A PERFORMANCE BOUND FOR MIMO-OFDM CHANNEL ESTIMATION AND PREDICTION

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ABSTRACT

The performance of a mobile MIMO-OFDM system depends on the ability of the system to accurately account for the effects of the frequency-selective time-varying channel at every symbol time and at every frequency subcarrier. In this paper, a vector formulation of the Cramér-Rao bound (CRB) for biased estimators and for functions of parameters is used to find a lower bound on the estimation and prediction error of such a system. Numerical simulations demonstrate the benefits of multiple antennas for channel estimation and prediction and illustrate the impact of calibration errors on estimation performance when using parametric channel models.

1. INTRODUCTION

Channel estimation is a central problem in the design of mobile multiple-input multiple-output (MIMO) orthogonal frequency division multiplexing OFDM, or MIMO-OFDM, systems [1, 2]. Since the MIMO-OFDM channel varies in both time and frequency, performance will depend on the ability of the system to accurately account for the effects of the variable channel at every frequency subcarrier and at every symbol time. Channel estimation in MIMO-OFDM systems is typically carried out using pilot symbols or tones to obtain estimates for a given subset of the time-frequency locations, followed by interpolation and prediction to determine the channel for the remaining times and frequencies [3–5]. Situations when it is advantageous to predict the channel include bridging the gap between the channel estimates and the current channel state, adaptive modulation, and power control.

This paper studies the theoretical performance of pilot-based channel interpolation and prediction for frequency-selective, time-fading, wireless MIMO-OFDM channels via the calculation of bounds for the interpolation and prediction error of the channel. Such bounds can serve as a standard for evaluating various estimation and prediction tech-

niques and may indicate characteristics that are necessary for optimal estimation and prediction performance. Our analysis of these bounds demonstrates that (1) better estimation and prediction performance can be obtained using MIMO systems, (2) parametric channel modeling is advantageous in terms of estimation and prediction performance, but (3) the presence of array calibration errors quickly degrades the performance of the parametric approach and necessitates the use of a more robust model. The lower bounds are derived using a vector formulation of the Cramér-Rao Bound (CRB) for functions of parameters, in a manner similar to previous work by [6–8]. A first bound assumes a model that employs directions of departure (DODs) and directions of arrival (DOAs) at the transmit and receive arrays, respectively. Instead of DOAs/DODs, the second bound uses a more robust spatial-signature representation. A comparison of the bounds highlights the advantages of using DOD and DOA information, and indicates how much calibration error may be tolerated before those advantages are lost.

The paper is organized as follows. Section 2 introduces the DOD/DOA and spatial-signature-based channel models. The performance bounds on the interpolation and prediction error are derived in Section 3. Numerical simulations of the bound are examined in Section 4, and concluding remarks are given in Section 5.

2. CHANNEL MODEL

The models considered in this paper are ray-based channel models, i.e., the models assume that the signal at the receiver is a sum of a finite number of copies of the transmitted signal, each copy experiencing its own attenuation, delay, and Doppler. The channel matrix at a particular time and frequency is given by

$$\mathbf{H}(\omega, t) = \sum_{l=1}^L \alpha_l \mathbf{a}_{r,t} \mathbf{a}_{t,l}^T e^{j((\omega_c - \omega)\tau_l - \omega_{d,l}t)} \quad (1)$$

where α_l is the complex scattering coefficient, $\mathbf{a}_{t,l}$ is the transmit array response vector of length M_t , the number of transmit antennas, $\mathbf{a}_{r,l}$ is the receive array response vector of length M_r , the number of receive antennas, $\omega_{d,l}$ is the Doppler frequency in rad/s, and τ_l is the delay in seconds, all for path l . Also, ω_c is the center or reference frequency of the band of interest, and L denotes the total number of signal paths. We use the above model over time intervals where the relative positions of the transmitter and receiver change by at most a few tens of wavelengths, and thus we assume that the given physical channel parameters are time-invariant. The time-varying phase due to the Doppler induces the multipath fading effect. This model is an extension of the narrow-band time-varying and wide-band time-invariant models of [7, 8]. One of the advantages of this model is that the channel is explicitly defined for every time and frequency. Thus, it is directly applicable to the MIMO-OFDM problem where information symbols are transmitted at particular times and frequencies.

