An Analysis of MUSIC and Root-MUSIC in the Presence of Sensor Perturbations

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Abstract—MUSIC and root-MUSIC are two important and popular algorithms for estimating the directions-of-arrival (DOA) of narrowband plane waves impinging on an array of sensors. Both techniques require a complete knowledge of the array response in all directions of interest. For the case of MUSIC, the array response is measured and stored. For root-MUSIC, the response is known analytically since a uniform linear array of identical sensors is assumed. In practice, the response of a real antenna array will differ from its nominal value (measured or predicted) due to measurement errors and changes in weather, the surrounding environment, and sensor location. The result of these modeling errors is that when the algorithms are actually applied, estimation performance may be seriously degraded. The goal of this paper is to analyze the performance of MUSIC and root-MUSIC when the response of the sensor array is perturbed from its nominal value. Theoretical expressions for the error in the DOA estimates are derived and compared with simulations performed for several representative cases.

I. INTRODUCTION

Within the class of so-called signal-subspace algorithms for direction of arrival (DOA) estimation, MUSIC [1] has been the most widely studied [2, 3]. Its popularity stems primarily from its generality; for example, in the DOA estimation problem it is applicable to arrays of arbitrary (but known) configuration and response, and can be used to estimate multiple parameters (e.g., azimuth, elevation, range, polarization, etc.) per source. The price paid for this generality is that the array response must be measured (a process known as array calibration) and stored for all possible combinations of source parameters. In practice, the calibration measurements are subject to error in both gain and phase, and the array response may change with time as a function of weather, the surrounding environment, location, etc. The result is that when the algorithm is actually applied, the array response may differ substantially from its nominal measured value and algorithm performance may be significantly degraded.

For the special case of a uniform linear array (ULA) of identical sensors, the array response for a particular DOA is Vandermonde. This analytical characterization of the physical array eliminates, in principle, the need for calibration. In addition, instead of a spectral search as required by MUSIC, the DOA estimates may be obtained by rooting a certain symmetric polynomial. This procedure is known as the root-MUSIC algorithm [4, 5], and is preferred over MUSIC for cases involving ULAs. While no calibration per se is required for root-MUSIC, deviations in the sensor array from its nominal ULA structure will lead, as in the case of MUSIC, to degraded parameter estimates.

Most of the analysis performed to date for MUSIC and root-MUSIC [2, 3, 5] has been concerned with the effects of estimating the array covariance matrix by its sample average. While techniques have been proposed to mitigate the effects of antenna modeling errors for these algorithms [6, 7, 8, 9], little work has focused on obtaining analytical expressions for the error induced in the DOA estimates by such perturbations. In this paper, such a sensitivity analysis is undertaken. The analysis is applicable to both angle-dependent and angle-independent sensor errors, including gain and phase perturbations and mutual coupling effects. It is shown how these array errors lead to errors in the projection of the “noise-subspace” eigenvectors into the “signal-subspace.” Once this connection is made, arguments similar to those in [3, 5] are used to obtain expressions for the errors in the DOA estimates. Simulations are carried out for several representative examples to validate the analysis.

Friedlander [10] has also studied the performance of MUSIC in the presence of sensor errors. Although the analysis methodology used in [10] is somewhat different than that presented herein, both procedures can be shown to yield identical results for certain special cases.

II. MUSIC AND ROOT-MUSIC

We begin by briefly describing the unperturbed data model assumed for the narrowband DOA estimation problem, as well as the notion of signal and noise subspaces. Assume an m-element array of sensors, d narrowband far-field emitters, and define \( a(\theta) \in \mathbb{C}^m \) to be the \( m \times 1 \) response for a narrowband emitter at DOA \( \theta \).

