect of these errors on the MUSIC cost function is quantified, and general expressions are developed for the rms error of the DOA estimates. Several specific perturbation models are examined in detail in Section IV, and their corresponding physical interpretations are given. The error expressions for these models are also compared with that obtained in [8] for the finite sample approximation, and their relative contribution to the error in the DOA estimates is studied. Section V extends the analysis to a weighted version of MUSIC. It is shown that for a particular array perturbation model, an optimal weighting is possible that minimizes the rms estimation error. Finally, in Section VI, simulation results are presented for several representative cases to validate the theoretical expressions and illustrate how an analysis of this type might be used in the antenna system design process.

A. Nomenclature

The notational conventions listed below will be followed throughout the paper.

\[ \mathbb{C}^m \quad \text{m-dimensional complex vector space.} \]
\[ (\cdot)^{\dagger} \quad \text{Transpose.} \]
\[ (\cdot)^\ast \quad \text{Hermitian transpose.} \]
\[ \text{Re} \{ \cdot \} \quad \text{Real part.} \]
\[ \text{Im} \{ \cdot \} \quad \text{Imaginary part.} \]
\[ (\cdot)^{\ast} \quad \text{Conjugate.} \]
\[ X \odot Y \quad \text{Schur (Hadamard) product: } (X \odot Y)_{ik} = X_{ik} Y_{ik}. \]
\[ Y^\dagger \quad \text{Pseudoinverse of a full-rank matrix: } Y^\dagger = (Y^*Y)^{-1}Y^*. \]
\[ \hat{Y} \quad \text{An estimate of the quantity } Y. \]
\[ \tilde{Y} \quad \text{The error incurred by the estimate } \hat{Y}: \tilde{Y} = \hat{Y} - Y. \]
\[ \text{diag} \{ y \} \quad \text{Diagonal matrix with the elements of vector } y \text{ on its diagonal.} \]
\[ \text{diag} \{ Y \} \quad \text{Vector formed from the diagonal elements of square matrix } Y. \]
\[ || \cdot || \quad \text{Matrix norm.} \]
\[ I \quad \text{Identity matrix.} \]
\[ \triangleq \quad \text{Equal to first order in the perturbation.} \]
\[ \mathcal{O}(\alpha) \quad \text{Terms in an equation of order } \alpha. \]
\[ \mathcal{E} \{ \cdot \} \quad \text{Expected value.} \]
\[ \text{span} \{ \cdot \} \quad \text{Range (column) space.} \]
\[ \text{det} \{ \cdot \} \quad \text{Determinant.} \]
\[ \text{tr} \{ \cdot \} \quad \text{Trace.} \]

II. THE DATA MODEL AND MUSIC

We begin by briefly describing the unperturbed data model assumed for the narrow-band DOA estimation problem. Though for simplicity our discussion is confined to the single-parameter-per-source case (e.g., azimuth angle only), the analysis presented in the next section is easily extended to the multiple parameter case (e.g., azimuth and elevation angles, etc.).

Assume an \( m \)-element array of sensors, \( d \) narrow-band far-field signal sources, and define \( a(\theta) \in \mathbb{C}^m \) to be the complex array response for a source with direction-of-arrival \( \theta \). The array manifold is defined to be the set \( \mathcal{A} = \{ a(\theta): \theta \in \Theta \} \) for some region \( \Theta \) in DOA space. The set \( \mathcal{A} \) is assumed to be known, either analytically or via some calibration procedure. The array manifold is also assumed to be unambiguous; that is, any collection of \( d < m \) vectors from \( \mathcal{A} \) forms a linearly independent set.

The outputs of the \( m \) array elements at time \( t \) are stacked in a vector \( \mathbf{x}(t) \in \mathbb{C}^m \). Under the assumptions that the signal waveforms are narrow band and that the array elements are linear devices, the array output \( \mathbf{x}(t) \) may be written as

\[ \mathbf{x}(t) = \mathbf{A} \mathbf{s}(t) + \mathbf{n}(t) \]

where \( \mathbf{s}(t) \in \mathbb{C}^d \) is the amplitude and phase of the signals...
at time \(t\), \(n(t)\) is additive noise, and where

\[ A = [a(\theta_1), \ldots, a(\theta_d)]. \]

If no noise were present, the observations \(x(t)\) would be confined entirely to the \(d\)-dimensional subspace of \(\mathbb{C}^m\) defined by the span of \(A\). Determining the DOA’s for the no-noise case is simply a matter of finding the \(d\) unique elements of \(\mathcal{G}\) that intersect this subspace. A different approach is of course necessary in the presence of noise, since the observations are “full-rank.” The approach of MUSIC and other subspace-based methods is to first estimate the dominant subspace of the observations, and then find the elements of \(\mathcal{G}\) that are in some sense closest to this subspace.

The subspace estimation step is typically achieved by performing an eigendecomposition on the covariance matrix \(R\) of the received data. Assuming the noise and signals are uncorrelated, and for the moment that the noise is spatially white, we have

\[ R = E[x(t)x^*(t)] = SAS^* + \sigma^2 I \]  

where \(S\) is the covariance matrix of the emitter signals and \(\sigma^2\) is the noise power in each channel. For MUSIC to be applicable, the emitter covariance \(S\) is required to be full rank \(d\) (no unity correlated signals). Using the model of (1), it is easily shown that the eigendecomposition of \(R\) has the following form:

\[ R = \sum_{i=1}^{m} \lambda_i E_i E_i^* \]  

where \(E_i = [e_1, \ldots, e_d], E_n = [e_{d+1}, \ldots, e_m],\) and \(\lambda_1 \geq \cdots > \lambda_{d+1} = \cdots = \lambda_m = \sigma^2.\) The eigenvectors \(E = [E_1, E_n]\) can be assumed to form an orthonormal basis; i.e., \(EE^* = E^*E = I.\)

The span of the \(d\) vectors \(E_i\) defines the so-called signal subspace, and the orthogonal complement spanned by \(E_n\) defines the noise subspace. This terminology is a consequence of the fact that span \(\{E_i\}\) = span \(\{A\} \perp\) span \(\{E_n\}.\) Thus, provided \(R\) is available, the low-rank subspace required to determine the DOA’s can be found even when noise is present. One way of exploiting this property would be to find vectors on the array manifold that have zero projection in the noise subspace; i.e., by finding the zeros of the function

\[ f(\theta) = \frac{a^*(\theta) E_n E_i^* a(\theta)}{a^*(\theta) a(\theta)} = \text{tr} (P_{E_n} E_i^* E_i) \]  

where \(P_{E_n}\) is defined as the projection matrix \(P_{E_n} = a(\theta) [a^*(\theta) a(\theta)]^{-1} a^*(\theta).\) However, due to various sources of error that arise in practice, only an estimate \(\hat{E}_n\) of the noise subspace eigenvectors is available. Consequently, we are unable to evaluate \(f(\theta),\) and an alternative approach is necessary. The approach of the MUSIC algorithm is to estimate the DOA’s as those values of \(\theta\) that minimize the following approximation to (3):

\[ \hat{f}(\theta) = \text{tr} (P_{E_n} \hat{E}_n E_i^*). \]  

If the additive noise is not spatially white, its covariance \(E[n(t)a^*(t)] = \sigma^2 \Sigma\) is not the identity,1 and the minimization of \(f(\theta)\) must be carried out in the \(\Sigma\)-metric. Assuming \(\Sigma\) is known, this minimization can be done either via a generalized eigendecomposition of the pair \((R, \Sigma),\) or via an eigendecomposition of the prewhitened covariance

\[ \Sigma^{-1/2} R \Sigma^{-1/2} = \Sigma^{-1/2} A S A^* \Sigma^{-1/2} + \sigma^2 I \]  

where all quantities are defined similarly to those of (2). In the latter case, the subspace relationships are span \(\{E_i\}\) = span \(\{\Sigma^{-1/2} A\} \perp\) span \(\{E_n\}\), and the MUSIC functional becomes

\[ f(\theta) = \text{tr} (P_{E_n} \hat{E}_n E_i^*) \]  

where \(P_{E_n} = \Sigma^{-1/2} a(\theta) [a^*(\theta) \Sigma^{-1/2} a(\theta)]^{-1} a^*(\theta) \Sigma^{-1/2}.\) This more general formulation of MUSIC will be used throughout most of the remainder of the paper.