2.1. DOD/DOA Model

For the DOD/DOA model, we assume that the array response vectors $\mathbf{a}_{t,l}$ and $\mathbf{a}_{r,l}$ in (1) are functions of the DOD and DOA, respectively, of signal path l . This model is valid for any array geometry; for example, a uniform linear array may be described using the Vandermonde structure

$$\mathbf{a}_{t,l}^T = [1 \quad e^{-j\Omega_{t,l}} \quad e^{-j2\Omega_{t,l}} \quad \dots \quad e^{-j(M-1)\Omega_{t,l}}] \quad (2)$$

where $\Omega_{t,l} = kd \sin \phi_{t,l}$ is the solid angle of path l , k is the wave number, d is the separation between antenna elements, and $\phi_{t,l}$ is either the DOD of path l at the transmitter or the DOA of path l at the receiver. While we use a scalar direction parameter to describe the DOD or DOA, this approach is easily extended to cases where the array response vectors depend on multiple parameters, including azimuth and elevation angles, polarization states, and so forth.

In general, geometric models such as the ULA model are idealized, and the actual array response vectors will be somewhat different due to various types of uncertainties (antenna position errors, mutual coupling, etc.) that we refer to as calibration errors. To account for the effects of these errors on system performance, we include array calibration error information in our model by representing the array responses in (1) as the sum of the nominal ideal response and a perturbation term: $\mathbf{a}_{t,l} + \mathbf{v}_{t,l}$ and $\mathbf{a}_{r,l} + \mathbf{v}_{r,l}$. With this addition, the model of (1) becomes

$$\begin{aligned} \mathbf{H}(\omega, t) &= \sum_{l=1}^L \alpha_l (\mathbf{a}_{r,l} + \mathbf{v}_{r,l}) (\mathbf{a}_{t,l} + \mathbf{v}_{t,l})^T e^{j\gamma_l(\omega, t)} \\ &= (\mathbf{A}_r + \mathbf{V}_r) \mathbf{X} \mathbf{W}(\omega, t) (\mathbf{A}_t + \mathbf{V}_t)^T \end{aligned} \quad (4)$$

where $\gamma_l(\omega, t) = e^{j((\omega_c - \omega)\tau_l - \omega_{d,l}t)}$,

$$\mathbf{A}_r = [\mathbf{a}_{r,1} \quad \mathbf{a}_{r,2} \quad \dots \quad \mathbf{a}_{r,L}], \quad (5)$$

$$\mathbf{X} = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_L), \quad (6)$$

$$\mathbf{W}(\omega, t) = \text{diag}(e^{j\gamma_1(\omega, t)}, \dots, e^{j\gamma_L(\omega, t)}), \quad (7)$$

$$\mathbf{A}_t = [\mathbf{a}_{t,1} \quad \mathbf{a}_{t,2} \quad \dots \quad \mathbf{a}_{t,L}], \quad (8)$$

and the calibration error matrices $\mathbf{V}_t$ and $\mathbf{V}_r$ are defined in a similar manner as $\mathbf{A}_t$ and $\mathbf{A}_r$. Stacking the columns of the $M_r \times M_t$ channel matrix $\mathbf{H}(\omega, t)$ results in

$$\mathbf{h}(\omega, t) = ((\mathbf{A}_t + \mathbf{V}_t) \otimes (\mathbf{A}_r + \mathbf{V}_r) \mathbf{X}) \text{vec}(\mathbf{W}(\omega, t)) \quad (9)$$

where $\otimes$ is the Kronecker product and $\text{vec}(\mathbf{A})$, the vectorization operator, stacks the columns of $\mathbf{A}$.

The channel model in (9) is parameterized by the L -element vectors $\text{Re}[\alpha]$, $\text{Im}[\alpha]$, τ , ω , Ω_t , and Ω_r . We will represent all of these parameters using the vector Θ . Note that the number of parameters depends only on the number of paths L , not on the sizes of the antenna arrays M_t and M_r .

2.2. Vector Spatial Signature Model

The DOD/DOA model assumes specific array configurations that depend on the parameters Ω_t and Ω_r . The estimation of these parameters can be difficult and is very sensitive to calibration errors. We can avoid these problems with the use of a more general model in which the angle (and position) dependent array responses and scattering coefficients of the DOD/DOA model are replaced by unstructured vectors, termed *spatial signatures*. In this case, the model of (1) becomes

$$\mathbf{H}(\omega, t) = \sum_{l=1}^L \mathbf{a}_{r,l} \mathbf{a}_{t,l}^T e^{j((\omega_c - \omega)\tau_l - \omega_{d,l}t)} \quad (10)$$

$$= \mathbf{A}_r \mathbf{W}(\omega, t) \mathbf{A}_t^T \quad (11)$$

or, in vectorized form,

$$\mathbf{h}(\omega, t) = (\mathbf{A}_t \otimes \mathbf{A}_r) \text{vec}(\mathbf{W}(\omega, t)). \quad (12)$$