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The array manifold is defined to be the set \( \mathcal{A} = \{ a(\theta) : \theta \in \Theta \} \) for some region \( \Theta \) in DOA space. The set \( \mathcal{A} \) is assumed to be known, either analytically or via some calibration procedure. The output \( \mathbf{z}(t) \in \mathbb{C}^{m} \) of the array at time \( t \) is given by
\[
\mathbf{z}(t) = \mathbf{A}s(t) + \mathbf{n}(t),
\]
where \( s(t) \in \mathbb{C}^{d} \) is the amplitude and phase of the signals at time \( t \), \( \mathbf{n}(t) \) is additive noise, and where
\[
\mathbf{A} \stackrel{\text{def}}{=} [a(\theta_1) \cdots a(\theta_d)].
\]

The decomposition of \( \mathbb{C}^{m} \) into signal and noise subspaces is typically achieved via an eigendecomposition of the corresponding covariance matrix \( \mathbf{R}_{xz} \). Assuming for simplicity that the noise is spatially white and uncorrelated with the signals, we have
\[
\mathbf{R}_{xz} = \mathcal{E}\{\mathbf{z}(t)\mathbf{z}^H(t)\} = \mathbf{A}\mathbf{R}_{xx}\mathbf{A}^* + \sigma^2 \mathbf{I},
\]
where \( \mathcal{E} \) denotes expectation, \( \mathbf{R}_{xx} \) is the covariance matrix of the emitter signals, and \( \sigma^2 \) is the noise variance at each sensor. The covariance \( \mathbf{R}_{xz} \) is assumed to be full rank \( d \) (no unity correlated signals) and the columns of \( \mathbf{A} \) are assumed to be linearly independent. The eigendecomposition of \( \mathbf{R}_{xz} \) has the following form:
\[
\mathbf{R}_{xz} = \sum_{i=1}^{m} \lambda_i \mathbf{e}_i^H \mathbf{e}_i = \mathbf{E}_x \mathbf{A}^* \mathbf{E}_s^* + \sigma^2 \mathbf{E}_n \mathbf{E}_n^H
\]
where \( \mathbf{E}_x = [\mathbf{e}_1 \cdots \mathbf{e}_d] \), \( \mathbf{E}_n = [\mathbf{e}_{d+1} \cdots \mathbf{e}_m] \), and \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > \lambda_{d+1} = \cdots = \lambda_m = \sigma^2 \).

The span of the \( d \) eigenvectors \( \mathbf{e}_i \) defines the signal subspace, and the orthogonal complement spanned by \( \mathbf{E}_n \) defines the noise subspace. This terminology is a consequence of the fact that \( \text{span}(\mathbf{E}_x) = \text{span}(\mathbf{A}) \perp \text{span}(\mathbf{E}_n) \). The basis of the MUSIC algorithm is that, since \( \mathbf{A}^*\mathbf{E}_n = 0 \), \( i = 1, \ldots, d \), with a perfect covariance measurement the DOAs could be obtained by finding the zeros of the following function, often referred to as the *null-spectrum*:
\[
f(\theta) \stackrel{\text{def}}{=} a^*(\theta)\mathbf{E}_n^H \mathbf{E}_n a(\theta).
\]

Since in practice \( f(\theta) \neq 0 \) due to errors in \( \mathbf{E}_n \) or \( a(\theta) \), the MUSIC algorithm estimates the DOAs at those values of \( \theta \) which minimize \( f(\theta) \).

For the special case of a uniform linear array (ULA) of identical sensors, the array manifold is comprised entirely of Vandermonde vectors, i.e., the array response in direction \( \theta \) may be written as
\[
a(\theta) = n(\theta) \stackrel{\text{def}}{=} [1 : z : z^2 : \cdots : z^{m-1}]^T,
\]
where \( z = \exp[-j2\pi \Delta \sin(\theta)/\lambda] \). \( \Delta \) is the inter-element separation, and \( \lambda \) is the wavelength of the signal. Consequently, the function \( f(\theta) \) may be written as a polynomial in \( z \) of order \( 2m-2 \), as follows:
\[
f(\theta) = a^*(\theta)\mathbf{E}_n^H \mathbf{E}_n a(\theta) = \sum_{i=d+1}^{m} c_i(z^{-1})c_i^*(z),
\]
where \( c_i(z) \stackrel{\text{def}}{=} c_i^* n(\theta) \) and \( c_i(z^{-1}) \stackrel{\text{def}}{=} a_i^* a(\theta) \). Note that, since only values of \( z \) on the unit circle are of interest, the simplifying assumption \( a_i^*(z) = a_i^*(z^{-1}) \) is made. This polynomial will clearly have roots on the unit circle at values of \( z \) which correspond to the true angles of arrival. However, in the presence of noise or modeling errors, the roots corresponding to the true DOAs will no longer lie on the unit circle. To isolate the \( d \) roots of interest from the \( 2m-2 \) zeros of the polynomial, the root-MUSIC algorithm simply chooses the roots with modulus nearest unity from among those inside the unit circle (only roots inside the unit circle need be considered since all roots occur in conjugate reciprocal pairs).