For the special case of a uniform linear array (ULA) of identical sensors, a variation of the MUSIC algorithm known as root-MUSIC is often employed. The array manifold for a ULA is comprised entirely of Vandermonde vectors; i.e., the array response in direction \(\theta\) may be written as

\[ a(\theta) = a_r(z) = [1, z, z^2, \ldots, z^{m-1}]^T \]

where \(z = \exp \{-j/(2d \Delta \sin (\theta)/\lambda)\}\), \(\Delta\) is the interelement separation, and \(\lambda\) is the wavelength of the signal. Consequently, the function \(f(\theta)\) may be written as a polynomial in \(z\) of order \(2m - 2\), as follows:

\[ f(\theta) = a^{\ast}(z) \Sigma^{-1/2} E_n E_i^* \Sigma^{-1/2} a_r(z) \]

where \(a^{\ast}(z) = \Sigma^{-1/2} a(\theta) \Sigma^{-1/2} a_r(z)\) and \(e_i(z^{-1}) = a^{\ast}(z^{-1}) \Sigma^{-1/2} a_r(z)\) is made.

With a perfect covariance measurement, the root-MUSIC polynomial will clearly have \(d\) roots on the unit circle at values of \(z\) that correspond to the true angles of arrival. However, noisy or perturbed measurements cause the roots corresponding to the true DOA’s to be perturbed away from the unit circle. To isolate the \(d\) roots of interest from among the \(2m - 2\) zeros of the polynomial, the root-MUSIC algorithm simply chooses the roots with modulus nearest unity from among those inside the unit circle (only roots inside the unit circle need be considered since all

1When \(\Sigma = I\), the noise power \(\sigma^2\) is defined so that \(\det (\Sigma) = 1.\)
roots occur in para-Hermitian pairs). Root-MUSIC is usually preferred over MUSIC for applications involving ULA’s since it does not exhibit a loss-of-resolution threshold effect. Loss-of-resolution occurs for MUSIC when $f(\theta)$ has fewer minima than the number of sources $d$.

In their recent analysis of MUSIC, Stoica and Nehorai [8] used a first-order expansion of the derivative of (3) with respect to $\theta$ to obtain error expressions for the DOA estimates. Though their goal was specifically an asymptotic analysis of the finite sample approximation, this analysis technique may be extended to apply to any source of error in the noise subspace eigenvectors. In the next section, we show how the DOA estimation error due to array and noise covariance perturbations may be obtained using this type of first order analysis.

III. A First-Order Error Analysis

Errors in the MUSIC and root-MUSIC DOA estimates can arise from any of a number of sources. The most important of these include:

1) finite sample effects;
2) an imprecisely known noise covariance $\Sigma$; and
3) a perturbed array manifold $A$.

Finite sample effects occur since a perfect covariance measurement $R$ cannot be obtained. In practice, the sample covariance $\hat{R}$ defined by

$$\hat{R} = \frac{1}{N} \sum_{k=1}^{N} x(t_k)x^*(t_k)$$

is used to estimate $R$. For finite $N$, the signals and noise have not had "time" to decorrelate, and the noise covariance has not yet converged to its limiting value. When $N$ is large or the SNR is high, finite sample effects may be neglected. There are in fact many applications for which the limiting factor in performance is not due to finite sample effects, but rather to the model errors of items 2 and 3.

A. A General Framework for Model Errors

To isolate the effects of these model errors on the DOA estimates, it will be assumed that the finite sample effects due to additive noise are negligible and that an exact measurement of the perturbed covariance $\hat{R}$ is available. A very general model for $\hat{R}$ is as follows:

$$\hat{R} = (I + \Delta)[(A + \tilde{A})S(A + \tilde{A})^* + \sigma^2(\Sigma + \tilde{\Sigma})(I + \Delta)^*$$

where the matrices $\Delta$, $\tilde{A}$, and $\tilde{\Sigma}$ are the result of various types of model perturbations. The matrix $\Delta$ contains errors that affect both the signal and noise components of the data. Such errors include gain imbalances in the receiver electronics, channel crosstalk, and mutual coupling effects. The matrix $\tilde{A}$ represents the error in the nominal array response, and incorporates the effects of imprecisely known sensor locations, perturbations in the antenna amplitude and phase patterns, and signal-only mutual coupling. Finally, deviations of the additive noise statistics from $\Sigma$ are denoted by the Hermitian matrix $\tilde{\Sigma}$.

For both MUSIC and root-MUSIC, we are primarily interested in how the presence of $\Delta$, $\tilde{A}$, and $\tilde{\Sigma}$ affect the noise subspace. Let $E_\alpha = E_{\alpha + \tilde{\alpha}}$ represent the perturbed noise subspace eigenvectors, and assume that $E_\alpha$ has been normalized so that $E_\alpha E_\alpha^* = I$. Since $E_\alpha$ is also normalized, the perturbation $E_\alpha$ will in general have components in both the true signal and noise subspaces. As will be seen shortly, however, only those components of $E_\alpha$ in the signal subspace will contribute to the estimation error.

To establish a link between the error terms of (8) and $E_{\alpha + \tilde{\alpha}}$, we examine the noise eigenvectors of $\Sigma^{-1/2}R\Sigma^{-1/2}$, defined by

$$\Sigma^{-1/2}R\Sigma^{-1/2}E_\alpha = E_\alpha (\sigma^2 I + \tilde{\alpha})$$

where $\tilde{\alpha}$ represents the perturbed noise eigenvectors. Expanding this equation using the model of (8) and eliminating second-order error terms (e.g., terms of order $O(||\Delta||^2)$, $O(||\Delta||||\Delta||)$, etc.) and terms involving $A^*\Sigma^{-1/2}E_\alpha = 0$ leads to the following first-order approximation:

$$\Sigma^{-1/2}R\Sigma^{-1/2}E_\alpha = E_\alpha (\sigma^2 I + \tilde{\alpha})$$

Multiplying on the left by $A^*\Sigma^{-1/2}$ and transposing the result then yields

$$E_\alpha^* \Sigma^{-1/2}A = -E_\alpha^* \Sigma^{-1/2}$$

where the error matrix $\Sigma$ is defined as

$$\Sigma = \tilde{\alpha} + \Delta\Sigma + \sigma^2(\Sigma + \Delta\Sigma)^*(I + \Delta)^* \Sigma^{-1/2}$$

It is clear from (10) that only those model errors that give rise to components outside the signal subspace (i.e., those that result in errors in the DOA estimates) will produce a noise eigenvector perturbation $\tilde{E}_\alpha$ with components outside the noise subspace.

Multiplying (10) on the left by $\tilde{E}_\alpha$ and using the fact that $E_\alpha^* \Sigma^{-1/2}A = E_\alpha^* \Sigma^{-1/2}$ leads to the following first-
order relationships:
\[
\hat{E}_n \Sigma^{-1/2} \Delta a(\theta) = -E_n \hat{E}_n^* \Sigma^{-1/2} \xi_i,
\]
\[
i = 1, \ldots, d
\]  
(12)
where the error vector \(\xi_i\) for the \(i\)th source is defined as the \(i\)th column of \(\Sigma\):
\[
\xi_i = a(\theta) + \Delta a(\theta) + \sigma^2 (\hat{L} + \Sigma \Delta \hat{a} + \Delta \Sigma)
\]
\[
\cdot \Sigma^{-1/2} (\Sigma^{-1/2} A)^t \Sigma_i^{-1}
\]  
(13)
and \(\Sigma_i^{-1}\) denotes the \(i\)th column of \(\Sigma^{-1}\). Thus, the projection of the true steering vector \(\Sigma^{-1/2} a(\theta)\) into the perturbed noise subspace is approximately equal to the projection of the perturbation vector \(\Sigma^{-1/2} \xi_i\) into the true noise subspace. This is the key relationship used in the sequel to develop error expressions for the DOA estimates.

**B. Error Expressions**

Following the approach of [8], an expression for \(\hat{\theta}_i - \theta_i\) will be obtained via a first-order expansion of \(\hat{f}(\theta) = \hat{f}(\theta_i + \hat{f}'(\theta_i) \theta_i - \theta_i\) for small enough errors we may write
\[
0 = \hat{f}'(\theta_i) \approx \hat{f}(\theta_i) + \hat{f}'(\theta_i) \hat{f}(\theta_i - \theta_i)
\]  
(14)
where
\[
\hat{f}'(\theta_i) \overset{\text{def}}{=} \hat{f}(\theta_i) |_{\theta = \theta_i}
\]
Before solving for the error \(\hat{\theta}_i - \theta_i\) in (14), we will find first-order approximations for each term of the sum.