The vectors $\mathbf{a}_{t,l}$ and $\mathbf{a}_{r,l}$ are not functions of DOD or DOA, but instead abstractly represent the array and channel responses for path l with delay τ_l and Doppler $\omega_{d,l}$. Note that, while simpler to estimate and insensitive to calibration errors, this vector spatial signature (VSS) model approximates the array response vectors as being frequency independent, which is not true for wideband signals (note that in the DOD/DOA case, the response vectors are a function of the wavenumber). The VSS model is parameterized by the L -element τ and ω , the LM_t -element $\text{Re}[\mathbf{a}_t]$ and $\text{Im}[\mathbf{a}_t]$, and the LM_r -element $\text{Re}[\mathbf{a}_r]$ and $\text{Im}[\mathbf{a}_r]$, where $\mathbf{a}_t = \text{vec}(\mathbf{A}_t)$ and $\mathbf{a}_r = \text{vec}(\mathbf{A}_r)$. Once again, we will represent all of these parameters using the vector Θ . Unlike the DOD/DOA model, the number of parameters depends on M_t and M_r , as well as L .

3. LOWER BOUND ON ESTIMATION/PREDICTION ERROR

Assume that, by means of pilot symbols or otherwise, a series of N_M channel measurements $\tilde{\mathbf{h}}(\omega_m, t_m)$ are available at time-frequency pairs $(\omega_1, t_1), \dots, (\omega_{N_M}, t_{N_M})$. These measurements are imperfect due, for example, to noise and interference present along with the training data. Thus, we model the channel measurements as a sum of the true channel $\mathbf{h}(\omega, t)$ and a Gaussian noise term due to estimation error, so that the channel measurement at (ω_m, t_m) is given by

$$\tilde{\mathbf{h}}(\omega_m, t_m) = \mathbf{h}(\omega_m, t_m) + \mathbf{n}(\omega_m, t_m) \quad (13)$$

where the $M_t M_r \times 1$ Gaussian noise term is distributed as $\mathbf{n}(\omega_m, t_m) \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_m)$. These noisy channel “samples” are what would be used to estimate the channel for other values of (ω, t) . Stacking the N_M measurements into a single vector, we have

$$\tilde{\mathbf{h}} = [\tilde{\mathbf{h}}(\omega_1, t_1)^T \dots \tilde{\mathbf{h}}(\omega_{N_M}, t_{N_M})^T]^T = \mathbf{h} + \mathbf{n} \quad (14)$$

where $\mathbf{n}$ is a $N_M M_t M_r$ -length stacked noise vector with covariance $\mathbf{C}$. Although our analysis is general enough to accommodate an arbitrary covariance matrix $\mathbf{C}$, for simplicity in presentation we will assume the channel measurement error to be spatially and temporally white, so that $\mathbf{C} = \sigma \mathbf{I}$ where $\mathbf{I}$ is an $N_M M_t M_r \times N_M M_t M_r$ identity matrix and σ is the variance. In computing the channel estimation bounds we will assume that σ is an unknown parameter that must itself be estimated. Thus, σ must be added as a parameter to the models in the previous section.

At any particular time and frequency (ω, t) , the estimation error may be expressed as

$$\mathbf{e}(\omega, t) = \mathbf{h}(\omega, t; \hat{\Theta}) - \mathbf{h}(\omega, t; \Theta). \quad (15)$$

For clarity, we have explicitly included the Θ and $\hat{\Theta}$ dependence in (15). However, for notational simplicity, we continue to omit this dependence elsewhere and write the channel estimate as $\hat{\mathbf{h}}(\omega, t)$. In what follows, we find lower bounds on the error covariance matrix of estimators $\hat{\mathbf{h}}$ via the Cramér-Rao bound (CRB). Using a vector formulation of the CRB for biased estimates and for functions of parameters (similar to those found in [9]), the bound may be written as

$$E[\mathbf{e}(\omega, t)\mathbf{e}(\omega, t)^H] \geq \mathbf{H}'\mathbf{B}\mathbf{H}'^H + \mathbf{b}\mathbf{b}^H \quad (16)$$

where the matrix inequality $\mathbf{F} \geq \mathbf{G}$ indicates that the matrix difference $\mathbf{F} - \mathbf{G}$ is positive semi-definite, $\mathbf{B}$ is the CRB matrix with respect to the P parameters Θ , and $\mathbf{H}'$ is the Jacobian matrix

$$\mathbf{H}' = \begin{bmatrix} \frac{\partial(\mathbf{h}(\omega, t) + \mathbf{b}(\omega, t))}{\partial \theta_1} & \dots & \frac{\partial(\mathbf{h}(\omega, t) + \mathbf{b}(\omega, t))}{\partial \theta_P} \end{bmatrix}. \quad (17)$$

