A TIME DOMAIN METHOD FOR JOINT ESTIMATION OF TIME DELAYS, DOPPLER SHIFTS AND SPATIAL SIGNATURES

Andreas Jakobsson
Systems and Control Group
Box 27, SE-751 03 Uppsala
Sweden.

ABSTRACT

This paper presents an efficient algorithm for estimating the time delays, Doppler shifts and spatial signatures of a known waveform received via several distinct paths by an array of antennas. The algorithm is based on the Signal Subspace Fitting approach. Unlike the approach recently presented in [1], this paper uses a more accurate time domain model for the problem. Simulation examples are included to illustrate the algorithm's performance relative to the Cramer-Rao bound (which is also derived).

1. INTRODUCTION

The problem of estimating the time delays, Doppler shifts and spatial signatures of a known signal is a central problem in many fields including radar, sonar and communications. The recent literature considers primarily the case of multiple backscatters of non-moving reflectors (see, e.g., [2, 3, 4, 5]). However, in the applications mentioned the scatterers are often rapidly moving, and taking the Doppler shift into account provides more accurate time delay estimates, as well as information about the position and relative motion of the reflecting objects. This more involved problem has attracted relatively little attention in the literature (see, e.g., [1, 6, 7, 8]). In [1] a commonly used approximate Doppler shift model is used, and the time delays are estimated by transforming this data model to the frequency domain. The model used in this paper avoids using this approximate Doppler model, and consequently yields more accurate results.

This work was supported in part by the Swedish Institute, the Royal Swedish Academy of Sciences, and by the Office of Naval Research under grant N00014-96-1-0954.

1 Andreas Jakobsson is currently on leave at Dept. of Elec. & Comp. Engineering, Brigham Young University, Provo, UT 84602.

Suppose an antenna receives several scaled, time-delayed, and Doppler-shifted copies of a known transmitted baseband signal. The received signals could, for instance, be the echoes from a pulse transmitted by an active sonar, or they could result from a training sequence sent over a multipath communication channel. In either case, we may model the received signal, $s_R(t)$, as a delayed and time scaled version of the transmitted signal, $s(t)$, (see, e.g., [9]):

$$s_R(t) = s(t - \tau + \epsilon t)$$

where $\tau$ and $\epsilon$ are the time-delay and Doppler parameters defined as

$$\tau = \frac{2R_0}{c}$$
$$\epsilon = -\frac{2\dot{R}}{c}$$

and $c$, $R_0$ and $\dot{R}$ are respectively the propagation speed, the range at $t = 0$ and the range rate.

Given an array of $m$ antennas, the output of the array can be expressed in vector form as

$$y(t) = x(t) + n(t)$$

where

$$x(t) = \sum_{k=1}^{d} a_k \delta(t - \tau_k + \epsilon_k t),$$

and $d$ and $a_k$ represent the number of different multipath arrivals and the spatial signature of the $k$:th arrival, respectively. The additive noise vector, $n(t)$, is assumed to be a zero mean temporally and spatially white noise process with covariance $\sigma^2I$. The standard narrowband assumption is employed here; i.e., the propagation time of the signal across the array is assumed to be much less than the reciprocal of the signal bandwidth. To simplify the problem, we do not use an explicit parameterization of the spatial response.
in terms of directions of arrival (DOA), but instead treat the elements of \( \mathbf{a}_k \) as deterministic parameters to be estimated. This allows us to consider a cluster of coherent arrivals that share a given time delay and Doppler shift, without the necessity of estimating the number of such arrivals nor their individual DOAs and amplitudes. In addition, this assumption eliminates the need for an accurately calibrated array.

Under the assumption that the Doppler shifts are "small", it is possible to simplify the dependence of (5) on the Doppler parameters by neglecting the higher order terms in a Taylor series expansion of \( s(t - \tau_k + \epsilon_k t) \):

\[
s(t - \tau_k + \epsilon_k t) \approx s(t - \tau_k) + \epsilon_k t d(t - \tau_k)
\]

where \( d(t) = \frac{\partial s(t)}{\partial t} \).

Assuming that \( x(t) \) is an \( m \times 1 \) column vector, and that a total of \( N \) snapshots are collected from the array at time instances \( t_1, \ldots, t_N \), the data may be arranged in matrix form as

\[
Y = (S + T \mathbf{D} \mathbf{A}) \mathbf{A} + \mathbf{N}
\]

where \( \mathbf{D} = [ \mathbf{d}_1 \ldots \mathbf{d}_d ] \).