To evaluate \(\hat{f}'\), we need the following easily proved result:
\[
\frac{\partial P_{\Sigma, \theta}}{\partial \theta} = P_{\Sigma, \theta} (\Sigma^{-1/2} d(\theta) (\Sigma^{-1/2} a(\theta))^t)
\]
\[
+ \{\Sigma^{-1/2} a(\theta)^t\}^t d(\theta) (\Sigma^{-1/2} a(\theta))^t P_{\Sigma, \theta}
\]
where the definitions
\[
d(\theta) = \frac{\partial a(\theta)}{\partial \theta}
\]
\[
P_{\Sigma, \theta} = I - P_{\Sigma, \theta}
\]
have been used. With these expressions and the relationship \(E_n^* \Sigma^{-1/2} a(\theta) = 0\), the first derivative may be written as
\[
\hat{f}'(\theta_i) = 2 \Re \{d(a(\theta)) \Sigma^{-1/2} P_{\Sigma, \theta} \hat{E}_n^* \hat{E}_n \Sigma^{-1/2} \hat{a}(\theta) \}
\]  
(15)
\[
= 2 \Re \{d(a(\theta)) \Sigma^{-1/2} E_n^* E_n \Sigma^{-1/2} \hat{a}(\theta) \}
\]
\[
+ O(\|E_n\|^3)
\]
\[
\pm \frac{2 \Re \{d(a(\theta)) \Sigma^{-1/2} E_n^* E_n \Sigma^{-1/2} \hat{a}(\theta) \}}{a(\theta) \Sigma^{-1/2} a(\theta)}
\]
(16)
Since the second derivative appearing in (14) is multiplied by the error term \(\hat{\theta}_i - \theta_i\) which is assumed to be small, we make the following first-order approximation:
\[
\hat{f}'(\theta_i) \hat{f}'(\theta_i) \approx f'\Sigma(\theta_i) \hat{f}'(\theta_i)
\]
which together with (14) implies
\[
\hat{f}'(\theta_i) \approx \hat{f}'(\theta_i) \theta_i - \theta_i
\]  
(17)
It is straightforward to show that
\[
\hat{f}'(\theta_i) = 2 \Re \left\{ \frac{d(a(\theta)) \Sigma^{-1/2} E_n^* E_n \Sigma^{-1/2} a(\theta)}{a(\theta) \Sigma^{-1/2} a(\theta)} \right\}
\]  
(18)
and that the combination of (12), (15)–(17) lead to the following first-order expression for the estimation error:
\[
\hat{\theta}_i - \theta_i \approx \frac{\Re \{d(a(\theta)) \Sigma^{-1/2} E_n^* E_n \Sigma^{-1/2} a(\theta) \}}{d(a(\theta)) \Sigma^{-1/2} E_n^* E_n \Sigma^{-1/2} d(a(\theta))}
\]  
(19)
Rao and Hari have shown [9] that the perturbation in \(\theta_i\) for root-MUSIC can be written as
\[
\hat{\theta}_i - \theta_i \approx \frac{-\hat{f}'(\theta_i)}{2d(a(\theta)) \Sigma^{-1/2} E_n^* E_n \Sigma^{-1/2} d(a(\theta))}
\]
(20)
From the arguments given above, it is clear that (18) and (19) are equivalent. Thus, for small perturbations, the error expressions developed in this section are valid for both MUSIC and root-MUSIC.

Given any particular deterministic perturbation \(\xi\), the DOA error \(\hat{\theta}_i - \theta_i\) could of course be evaluated directly without using the approximation of (18). However, a more useful approach would be to assume that \(\xi\) is not known precisely, but instead is a realization of some known perturbation model. In this approach, one is not interested in a particular value of the estimation error \(\hat{\theta}_i - \theta_i\), but rather some averaged measure of it. In this paper, we will assume that the perturbation model is specified in probabilistic terms (i.e., \(\xi\) is random), and obtain expressions for the resulting bias and variance of \(\hat{\theta}_i - \theta_i\).

For cases where \(\xi\) is zero mean, it is clear that \(E(\hat{\theta}_i - \theta_i) = 0\). This condition will be assumed for all cases considered in this paper, so the bias term will not be addressed further. As for the variance (or, in this case, rms value) of the estimation error for MUSIC, it is easily
shown that
\[
C_{\theta i, k} = \mathbb{E}\{ (\hat{\theta}_i - \theta_i) (\hat{\theta}_k - \theta_k) \}
\]
\[
= \operatorname{Re} \left\{ \frac{f_i^* E_\Sigma^{-1/2}(C_{\xi i} \Sigma^{-1/2} E_i f_i + C_{\xi i}^R \Sigma^{-1/2} E_i f_i)}{2(f_i^* f_i)} \right\}
\]
where the following definitions have been made:
\[
f_i = E_\Sigma^{-1/2} d(\theta_i)
\]
\[
C_{\xi i} = \mathbb{E}\{ \xi_i \xi_i^* \}
\]
\[
C_{\xi i}^R = \mathbb{E}\{ \xi_i \xi_i^* \}
\]
Thus, the error covariance of MUSIC’s DOA estimates can be computed for any scenario where the perturbation covariances \( C_{\xi i} \) and \( C_{\xi i}^R \) may be evaluated. This will in turn be possible provided that the second order statistics of the elements of \( \Lambda, \Delta, \text{ and } \Sigma \) are known.

In Section IV, expressions are given for \( C_{\xi i} \) and \( C_{\xi i}^R \) for several special perturbation models. It will be seen that a more convenient compact representation of (20) is possible for all of these examples. Before moving on to them however, we briefly discuss how performance bounds are obtained when random model errors are present.

C. Performance Bounds

It is important to determine not only the absolute estimate variance as in (20), but also how this variance compares with the lowest achievable by any algorithm. For unbiased estimators, this is typically done by examining the Cramér–Rao bound (CRB) for the problem at hand. The CRB is obtained by taking the inverse of the Fisher information matrix \( J_F \), whose elements are defined by
\[
J_{F, i k} = -\mathbb{E}_n \left\{ \frac{\partial^2 \log \{ p(x|n) \}}{\partial n_i \partial n_k} \right\}
\]
where \( n \) is a vector of (real-valued) parameters to be estimated, and \( p(x|n) \) is the probability density for a single observation from the array given the parameters \( n \). For \( N \) independent observations, the covariance of the parameter estimate \( n \) satisfies
\[
\mathbb{E}\{ (\hat{n} - n)(\hat{n} - n)^T \} \geq \text{CRB} = (NJ_F)^{-1}.
\]
Under the assumption that there are no model errors (i.e., that \( n \) contains no random coefficients), and that \( s(t) \) and \( n(t) \) are independent, stationary, temporally white, zero-mean Gaussian random processes, it may be shown [24] that the elements of \( J_F \) are given by
\[
J_{F, i k} = \operatorname{tr} (R^{-1} R R^{-1} R_i)
\]
where \( R \) denotes the derivative of \( R \) with respect to the \( i \)th element of \( n \). Under the above assumptions, the parameter vector \( n \) for MUSIC will contain the DOA’s \( \theta_1, \ldots, \theta_d \), the noise variance \( \sigma^2 \), and elements of \( \text{Re} \{ S \} \) and \( \text{Im} \{ S \} \).

In the problem under consideration in this paper, the estimates are affected by the presence of various errors parameterized by the matrices \( \tilde{A}, \Delta, \text{ and } \Sigma \). Consequently, the parameter vector \( \eta \) must be augmented to include any nonzero entries from these matrices. Since these errors are assumed to be random, an additional term is also required in the equation for the Fisher information matrix [25]:
\[
J_F = J_C + J_P
\]
where the elements of \( J_C \) and \( J_P \) are defined by
\[
J_{C, i k} = -\mathbb{E}_n \left\{ \frac{\partial^2 \log \{ p(x|n) \}}{\partial n_i \partial n_k} \right\}
\]
\[
= \mathbb{E}_n \{ \operatorname{tr} (R^{-1} R R^{-1} R_i) \}
\]
\[
J_{P, i k} = -\mathbb{E}_n \left\{ \frac{\partial^2 \log \{ p(n) \}}{\partial n_i \partial n_k} \right\}
\]
and \( p(n) \) is the prior distribution of \( n \). The matrix \( J_C \) represents the information contained in the data, while \( J_P \) represents the \textit{a priori} information of the random parameters. The bound of (21) for \( N \) independent snapshots then becomes
\[
\mathbb{E}\{ (\hat{n} - n)(\hat{n} - n)^T \} \geq \text{CRB} = (NJ_C + J_P)^{-1}.
\]
In this analysis, we will use (25) in a somewhat modified form. First, since we are neglecting finite sample effects and including only the effects of model errors, the appropriate lower bound is obtained by taking the limit
\[
\lim_{N \to \infty} (J_P)^{-1} = \lim_{N \to \infty} (NJ_C + J_P)^{-1}.
\]
Note that even though \( J_F \) is affinely proportional to \( N \), taking the limit of its inverse does not produce a zero result since, for problems involving unidentifiable model errors, \( J_C \) is rank deficient and \( J_P \) has components in its nullspace. For some examples of the structure of \( J_C \) and \( J_P \) for various perturbation models, the interested reader is referred to [26].