The vector $\mathbf{b}$ is a time and frequency dependent bias term that accounts for the effects of the calibration errors on the bound. Matrix $\mathbf{B}$, the CRB for Θ , may be calculated by applying Bangs formula [10]

$$[\mathbf{B}^{-1}]_{ij} = \text{Tr} \left[\mathbf{C}^{-1} \frac{\partial \mathbf{C}}{\partial \theta_i} \mathbf{C}^{-1} \frac{\partial \mathbf{C}}{\partial \theta_j} \right] + 2\text{Re} \left[\frac{\partial \mathbf{h}^H}{\partial \theta_i} \mathbf{C}^{-1} \frac{\partial \mathbf{h}}{\partial \theta_j} \right] \quad (18)$$

to the sampled channel model $\mathbf{h}$.

Once the CRB is known, the sum of variances of the elements of estimation error vector $\mathbf{e}(\omega, t)$ may be bounded by

$$E[\|\mathbf{e}(\omega, t)\|_2^2] \geq \text{Tr}[\mathbf{H}'\mathbf{B}\mathbf{H}'^H + \mathbf{b}\mathbf{b}^H] \quad (19)$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Note that even though $\mathbf{B}$ depends on the N_M channel measurements, this expression is valid for any (ω, t) pair. That is, once the model parameters Θ are estimated, the bound may be evaluated for any (ω, t) .

3.1. DOD/DOA CRB

For the DOD/DOA model, the bias is given by

$$\mathbf{b}(\omega, t) = (\mathbf{A}_t \otimes \mathbf{A}_r \mathbf{X}) \text{vec}(\mathbf{W}(\omega, t)) - \mathbf{h}(\omega, t). \quad (20)$$

This represents a channel estimator that attempts to estimate the channel as though there were no calibration errors present.

Using the tools given in (16)-(18), the CRB and the bound on the estimation error may now be found. The evaluation of the derivatives in (17) and (18) is straightforward, but also involved and not necessarily insightful. Therefore, in the interest of space, the derivation of the CRB will not be given in this paper. The authors may be contacted if the reader is interested in further detail.

3.2. VSS CRB

In the VSS model, calibration errors, if present, are incorporated into the spatial signature and do not produce a bias, i.e., $\mathbf{b} = \mathbf{0}$. The equations given above may now be used to find the CRB and estimation error bound. As with the DOD/DOA model, the derivation of the results are not included, but may be obtained from the author.

4. NUMERICAL SIMULATIONS

Given the CRBs described in the previous section, we now explore the limiting performance of MIMO-OFDM channel estimation and prediction by numerically evaluating the derived bounds for a few scenarios. In the examples we use the following scalar Root Mean Square Error (RMSE) performance measure:

$$\sqrt{\frac{E[\|\mathbf{e}(\omega, t)\|_2^2]}{E[\|\mathbf{H}(\omega, t)\|_F^2]}} \geq \sqrt{\frac{\text{Tr}[\text{CRB}(\omega, t)]}{LM_t M_r}} \quad (21)$$

where the simplified right-hand denominator is derived by assuming independent zero-mean Gaussian path amplitudes α . We refer to the quantity on the right hand of (21) as the normalized error bound.