3. SIGNAL SUBSPACE FITTING

In this section, we present a computationally efficient Signal Subspace Fitting (SSF) algorithm that approximates the maximum likelihood estimator of \( \tau \), \( \epsilon \), and \( \mathbf{A} \). The estimator is derived similarly to the SSF estimator presented in [1] and is thus presented in a somewhat condensed form. The SSF estimates of the delays and Doppler shifts can be found by minimizing [10, 11, 12]

\[
V_{SSF}(\tau, \omega) = \text{tr} \left\{ \Pi_Q^2 \tilde{\mathbf{E}}_s \mathbf{W} \tilde{\mathbf{E}}_s^* \right\},
\]

where \( \text{tr} \{ \cdot \} \) and \( \{ \cdot \}^* \) denote the trace operator and the conjugate transpose, \( \Pi_Q \) is the orthogonal projection matrix

\[
\Pi_Q = I - Q (Q^* Q)^{-1} Q^*,
\]

\( \tilde{\mathbf{E}}_s \) is the matrix whose columns are the left singular vectors corresponding to the \( d \) largest singular values of \( Y \), and \( \mathbf{W} \) is a diagonal weighting matrix. In the simulations presented later, we use the stochastic ML weighting

\[
\mathbf{W} = (\tilde{\mathbf{A}}_s - \sigma^2 \mathbf{I})^2 \tilde{\mathbf{A}}_s^{-1},
\]

where \( \tilde{\mathbf{A}}_s \) is a diagonal matrix formed from the \( d \) largest squared singular values of \( Y \), and \( \sigma^2 \) is a consistent estimate of the noise variance (obtained, for example, as the average of the \( m - d \) smallest non-zero squared singular values of \( Y \)).

The Doppler parameters can be explicitly estimated using SSF, but only for the case where \( d < N/2 \), which is not a serious restriction in most cases [1]. Let \( \mathbf{C} = [ \mathbf{S} \quad -T \mathbf{D} ] \), and suppose that \( d < N/2 \) and that \( \mathbf{C} \) has full column rank. Introduce

\[
\mathbf{V}^{-1} = [ \mathbf{A} \quad \mathbf{I} ] (\mathbf{C}^* \mathbf{C})^{-1} [ \mathbf{A} \quad \mathbf{I} ],
\]

and let \( \Gamma_{ij} \) be the \( d \times d \) blocks of the matrix

\[
\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12} & \Gamma_{22} \end{bmatrix} \triangleq \mathbf{C}^* \tilde{\mathbf{E}}_s \mathbf{W} \tilde{\mathbf{E}}_s^* \mathbf{C}^*.
\]

where \( \cdot^\dagger \) denotes the pseudo-inverse. Also, let \( \mathbf{a} \) be the vector formed from the real part of the diagonal elements of \( \Gamma_{12} \), and define

\[
\Omega = \Re \left( \tilde{\mathbf{V}} \odot \Gamma_{11}^\dagger \right)
\]

\[
\rho = \text{tr} \left\{ \Gamma_{22} \tilde{\mathbf{V}} \right\},
\]

where \( \Re \{ \cdot \} \) and \( \odot \) denote the real part and the Schur-Hadamard product, respectively. Then, minimization of (19) with respect to \( \omega \) yields (see [1] for more details)

\[
\hat{\epsilon} = -\Omega^{-1} \alpha.
\]

Inserting (27) into the cost function leads to the following criteria for estimating \( \tau \):

\[
\hat{\tau} = \arg \min_{\tau} \rho - \alpha^T \Omega^{-1} \alpha - \text{tr} \left\{ \mathbf{C}^* \tilde{\mathbf{E}}_s \mathbf{W} \tilde{\mathbf{E}}_s^* \right\}.
\]
Thus the SSF algorithm is implemented by first performing
the $d$-dimensional search over $\tau$ in (28), and by then solv-
ing for $\hat{e}$ using (27). Note that the computation required to
evaluate the SSF criterion can be significantly simplified by
performing the trace calculation in (28) as

$$\text{tr} \left\{ CC^\dagger \hat{E} \hat{W} \hat{E}^\dagger \right\} = \text{tr} \left\{ \left( C^\dagger \hat{E} \hat{W} \right) \left( \hat{E}^\dagger C \right) \right\}. \quad (29)$$

It should be noted here that the algorithm requires an
initial estimate of the multipath parameters. In the simu-
lations presented in the next section we have used an ini-
tial Doppler estimate of zero, and a fast ESPRIT-based es-
timator presented in [5] for estimating time-delays in cases
when the Doppler offset is zero. Our empirical results indi-
cate that this approach still gives reasonable time delay
estimates even when the Doppler is non-zero but small. The
fact that the algorithm yields the desired estimates in closed
form (i.e., without search) makes it an attractive alternative
for initialization.