The second modification to (25) we will employ stems from the difficulty of performing the expectation of a complicated nonlinear function of the random parameters as in (23). Consequently, we will follow the approach of Zhu and Wang [26] and approximate (23) as
\[
J_C = \hat{J}_C = \operatorname{tr} (R^{-1} R R^{-1} R_i)|_{n = \hat{n}}
\]
where \( \hat{n} \) is the parameter vector that generated \( \hat{R} \). This amounts to approximating the density functions of the random parameters as point masses, and is a reasonable approach for small perturbations. Since to simplify the CRB computation we are making an optimistic assump-
tion about the random parameters, the actual bound will
correspond to a slightly higher variance than that of the
approximate bound that results from using (26). In the
simulation studies of Section VI, the performance of MU-
SIC will be compared with the approximate CRB that
results from using (26).

IV. SOME SPECIAL PERTURBATION MODELS

In the previous section, we saw that the rms error of
the MUSIC DOA estimates can be evaluated for a partic-
ular perturbation model provided that the error co-
variances \( C_{ξ1} \) and \( C_{ξ2} \) can be computed. Below, we show
what form these matrices take for several special error
models. To simplify notation and enable us to obtain
compact matrix expressions for the estimation error, the
following definitions will be used:

\[
D = [d(θ_1), \ldots, d(θ_3)]
\]

\[
F = [f_1, \ldots, f_d] = E_x^*Σ^{-1/2}D
\]

\[
H = F^*F = D^*Σ^{-1/2}E_xE_x^*Σ^{-1/2}D
\]

\[
θ_0 = [θ_1, \ldots, θ_3]^T
\]

\[
θ = [θ_1, \ldots, θ_3]^T.
\]

For some of the models considered below, \( C_{ξ1} \) and \( C_{ξ2} \) may be expressed in the following general form:

\[
C_{ξ1} = \text{diag} \{K_i\}B_1\text{diag} \{K_i\}
\]

\[
C_{ξ2} = \text{diag} \{K_i\}B_2\text{diag} \{K_i\}
\]

(27)

(28)

where \( B_1 \) and \( B_2 \) correspond to the covariance of some perturbation parameter, and \( K_i \) represents the \( i \)th row of a matrix \( K \) that is determined by the particular error model under consideration. In such cases, combining the expres-
sions of (27) and (28) with that of (20) leads to the fol-
lowing expression for the covariance of the MUSIC DOA
estimates:

\[
C_{MU}^{def} = E\{(θ - θ_0)(θ - θ_0)^T\}
\]

\[
= \frac{1}{2}(H \otimes I)^{-1}[\text{Re}\{(F^*E_x^*Σ^{-1/2} \otimes K)
\]

\[
\cdot B_1(K^* \otimes Σ^{-1/2}E_x) + (F^*E_x^*Σ^{-1/2} \otimes K)
\]

\[
\cdot B_2(K^T \otimes Σ^{-1/2}E_x)\}] (H \otimes I)^{-1}.
\]

(29)

In the examples below where (29) holds, \( C_{MU} \) will be
described by specifying what the values the matrices \( K, B_1, \) and
\( B_2 \) assume in each case. In the other examples, \( C_{ξ1} \),
\( C_{ξ2} \), and \( C_{MU} \) will be specified directly.

Since array perturbations tend to be a limiting factor in
algorithm performance more than noise model errors, and
since the results in the next section will specifically apply
to the case where \( Λ = Σ = 0 \) and consequently \( ξ = d(θ) \), we choose to consider the array error case separately
and in more detail. There is really no loss of generality in
doing this since it will always be assumed that the error
terms \( A, Λ, \) and \( Σ \) are independent of one another.

A. Array Model Errors

An example of a perturbed sensor array is depicted in
Fig. 1. The array is nominally assumed to be composed of
uniformly spaced identical elements; i.e., each sensor
is assumed to have identical response, the signal condi-
tioning electronics (e.g., filter gain and phase response,
automatic gain controls (AGC), etc.) are assumed to per-
form identically, and the analog-to-digital (A/D) conver-
vers are assumed to be synchronized. However, as shown
in the figure, the sensors are not identical (their beampat-
terns are different), and their positions are not uniform.
In addition, the filter and AGC characteristics will not
be uniform from receiver to receiver, the A/D converters
will not be exactly in phase, and there may be uncalibrated
or nonuniform mutual coupling present. All of these factors
combine in varying degrees to produce the array pertur-
bation \( A \).

There are a variety of models that could be used to de-
scribe \( A \). A particularly simple model that has been widely
used [18], [19], [26], [27] is to assume that the columns
of \( A \) are independent zero-mean complex Gaussian ran-
don vectors with known covariance:

\[
\bar{a}(θ) \sim \mathcal{CN}(0, B_i), \quad E\{\bar{a}(θ)a^T(θ)\} = 0,
\]

\[
i = 1, \ldots, d.
\]

(30)

If the errors are independent from sensor to sensor, \( B_i \) is
clearly diagonal. Off-diagonal terms indicate sensor-to-
sensor correlations that result, for example, if there are
uncalibrated mutual coupling effects, or if some sensors
tend to perturb uniformly (such as identical or adjacent
elements). Because of the simple error expressions that
result from using (30), this model will be especially use-
ful later when analyzing weighted versions of MUSIC and
comparing the sensitivity of different algorithms. Though
convenient for such performance comparisons, this model
is not as ideal for performance analysis since, for ex-
ample, it is difficult to describe the gain and phase errors
independently.

To see how a more general model may be obtained,
suppose that \( a_i \) and \( e^{jθ} \) are respectively the nominal gain
and phase response of the \( k \)th sensor to a signal from di-
rection \( θ \); i.e., \( a_i(θ) = a_i e^{jθ} \). The perturbed response
\( \bar{a}_i(θ) = a_i(θ) + \hat{a}_i(θ) \) with separate gain and phase errors
\( \hat{a}_i \) and \( \hat{θ}_i \) may be written as

\[
\bar{a}_i(θ) = (a_i + \hat{a}_i) e^{j(θ + \hat{θ}_i)}
\]

\[
= a_i e^{jθ} \left( 1 + \frac{\hat{a}_i}{a_i} \right) e^{j\hat{θ}_i}
\]

\[
\bar{a}_i(θ) = a_i(θ) + γ_θ a_i(θ)
\]

(31)

*Mutual coupling occurs when a nominally passive collector acts as a
transmitter and retranslates some of its received energy.
where \( \gamma_i(\theta) \) is defined as \( (1 + a_i/a_k) e^{j\phi_i} - 1 \). The \( m \times d \) matrix \( \bar{A} \) can thus be described by the equation

\[
\bar{A} = [\Gamma_1 a(\theta_1), \Gamma_2 a(\theta_2), \ldots, \Gamma_d a(\theta_d)]
\]

(32)

where \( \Gamma_i = \text{diag} \{ \gamma_i(\theta_1), \gamma_i(\theta_2), \ldots, \gamma_i(\theta_d) \} \). Though for simplicity we have written \( \Gamma_i \) as a diagonal matrix, the off-diagonal elements will be nonzero for cases involving uncalibrated mutual coupling (cf. [15]).

The models of (30) and (32) are general in the sense that they apply to both angle-independent and angle-dependent sensor errors. For the angle-dependent case, \( B_i \neq B_k \) and \( \Gamma_i \neq \Gamma_k \) for \( i \neq k \). If the deviations from the nominal response are due to bulk delay and gain errors in the antenna receiver electronics, or if the sources are grouped closely in angle, the errors may be assumed to be angle-independent. Under this assumption, (32) may be written as

\[
\bar{A} = \Gamma A. \quad (33)
\]

However, in situations where the perturbations are due to imprecise knowledge of sensor locations, or where the sensor gain and phase patterns do not distort uniformly in \( \theta \), the more general angle-dependent assumption is more realistic.