In the simulations, we assume the channel is given by the model in (3). The results are obtained by averaging over 500 independent channel realizations with $L = 6$, the measurement noise power equal to -20 dB per receive antenna, and the other parameters selected as follows. The scattering parameters α are generated as independent circular-symmetric complex Gaussian random variables distributed as $\alpha_l \sim \mathcal{CN}(0, 1)$. The path delays τ are selected from an exponential distribution such that approximately 98% of the τ_l fall in the delay range $0.042\mu s$ to $0.42\mu s$. The physical DODs and DOAs are drawn uniformly so that $\phi_{t,i}, \phi_{r,i} \sim U[0, 2\pi)$, and the solid angles are given by the formula $\Omega_{t,i} = kd \sin \phi_{t,i}$ with $d = \lambda_c/2$, where λ_c is the wavelength at ω_c and k is the wavenumber. The Doppler frequency of path l is $\omega_{d,l} = k \frac{\Delta s}{T_s} \sin \phi_{d,l}$, where $\phi_{d,l}$ is the angle between the propagation path l and the direction of array motion, and Δs and T_s are the distance and time separating consecutive channel measurements, respectively, so that $\frac{\Delta s}{T_s}$ is the rate of motion of the antenna array, which we choose as 5 m/s for the simulations. We assume $\phi_{d,l} \sim U[0, 2\pi)$. The pilot-based channel estimates are arranged in a grid with 16 measurements in frequency by 32 in time. Finally, the calibration errors are distributed as $\text{vec}(V_t) \sim \mathcal{CN}(0, \sigma_t \mathbf{I})$ and $\text{vec}(V_r) \sim \mathcal{CN}(0, \sigma_r \mathbf{I})$.

We begin by examining the impact of the array sizes M_t and M_r on the normalized error bounds. Figure 1 displays a frequency slice of the DOD/DOA and VSS error bounds for a SISO, a 1×2 single input multiple output (SIMO), and 2×2 and 3×3 MIMO configurations. Note that DOD/DOA and VSS models are not uniquely identifiable for the SISO case since the array parameters cannot be estimated. Practically, however, the DOD/DOA and VSS models reduce to the same identifiable SISO channel, and their performance is identical for this scenario. It is clear that significant gains in channel estimation performance may be achieved through the use of an increased number of antennas at the transmitter and receiver. The one exception to this in the plot is the 1×2 SIMO VSS system, whose bound is higher than for the SISO system as a result of the extra and unnecessary intermediate step of estimating the transmit array element. The SIMO DOD/DOA bound was formulated to omit this extra step, and therefore does not suffer the same penalty. Overall, these results are in harmony with those obtained with the wideband time-invariant bounds developed in [8].

Nearly identical results are seen in the estimation portion of the position slice in Fig. 2. Even greater benefits due to MIMO arrays are seen in the prediction portion of this plot, i.e., the region to the right of the zero-wavelength mark. The results indicate that both the DOD/DOA and

Fig. 1. Lower bound for frequency estimation.

VSS MIMO systems may be predicted much farther into the future than the corresponding SISO and SIMO systems. As was suggested in [7], this increase in performance is intuitively explained by noting that the larger arrays reveal more of the underlying channel structure, allowing for a better characterization of the channel parameters. This advantage is maintained even when the number of channel measurements N_M is adjusted to be proportional to $1/M_t$, allowing for a fairer comparison for the given receive CNR of -20 dB. Also included in this plot is an example of the average normalized error performance when 2D cubic interpolation is used to estimate the channel from the measurement segment. The low points in the curve correspond to the locations of the channel measurements in time. It is clear that estimation of the channel through a parametric approach offers dramatic gains over simple unstructured interpolation schemes.

In Fig. 3, we examine the sensitivity of the DOD/DOA and spatial signature models to calibration error. Each point in the curves in this plot represent the lowest bound point from the 3D error bound surfaces, which occur at $(0 \text{ MHz}, -5\lambda)$. The results in the figure demonstrate the robustness of the VSS model with respect to calibration errors; the VSS performs equally well regardless of the underlying array structure. This is a significant advantage of this model, particularly in situations when the array structure may be in doubt or calibration errors are present. The DOD/DOA model, on the other hand, is shown to be extremely sensitive to even small amounts of calibration error. Note that estimating the calibration error as part of the model estimation process results in performance at best equivalent to that of the VSS model. These results suggest that unless the calibration errors in a system can be accurately accounted for prior to channel estimation, the DOD/DOA model should be

Fig. 2. Lower bound for time estimation and prediction.

abandoned in favor of the VSS model for parametric channel estimation. Also included in the plot as a reference is the average normalized error performance achieved when using cubic interpolation to estimate the channel from the channel measurements.

Fig. 3. Effect of calibration error on lower bound.

5. CONCLUSIONS

This paper presented lower bounds on channel estimation and prediction performance for mobile wideband MIMO-OFDM systems. The bounds, derived using a special formulation of the CRB, demonstrate the potential benefits of using antenna arrays in OFDM systems. They also support the conclusion that, when suitable, the use of parametric channel modeling provides a significant advantage over un-

structured interpolation schemes in terms of estimation and prediction performance. Furthermore, our results illustrate that parametric channel models based on spatial signatures potentially offer a robust compromise between interpolation/prediction with unstructured models and models that require accurate array calibration.

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