4. SIMULATION RESULTS

In this section we study the performance of the estimator
as the signal-to-noise ratio (SNR) varies. Simulation data
was generated using (4) for two multipath signals ($d = 2$)
with time-delays $\tau = [0.5\ 3]^T$ and Doppler shifts
$\epsilon = [0.001\ 0.005]^T$. The data was corrupted by spa-
tially and temporally white circular Gaussian noise with
zero mean and standard deviation $\sigma$. The two columns of
the signature matrix, $A$, were given by the array response
of a 5-element, half-wavelength spaced uniform linear array
with DOAs $0^\circ$ and $20^\circ$. The signal sequence was chosen to
be the raised cosine pulse

$$s(t) = \frac{\text{sinc}(t/T)}{1 - (t/T)^2}.$$ 

For the simulations presented here, $T = 5$, and $N = 101$
samples are assumed to be taken from the array.

The root mean squared error (rMSE) of the time-delay
and Doppler estimates were calculated for the algorithm
based on 200 Monte Carlo trials for various SNR values.
The resulting rMSE for the first time delay estimate (the
second time-delay estimate behaves similarly) are plotted in
Figure 1, together with the appropriate Cramér-Rao Bound
(CRB). See Appendix A for a derivation of the CRB. As can
be seen from the figure, the SSF estimates are found to
be efficient at about SNR = 10 dB. For SNRs below 0 dB the
Doppler-shift estimate was found to be biased towards zero,
which is the reason why its rMSE is actually below the CRB
in this region.

![Figure 1: The rMSE of the first time-delay estimate, $\hat{\tau}_1$, for the SSF and the ESPRIT-based estimators as a function of the SNR.](image1)

Figure 2 shows the rMSE for the first Doppler shift esti-
mate with the corresponding CRB as a function of SNR. As
can be seen from the figure, the SSF estimates are found to
be efficient at about SNR = 0 dB. For SNRs below 0 dB the
Doppler-shift estimate was found to be biased towards zero,
which is the reason why its rMSE is actually below the CRB
in this region.

![Figure 2: The rMSE of the first Doppler-shift estimate, $\hat{\epsilon}_1$, for the SSF estimator as a function of the SNR.](image2)
A. THE CRAMÉR-RAO BOUND

In this Appendix, we derive the CRB for the current estimation problem. Specifically, we derive the elements of the Fisher information matrix, $I(\theta)$, whose inverse yields the CRB. By definition (see, e.g., [13], Appendix B):

$$[I(\theta)]_{k,p} = -\mathbb{E} \left[ \frac{\partial^2 \ln p(Y; \theta)}{\partial \theta_k \partial \theta_p} \right] $$

where

$$\theta = [T^T \ e^T \ \text{vec} \ \Re \{A\}]^T \ (\text{vec} \ \Im \{A\})^T $$

and $\mathbb{E}[\cdot], \Re \{A\}$ and $\text{vec}(\cdot)$ denote expectation, the imaginary part of the spatial signature matrix $A$, and the vectorization operator, respectively. From (4) the log-likelihood function is given by

$$\ln p(Y; \theta) = \prod_{t=t_1}^{t_N} \ln p(y(t); \theta) = K - \frac{1}{\sigma^2} \sum_{t=t_1}^{t_N} \|e(t)\|^2 ,$$

where $K$ is a constant (with respect to $\theta$), $\| \cdot \|$ denotes the Euclidean norm, and

$$e(t) \triangleq y(t) - \mathbb{E}[x(t)] \triangleq y(t) - \mu(t) .$$

Thus, by straightforward differentiation, we get

$$[I(\theta)]_{k,p} = \mathbb{E} \left[ \sum_{t=t_1}^{t_N} \frac{\partial^2}{\partial \theta_k \partial \theta_p} \left\{ \frac{1}{\sigma^2} \|e(t)\|^2 \right\} \right]$$

$$= \frac{2}{\sigma^2} \sum_{t=t_1}^{t_N} \mathbb{E} \left\{ \frac{\partial \mu(t)^*}{\partial \theta_k} \frac{\partial \mu(t)}{\partial \theta_p} \right\}$$

where the partial derivatives with respect to $\tau_k$ and $\epsilon_k$ are given by

$$\frac{\partial \mu(t)}{\partial \tau_k} = -a_k d(t - \tau_k + \epsilon_k t)$$

$$\frac{\partial \mu(t)}{\partial \epsilon_k} = a_k d(t - \tau_k + \epsilon_k t).$$

The partial derivatives with respect to the real and imaginary parts of the $k,p$:th element of $A$ can also easily be found.

B. REFERENCES