In practice, the response of a given sensor is typically known to within some tolerance in gain and phase that

\[\text{accounts for variations in the construction of the sensor and the conditions under which it is to operate. This tolerance may be specified as limits above and below some nominal response, or as an expected deviation around the nominal. Consequently, as mentioned in the previous section, we will assume in this analysis that \( \bar{A} \) is specified in probabilistic terms (e.g., the mean and variance of the elements of \( \bar{A} \) are assumed known). This assumption has already been implicitly made in the model of (30). In this framework, one may think of the sensor array as one realization from the probability space of arrays specified by \( A \) and the distribution of \( \bar{A} \). As such, for each simulation study conducted in the next section, Monte Carlo trials will be performed over a large number of arrays “drawn” from the distribution specified by \( A \) and \( \bar{A} \).}

When considering array errors only, the covariances of the error vectors \( \xi_1, \ldots, \xi_d \) reduce to

\[
C_{\xi_{11}} = E\{a(\theta)a^*(\theta)\}
\]

\[
C_{\xi_{12}} = E\{a(\theta)a^*(\theta)\}
\]

Thus, it is assumed that we know not only the variance of the array perturbations at each sensor and in each signal direction, but also the amount of correlation between the errors from angle to angle (such as might be expected if two signals are very nearly coincident). The simple model of (30) assumes that our knowledge of the perturbed array response is specified directly in terms of these matrices. Some examples are provided below to illustrate their structure for cases where the errors are described in terms of physical quantities. In all cases described below, and throughout the remainder of the paper, it will be assumed that all error terms are zero-mean random variables.

1) Case 1: Simple Model with I.I.D. Gaussian Errors: If under the model of (30) the errors are independent and identically distributed (i.i.d.) from sensor to sensor and from angle to angle, then \( C_{\xi_{12}} = 0 \) and \( C_{\xi_{11}} = B = \sigma_n^2 I \delta_{ik} \), where \( \sigma_n^2 \) represents the variance of the perturbation at each sensor and \( \delta_{ik} \) is the Kronecker delta function. This case corresponds to adding an independent, circular complex Gaussian random variable of variance \( \sigma_n^2 \) to the response at each element of the array and in each signal direction. Though the sensor errors themselves are angle dependent, their statistics under this model are independent of \( \theta \). For this simple case, the covariance of the MUSIC estimates as given by (20) can be written in the following simple compact form:

\[
C_{\text{MU}} = \frac{\sigma_n^2}{2} (H \odot I)^{-1} (F^* E_n^* \Sigma^{-1} E_n F \odot I) (H \odot I)^{-1}.
\]

(34)

Note that if \( \Sigma = I \), (34) reduces to

\[
C_{\text{MU}} = \frac{\sigma_n^2}{2} (D^* P_d^2 D \odot I)^{-1}
\]

(35)

where \( P_d \) is defined as \( I - A^* A^{-1} A^* = E_n E_n^* \).
The variance $\sigma^2_a$ of the perturbation determines the amount of deviation of the gain and phase response from their nominal values. To quantify this deviation, the relative gain error amplitude (RGEA) and phase error amplitude (PEA) for the response of the kth sensor in direction $\theta_k$ are defined to be $a_{\text{g}} / a_k(\theta_k)$ and $180(\sigma_a / \pi)$ degrees, respectively. The RGEA may occasionally be given in decibels, in which case it is defined as $20 \log_{10}(a_{\text{g}} / a_k(\theta_k))$.

2) Case 2: Separate Gain and Phase Errors: Next consider the more general model of (32), where each sensor is subject to random, individually specified gain and phase perturbations. Assume that the errors are angle independent, and that there is no mutual coupling; i.e., $A = \Gamma A$, where $\Gamma$ is diagonal. For purposes of notation, define

$$g = [a_1, \ldots, a_n]^T$$
$$\hat{g} = [\hat{a}_1, \ldots, \hat{a}_n]^T$$

and

$$E\{gg^*\} = E\{\hat{g}\hat{g}^T\} = B_e$$
$$E\{\hat{g}\hat{g}^*\} = E\{\hat{g}\hat{g}^T\} = B_e$$
$$A = \Gamma^T.$$

Assuming that $g$ and $\hat{g}$ are uncorrelated, it can be shown that to first order, the covariance of the DOA estimates is given by (29), with

$$B_x = B_e + (B_e \otimes B_e)$$
$$B_y = B_e - (B_e \otimes B_e)$$
$$K = A^T.$$

If the errors are identically distributed from sensor to sensor, then $B_e = \sigma^2_e I$ and $B_e = \sigma^2_e I$, and we may define the RGEA and PEA to be $a_{\text{g}} / a_k(\theta_k)$ and $\sigma_a$ degrees, respectively.

3) Case 3: Mutual Coupling: As mentioned earlier, for situations involving uncalibrated mutual coupling, the matrix $\Gamma$ is assumed to have non-zero off-diagonal elements. For instance, consider an array where the response of each sensor is angle-independently coupled with that of $p$ other sensors, in addition to having i.i.d. perturbations of the type described in case 2. Each row of $\Gamma$ will thus have $p + 1$ nonzero entries, $p$ due to coupling and one due to the perturbed response of the sensor itself. A specific example would be a perturbed circular array in which each sensor leaks some of its received signal to immediately adjacent sensors on the array's perimeter (i.e., $p = 2$). If for simplicity we assume that the off-diagonal elements of $\Gamma$ are circular complex Gaussian random variables of equal variance $\sigma^2_{\alpha}$, then the expression for $C_{\text{MU}}$ of the previous example is applicable here by simply replacing $B_e$ with $B_e + p \sigma^2_e I$.

4) Case 4: Random Perturbations in Sensor Locations: For this case, assume an arbitrary array of identical unit-gain omnidirectional sensors in the $x$-$y$ plane with randomly perturbed sensor locations. The perturbed array response can be written as

$$\hat{a}(\theta) = \left[ \exp\left(\frac{j2\pi}{\lambda} (x_1 + \hat{x}_1) \sin \theta + (y_1 + \hat{y}_1) \cos \theta\right), \ldots, \exp\left(\frac{j2\pi}{\lambda} (x_m + \hat{x}_m) \sin \theta + (y_m + \hat{y}_m) \cos \theta\right) \right]^T$$

where $(x_i, y_i)$ are the nominal coordinates of the $i$th sensor and $(\hat{x}_i, \hat{y}_i)$ are the corresponding position errors. The perturbation term is given by $\hat{a}(\theta) = \Gamma a(\theta)$, where

$$\Gamma = \text{diag}\left\{\exp\left(\frac{j2\pi}{\lambda} [x_1 \sin \theta_1 + y_1 \cos \theta_1]\right) - 1, \ldots, \exp\left(\frac{j2\pi}{\lambda} [x_m \sin \theta_m + y_m \cos \theta_m]\right) - 1\right\}.$$ Note that in this case, the perturbation matrix $\Gamma$ is angle dependent.

If we define the covariances of the position errors as

$$E\{xx^*\} = E\{ \hat{x} x^T \} = B_x$$
$$E\{yy^*\} = E\{ \hat{y} y^T \} = B_y$$

where $x = [x_1, \ldots, x_n]^T$ and $y = [y_1, \ldots, y_m]^T$, then the error covariances can be shown to be approximately given by

$$C_{\text{MU}} \approx \frac{4\pi^2}{\lambda^2} \text{diag} \{a(\theta_i)\} \{ \sin \theta_i \sin \theta_i \} B_x + (\cos \theta_i \cos \theta_i) B_y$$

$$C_{\text{MU}} \approx \frac{4\pi^2}{\lambda^2} \text{diag} \{a(\theta_i)\} \{ \sin \theta_i \sin \theta_i \} B_x + (\cos \theta_i \cos \theta_i) B_y$$

(37)

where the $x$- and $y$-components of the position errors have been assumed to be uncorrelated. A matrix expression for $C_{\text{MU}}$ is also possible in this case, but it is somewhat cumbersome and consequently will not be given here.

B. Noise and Channel Model Errors

In the narrow-band DOA estimation context considered here, the term "noise" refers to anything that contributes to the $\sigma^2 \Sigma$ term of the covariance $\hat{R}$. Noise may enter into the array measurement either externally through the sensor, or internally through the receiver electronics. The

Since the coefficient of mutual coupling between two sensors depends on their relative position, sensor location errors will in general produce uncalibrated mutual coupling effects and hence the matrix $\Gamma$ will in general be non-diagonal. However, in this case, we have assumed for simplicity that the elements of the array are uncoupled, and consequently $\Gamma$ is diagonal.
ternal component is typically due to thermal noise and quantization effects, and is usually well modeled as independent (though not necessarily identical) from channel to channel. External noise would naturally include random background radiation and clutter, but it might also be the product of any type of interference which elicits an array response that is significantly removed from the assumed manifold (e.g., near-field, wide-band, or distributed emitters under the assumption of far-field, narrow-band point sources). The external component of the noise is the most difficult to model since its source is usually not well understood and is often both time and location dependent. Consequently, in the absence of prior information to the contrary, the simplifying assumption $\Sigma = I$ is often made. Deviations of the noise from this simple model are not critical at high SNR since they contribute little to $\hat{R}$; however, at low SNR the degradation may be severe.

The two error terms $\Delta$ and $\hat{\Sigma}$ in the perturbation model of (8) allow one to separately account for deviations in both the internal and external components of the noise model, in addition to bulk channel errors that affect both the signal and noise components of the data. With a few simple exceptions, it is somewhat more difficult than in the array error case to connect a particular model for $\Delta$ and $\hat{\Sigma}$ with some underlying physical phenomenon. Expressions for the estimation error covariance $C_{\text{MU}}$ are provided below for three of the simpler cases. For ease of notation, it is assumed in these examples that $\Sigma = I$ and $A = 0$, and hence that

$$\xi = \Delta a(\theta) + \sigma^2(\hat{\Sigma} + \Delta^* + \Delta)A^*S^{-1}.$$

1) Case 5: Diagonal Noise Covariance Perturbation: Assuming noise of equal power in each receiver channel is a more coarse approximation than assuming it is independent of the noise in other channels, so a logical first-cut perturbation model for $\hat{\Sigma}$ is that it be a logical diagonal matrix. Since $\Sigma^* = \Sigma$, its diagonal elements are real. Thus, if we let $\delta$ represent $\text{diag}(\Delta)$ and define the covariance

$$E\{\delta^*\delta\} = E\{\delta^2\} = B_s.$$

For this error model, the covariance of the DOA estimates is given by (29) with

$$B_1 = B_2 = B_s$$

$$K = \sigma^2(A^*S^{-1})^T.$$ (38)

2) Case 6: Nondiagonal Noise Covariance Perturbation: Perturbations to the noise covariance due to a spatially colored noise field require a nondiagonal model for $\hat{\Sigma}$. As a simple example, consider the case where each element of $\hat{\Sigma}$ is nonzero. In particular, assume that the elements of $\hat{\Sigma}$ are independent, zero-mean random variables with variance $\sigma^2$. The off-diagonal elements of $\hat{\Sigma}$ are circular complex random variables, while the diagonal elements are real. It is straightforward to show that for this model

$$C_{\xi,1}^\delta = \sigma^2 a^4(S_{\xi}^* - A^*A)^{-1}S_{\xi}^{-1}I$$

$$C_{\xi,2}^\delta = \sigma^2 a^4A^*S_{\xi}^{-1}S_{\xi}^{-1}A^T$$

(39)

where $S_{\xi}^{-1}$ denotes the Hermitian transpose of $S_{\xi}^{-1}$. A simple matrix expression is possible for $C_{\text{MU}}$ in this case, as follows:

$$C_{\text{MU}} = \frac{\sigma^2 a^4}{2} (D^*P_{\hat{K}}^2 D \odot I)^{-1} \text{Re} \{ (D^*P_{\hat{K}}^2 D \odot I)^{-1} \odot (SA^*AS)^{-1} \}$$

(40)

where we have used the fact that $E_{\xi,2}^*C_{\xi,2} = 0$ for all $i$ and $k$.

3) Case 7: Channel Gain Imbalance: Bulk receiver gain errors are another reason the noise power in each channel may be different. In such cases, the perturbation to $\hat{R}$ enters via $\Delta$ rather than $\hat{\Sigma}$, and $\Delta$ is a real-valued diagonal matrix. This type of error not only affects the noise component of the signal, but also introduces an angle-independent modification of the antenna gain response. Thus, it is useful to draw a distinction between this and case 5. Let $\delta$ denote $\text{diag}(\Delta)$ and define the covariance

$$E\{\delta\delta^*\} = E\{\delta^2\} = B_s.$$

The error covariance $C_{\text{MU}}$ is again given by (29), with

$$B_1 = B_2 = B_s$$

$$K = (A + 2\sigma^2 A^*S^{-1})^T.$$ (41)

It is interesting to compare the error covariance for $C_{\text{MU}}$ in this case with that obtained using (36) and (38). Not surprisingly, just as a channel gain imbalance of this type may be thought of as a combined perturbation to the array gain response and the diagonal elements of the noise covariance, the matrix $K$ for this case is just a weighted sum of those obtained in cases 2 and 5.

Evaluating $C_{\xi,1}^\delta$ and $C_{\xi,2}^\delta$ for other perturbation models is not difficult, but is often notationally cumbersome. Consequently, we will restrict our attention in this paper to the models presented in this section. Since it is reasonable to assume that $\hat{\Sigma}$, $\Delta$, and $\hat{\Xi}$ are independent of one another, error covariances for models involving combinations of these parameters can be obtained by simply summing the covariances due to each considered separately.

C. Comparisons with Finite Sample Effects

All of the analytical error expressions above have been obtained assuming a perfect estimate of the perturbed covariance was available, i.e., that the number of snap-
shots of data \( N \) was infinite. It is expected that these expressions would still be approximately valid for large \( N \), though exactly how large depends on the particular scenario under consideration. To make this more precise, we will compare the DOA error for the model perturbations above with that for the case where \( N \) is finite and no model perturbations are present. In particular, we will derive conditions under which the RMS error due to the model perturbations of cases 1 and 6 exceeds that due to finite sample effects alone.

The finite sample performance of MUSIC has been studied by several researchers, and the covariance of the MUSIC DOA estimates has recently been shown [8] to be asymptotically (for large \( N \)) given by

\[
C_{\text{MUS}} = \frac{\sigma^2}{2N} (D^*P_A^*D \otimes I)^{-1} \left[ \text{Re} \{ (D^*P_A^*D) \odot (S^{-1} + \sigma^2 (SA^*AS)^{-1}) \} (D^*P_A^*D \otimes I)^{-1} \right]
\]

where the notation \( C_{\text{MUS}} \) is used in the finite sample case to distinguish it from the notation \( C_{\text{MU}} \) used above, and where it is assumed for simplicity that \( \Sigma = I \). Comparison of (42) with (35) and (40) reveals some striking similarities. These are examined in more detail below.

1) Array Model Errors: Comparing (35) and (42), we see that the error in the \( i \)th DOA estimate due to Gaussian perturbations in the array manifold exceeds that due to finite sample effects alone when

\[
\sigma^2_i > \frac{\sigma^2}{N} [S^{-1}]_{ii} + \sigma^2 [S^{-1}(A^*A)^{-1}S^{-1}]_{ii}
\]

where the notation \([ \cdot ]_i\) denotes the \( i \)th diagonal element of the matrix.

For the special case of a single source and an \( m \)-element array of sensors with unity gain in the direction of the source, (43) reduces to

\[
\sigma^2_i > \frac{m \cdot \text{SNR}}{mN \cdot \text{SNR}}
\]

where SNR represents the ratio of the signal and noise powers. In most instances \( m \cdot \text{SNR} \gg 1 \), so (44) is approximately independent of the number of elements in the array: \( \sigma^2_i > 1/(N \cdot \text{SNR}) \). Thus, for a scenario involving 100 snapshots of a signal at 20-dB SNR, array calibration errors become important when \( \sigma_i > 0.01 \). This corresponds to an RGEA and PEA of only 0.01 and 0.6°, respectively, illustrating how critical accurate calibration information is.

For two uncorrelated sources and an \( m \)-element unity gain antenna array, (43) becomes

\[
\sigma^2_i > \frac{1}{N \cdot \text{SNR}_i} + \frac{m}{N(m^2 - |a^*(\theta_i)a(\theta_i)|^2) \cdot \text{SNR}_i}
\]

where SNR\(_i\) is the signal-to-noise ratio for the \( i \)th source. In most cases, this expression will differ from the single-source approximation obtained above only when the two signals are nearly coincident (\( m^2 - |a^*(\theta_i)a(\theta_i)|^2 \ll 1 \)) or the SNR is very low (\( \text{SNR}_i \ll 1 \)).

2) Noise Model Errors: The finite sample error covariance of (42) and that of (40) for the case of Gaussian perturbations to the noise covariance model have very similar forms. Comparison of these two expressions reveals that the error in the \( i \)th estimate due to perturbations in the noise model will exceed that due to finite sample effects when

\[
\sigma^2_i > \frac{1}{N} + \frac{|S^{-1}|_i}{N \sigma^2 |S^{-1}(A^*A)^{-1}S^{-1}|_i}.
\]

(45)

For a single source and an \( m \)-element unity gain array, the inequality of (45) reduces to

\[
\sigma^2_i > \frac{1 + m \cdot \text{SNR}}{N} \approx \frac{m \cdot \text{SNR}}{N}.
\]

Thus, as one would expect, errors in the noise model only become important when the signal-to-noise ratio is low, or the amount of observed data is large. Equation (46) also implies that large arrays tend to reduce the relative effect of such errors. Small perturbations to the noise model are apparently “averaged out” over many sensors more rapidly than in the case of finite observations. This is a result of the fact that \( C_{\text{MU}} \) in (40) goes to zero as \( m \to \infty \), while \( C_{\text{MUS}} \) in (42) does not.

For two uncorrelated sources, the inequality of (45) becomes

\[
\sigma^2_i > \frac{1}{N} + \frac{(m^2 - |a^*(\theta_i)a(\theta_i)|^2) \cdot \text{SNR}_i}{mN}.
\]

V. An Optimal Weighting for MUSIC

An advantage of the simple matrix expressions obtained for \( C_{\text{MU}} \) in the previous section is that performance comparisons between different implementations of the algorithm are facilitated. In this section, for example, it will be demonstrated how a weighted version of the MUSIC algorithm can be employed to improve the quality of the DOA estimates under a particular perturbation model.

The following weighted version of MUSIC will be considered:

\[
\hat{f}_w(\theta) = \text{tr} \left( P_{\theta_i} \hat{E}_w^* W^* \right)
\]

(47)

where the weighting matrix \( W \) satisfies \( W = W^* > 0 \). For weighted MUSIC, the error covariance of (20) must be modified as follows:

\[
C_{\text{MU},w}(W) = \text{Re} \left\{ \frac{f^*_w W^*_{w} \Sigma^{+1/2}(C_{\text{MU}}^{\dagger})^{\dagger} \Sigma^{+1/2} E_w W_f + C_{\text{MU}}^{\dagger} \Sigma^{+1/2} E_w W_f}{2(f^*_w W_f)(f^*_w W_f)} \right\}
\]

(48)
We will restrict our attention in this section to the simple array perturbation model of (30), and we will assume that the statistics of the perturbation are independent of \( \theta \). For this special model, we obtain the following compact matrix expression for the covariance of the weighted MUSIC estimates:

\[
C_{MU}(W) = \frac{1}{2} \left( F^* W F^* \otimes I \right)^{-1} \left( F^* W Q W F^* \otimes I \right) \left( F^* W F^* \otimes I \right)^{-1}
\]

where

\[
Q = E_{a}^{n} \Sigma_{s}^{-1/2} B \Sigma_{s}^{-1/2} E_{a}^{n}
\]

(49)

We wish to choose \( W \) in order to minimize the variance of the DOA estimates; i.e., our goal is to find an optimal weighting \( W_{OPT} \) such that \( C_{MU}(W_{OPT}) \leq C_{MU}(W) \) for any \( W > 0 \). The following theorem is the key to finding such a weighting:

**Theorem 5.1:** Using (49) and previous definitions for \( F \) and \( Q \), the inequality

\[
C_{MU}(W) \geq \frac{1}{2} \left( (F^* Q^{-1} F) \otimes I \right)^{-1}
\]

(50)

holds, where \( Q \) is assumed to be full rank.

**Proof:** The inequality

\[
\begin{bmatrix}
(F^* Q^{-1} F) \otimes I & (F^* W F^* \otimes I) \\
(F^* W F^* \otimes I) & (F^* W Q W F^* \otimes I)
\end{bmatrix}
\]

is equivalent to (50) through use of the Schur complement. This inequality (and hence the theorem) follows from the fact that both

\[
\begin{bmatrix}
I & I \\
I & I
\end{bmatrix}
= \begin{bmatrix} I & I \\ I & I \end{bmatrix}
\]

and

\[
\begin{bmatrix}
F^* Q^{-1} F & F^* W \\
F^* W & F^* W Q W F^* \otimes I
\end{bmatrix}
= \begin{bmatrix} F^* Q^{-1} \\ F^* W \end{bmatrix} Q (Q^{-1} F W)
\]

are positive semidefinite (PSD) matrices, and from the well-known result that the Schur product of two PSD matrices is itself PSD (e.g., see [28]).

To minimize \( C_{MU}(W) \), it would be sufficient to find a \( W \) for which equality was obtained in (50). Such a weighting is possible, and comparison of (49) and (50) shows that this optimal weighting is given by

\[
W_{OPT} = Q^{-1} = (E_{a}^{n} \Sigma_{s}^{-1/2} B \Sigma_{s}^{-1/2} E_{a}^{n})^{-1}
\]

(52)

Thus, provided that the array perturbations satisfy the model of (30), this choice for \( W \) in (47) will result in DOA estimates of minimum variance. Note that the optimal weighting becomes important (i.e., \( W_{OPT} \neq I \)) only when either \( B \neq I \) or \( \Sigma \neq I \). It is also interesting to note here that MUSIC's performance cannot be improved by this type of weighting when finite sample effects and not model errors are considered; under these assumptions, the choice \( W = I \) has been shown to be optimal in the asymptotic analysis of [29].

Since only an estimate of \( E_{a}^{n} \) is available, we are forced to use the approximate weighting

\[
\hat{W}_{OPT} = (\hat{E}_{a}^{n} \Sigma_{s}^{-1/2} B \Sigma_{s}^{-1/2} \hat{E}_{a}^{n})^{-1}
\]

(53)

Fortunately, however, using \( \hat{W}_{OPT} \) in place of \( W_{OPT} \) in the first-order approximation of (15) introduces only terms of order \( O(\|E_{a}\|^{2}) \) and smaller. Thus,

\[
C_{MU} (\hat{W}_{OPT}) \approx C_{MU} (W_{OPT})
\]

(54)

and, to first-order, identical performance is achieved. Some simulation examples illustrating the advantage of using this weighted version of MUSIC will be given in the next section.

**VI. SIMULATION EXAMPLES**

To illustrate the usefulness of the results of the previous sections, and to validate the analytical error expressions obtained for MUSIC, simulation examples are presented in this section for several representative cases. A total of 1000 trials were conducted for each example, with the perturbed covariance \( \hat{R} \) generated for each trial using the error-free covariance \( R \) and the distribution of the perturbation under consideration. The sample rms error of the DOA estimates was then calculated and compared to that predicted by the corresponding theoretical expressions. In all of the following examples, the nominal gain of all sensors was assumed to be unity in the direction of the impinging signals, and it was assumed that the number of emitters \( d \) had been correctly determined. In addition, the noise was assumed to be spatially white with unit variance, except for the case where noise covariance perturbations were studied.

The first three examples are given primarily to demonstrate how well the theoretical expressions predict the performance of MUSIC and root-MUSIC under a variety of perturbations scenarios. These results are also compared with the approximate Cramér–Rao bound described in Section III-C to quantify what performance improvement might accrue through use of a more efficient algorithm. The last two examples illustrate the advantage of using the optimally weighted version of MUSIC, and also demonstrate how the analysis of this paper might be used in the antenna array design process. In particular, it is shown how one might analyze tradeoffs that arise when adding unreliable sensors to a given array in order to increase its aperture.

In the first case, a 12 element ULA with \( \lambda/2 \) interelement spacing was assumed. Two uncorrelated emitters at angles of 10° and 15° relative to broadside were simulated, each with a power level of 0 dB relative to the additive noise. An angle- and sensor-independent Gaussian phase perturbation was made to the array response, and
the performance of MUSIC and root-MUSIC was evaluated as a function of the variance of the perturbation. The results of the simulations are plotted in Fig. 2. The symbols $\sigma$ and $x$ represent the sample standard deviation of the MUSIC and root-MUSIC estimates, respectively, the dashed line represents the error predicted by (36), and the solid line represents the approximate CRB for this case. The dotted line indicates the finite sample error that would result for this source/array configuration if $N = 500$ snapshots were observed and there were no phase perturbation. Note the excellent agreement between predicted and simulated values, even for the rightmost case where the phase error was substantial ($\sigma_\phi = 30^\circ$). As predicted, MUSIC and root-MUSIC have identical second-order performance. Note also that for this case, the finite sample error for $N = 500$ is roughly equivalent to a standard deviation of about $4^\circ$ in calibration phase. For this particular scenario, MUSIC compares very favorably to the CRB, so any additional computational effort required to more closely approach the bound would probably not be warranted. This is especially true in light of the fact that the approximate CRB is somewhat optimistic.

For the next simulation example, we consider the effects of perturbations to the noise covariance as a function of signal correlation. The array and source configuration was identical to that of the previous case, except that the signal-to-noise ratio was increased to 5 dB. The nominal covariance of the noise was assumed to be the identity, and two types of perturbations were simulated. The first was a diagonal perturbation of covariance $B_e = \sigma^2 I$ (see case 5 of Section IV), and the second was the type of perturbation described in case 6 of Section IV where random errors of variance $\sigma^2$ are added to all elements of the noise covariance. Fig. 3 shows the predicted and measured performance of MUSIC and root-MUSIC for these two types of perturbations as a function of the correlation between the two signals when $\sigma = 0.2$. The solid and dashed curve correspond to the diagonal and nondiagonal perturbation cases, respectively, while the dotted line represents the error that would be incurred for $N = 500$ when finite sample effects alone are considered. Even though in this case the level of uncertainty in $\Sigma$ results in diagonal elements between 0.6 and 1.4 and off-diagonal terms of magnitude as large as 0.3 to 0.5, at low values of correlation the error in the DOA estimates is quite small. Performance does, however, degrade rapidly as the correlation coefficient approaches unity, much more rapidly in fact than for finite data. As before, the agreement between the simulated and theoretical estimation error is excellent in every case, and the performance of MUSIC and root-MUSIC is identical.

The performance degradation due to uncalibrated mutual coupling is studied in the next example. The configuration, correlation, and power level of the sources are identical to that of the first example, but for variety we have chosen to simulate a uniform 12 element circular array with a diameter of six wavelengths. Immediately adjacent sensors on the perimeter of the circle were coupled with angle- and sensor-independent complex Gaussian coupling coefficients of variance $\sigma_\phi^2$. An angle-independent gain perturbation with RGEA $\sigma_a = 0.01 = -40$ dB and an angle-independent phase perturbation with PEA $\sigma_\phi = 0.06^\circ$ were also made to the nominal array response in addition to the mutual coupling errors. The results of the simulations are displayed in Fig. 4. Notice that, as one would expect, the mutual coupling perturbation deteriorates performance only when its magnitude exceeds that of the gain and phase errors, and that the theoretical expressions accurately track the effects of increasing $\sigma_\phi$. The error due to mutual coupling dominates the finite sample error in this case when $\sigma_\phi > 0.03$, or in other words when leakage between adjacent sensors represents only about 3% of the nominal sensor output. The 16.2% failure rate shown on the plot indicates that, for the largest
simulated value of $\sigma_{\theta}$, MUSIC was unable to resolve two sources 162 times. Since these trials were eliminated from the RMS error calculation, the measured RMS error is somewhat lower than that predicted by theory.

As a fourth example, consider the array of Fig. 5 which is composed of a seven element ULA and two additional elements separated from either end of the ULA by six wavelengths. Suppose that the array is subject to the type of gain and phase perturbation described by the model of (30), and further suppose that the error covariance $B$ is given by

$$B = \text{diag} \{10^{-2}, 10^{-4}, \cdots, 10^{-4}, 10^{-2}\}.$$  \hfill (55)

In other words, the standard deviation of the array perturbation for the end elements is ten times greater than for the elements of the ULA. The actual values of the RGPA and PEA for the ULA are 0.01 and 0.57°, respectively, while those for the end elements are 0.1 and 5.7°. For this array, a relevant system design problem would be to how to appropriately tradeoff a) the performance improvement that results from using the large aperture provided by the end elements, and b) the performance degradation caused by the unreliable calibration information for these elements.

An example of the application of our analysis to such a problem, as well as the usefulness of the optimally weighted version of MUSIC, consider the simulation results presented in Fig. 6. The array of Fig. 5 was simulated with the perturbation described by (55). Two uncorrelated emitters were assumed, each of power 0 dB relative to the additive noise. The first emitter was fixed at 0° broadside, while the second was varied from 2° to 50° over several experiments. The dotted curve and the symbols $\circ$ and $\times$ represent the predicted and measured performance of MUSIC and root-MUSIC when using only the

7 element ULA and ignoring the end elements (no MUSIC result is shown for the case where the second source was at 2° since the algorithm failed to resolve the two sources in over half the trials). The solid curve and the symbol $\cdot$ denote the predicted and measured performance of MUSIC for the full array (including the end elements) without weighting, while the dashed line and $+$ denote the same for the weighted MUSIC algorithm described by (47) and (52).

When the sources are closely spaced, the smaller ULA does not provide enough aperture for MUSIC to accurately estimate the DOA's. As the second source is moved away from broadside, the performance of the ULA improves until at $\theta_2 = 8^\circ$ it does as well as the full unweighted array. For DOA's beyond 8°, using the information from the unweighted end elements actually degrades algorithm performance relative to just ignoring them. However, the lowest estimation error is achieved using the full optimally weighted array. In fact, the CRB is not shown independently on this plot since it was virtually equivalent to the performance predicted for weighted MUSIC. Thus, in this case at least, the weighted MUSIC algorithm appears to be efficient. In all cases, the measured rms errors are in excellent agreement with those predicted by theory.

In our final example, we present another simple antenna tradeoff study. Suppose for this case that we have a nominal $\lambda/2$ spaced ULA with perturbed element locations. Suppose further that the position errors increase in magnitude from one end of the array to the other, as de-
picted in Fig. 7. Note that the uncertainty in sensor location perpendicular to the array (denoted by the standard deviation $\sigma_\perp$) is assumed to be somewhat larger than that parallel to the array (denoted by $\sigma_\parallel$). This is the type of positioning error that might be encountered when using a towed array in an underwater environment. The sensors nearest the towing vessel would be expected to retain their linear, equispaced structure, while those toward the end of the array might be "whipped" back and forth due to the effects of being pulled through the water and the nonlinear motion of the vessel. As in the previous case, the system designer is faced with trading off reliable calibration information for an increased array aperture.

As in previous examples, two uncorrelated 0 dB emitters were simulated at 10° and 15° relative to broadside. The length of the ULA was varied from 3 to 43 over several experiments, and in each case random perturbations as described by Fig. 7 were made to the array. For simplicity, the errors were assumed to be independent from sensor to sensor; this is a somewhat unrealistic assumption for a towed array since one would expect adjacent array elements to have correlated position errors, but it serves nonetheless as a first approximation. The results of the simulation are displayed in Fig. 8, where the performance of both an unweighted and weighted version of MUSIC are displayed along with the approximate CRB. The weighting matrix used was $W = (\bar{E}_\perp^H \bar{B} \bar{E}_\perp)^{-1}$. Even though the claim of optimality in Section V applies only to the weighting described for the model of (30), we see that the advantage of using an appropriate $W$ is still very evident. In fact, as the length of the array increases, the performance of the unweighted algorithm actually deteriorates, while the weighted algorithm improves slightly. However, the analysis demonstrates that for this scenario, there is not much to be gained by using an array of more than 10 or 15 elements.

In this case it happened that $\bar{B} = 10 \bar{B} = 0$, so either $\bar{B}$ or $\bar{B}$ can be used for $W$ since a scaling of the cost function is unimportant.

VII. CONCLUDING REMARKS

We have presented in this paper a first-order perturbation analysis of the MUSIC and root-MUSIC algorithms under the assumption of various model errors. The analysis is applicable to errors in the array response due to variations in the sensor gain and phase characteristics, unmodeled mutual coupling effects, and sensor positioning errors, and to perturbations in the assumed underlying noise covariance. Theoretical expressions for the estimation error are obtained by linking the statistical fluctuations of these quantities with the eigenvectors of the noise subspace, and then in turn with the DOA estimates themselves. A byproduct of this analysis is the development of a weighted version of MUSIC that, under the assumption of independent random perturbations to the array response, achieves estimates of minimum variance. Simulation results were presented to demonstrate the validity of the error expressions and the advantage of the optimal weighting, and to illustrate the types of applications for which this analysis would be useful. In all simulation examples, there was excellent agreement between the predicted and measured DOA estimation errors.

The performance of MUSIC is also compared with an approximation to the Cramér–Rao bound under the assumption of random model errors. Such comparisons enable one to determine if the situation warrants the additional effort required to improve performance through use of information about the model errors. This was pointed out in two of the simulation studies, where the weighted MUSIC algorithm provided performance on or very near the lower bound. Any additional computation in these cases would have resulted in insignificant performance gains. In the last simulation example, however, there was a sizable difference between MUSIC's performance and the CRB.

In a companion paper [21], the performance of multidimensional subspace-fitting algorithms will be investi-
gated under similar types of model errors. These algorithms include the (deterministic) maximum likelihood (ML) method, multidimensional (MD)-MUSIC, weighted subspace-fitting (WSF), and ESPRIT. As a somewhat surprising result of this combined analysis, it will be shown that the weighted MUSIC algorithm has lower variance than deterministic ML, MD-MUSIC, and WSF, for the simple random array perturbation model.

REFERENCES


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