Although it can be shown that both MUSIC and root-MUSIC have the same asymptotic second-order performance for small amounts of data root-MUSIC has lower finite sample bias than MUSIC and does not exhibit a threshold effect (i.e., loss of resolution). Thus, for ULAs, root-MUSIC is usually the method of choice. In the perturbation analysis which follows, it will be shown that for small deviations in the array response, the expected error in the DOA estimates for MUSIC and root-MUSIC will also be identical. However, as before, because of MUSIC's bias and threshold effects, simulations demonstrate that root-MUSIC provides slightly better performance.

### III. Perturbation Model

To isolate the effects of sensor errors on the DOA estimates, it will be assumed that the finite sample effects due to additive noise are negligible. With this assumption, a general model for the perturbed covariance \( \mathbf{R} \) is
\[
\mathbf{R} = (\mathbf{A} + \hat{\mathbf{A}}) \mathbf{R}_{xz} (\mathbf{A} + \hat{\mathbf{A}})^* + \sigma^2 \mathbf{I},
\]
where \( \hat{\mathbf{A}} \) is the error in the nominal array response. Since arbitrary scaling of the columns of \( \mathbf{A} \) does not lead to estimate errors, one element in each column of \( \hat{\mathbf{A}} \) may be assumed to be zero.

For both MUSIC and root-MUSIC, we are primarily interested in how the presence of \( \hat{\mathbf{A}} \) affects the noise subspace. Pick a set of orthonormal vectors \( \mathbf{E}_n \) which span the noise subspace, and let \( \hat{\mathbf{E}}_n = \mathbf{E}_n + \hat{\mathbf{E}}_n \) represent the perturbed orthonormal spanning set. Note that the minimal eigenvalue of \( \hat{\mathbf{R}} \) will still be \( \sigma^2 \), and will still have multiplicity \( m-d \). Consequently, the spanning set of the noise subspace is indeterminant, and the perturbation \( \hat{\mathbf{E}}_n \) must be entirely in the signal subspace. The following sequence of equations establishes the link between \( \hat{\mathbf{A}} \) and \( \hat{\mathbf{E}}_n \):
\[
\mathbf{RE}_n = \sigma^2 \mathbf{E}_n
\]

\(^2\)At low information-to-noise ratios (INR), root-MUSIC does have a performance threshold that results when the algorithm chooses a spurious root of the polynomial. However, this effect is manifest at much lower INR than the threshold of MUSIC.
\[
\begin{align*}
\Rightarrow (A + \hat{A})R_{xx}(A + \hat{A})^* \tilde{E}_n &= 0 \\
\Rightarrow AR_{xx}A^* \tilde{E}_n + A^* \tilde{E}_n &= O(||\hat{A}||^2) \\
\Rightarrow A^* \tilde{E}_n &\approx -\hat{A}^* \tilde{E}_n. 
\end{align*}
\]

(5)

As one would expect, only those columns of \( \hat{A} \) with components in the noise subspace lead to errors in \( \tilde{E}_n \).

There are a variety of models that could be used to describe the matrix of sensor errors \( \hat{A} \). One possibility is to assume that each of the columns of \( \hat{A} \) is a zero-mean complex Gaussian random vector with known covariance. However, with this type of model it is difficult to control gain and phase errors independently and to incorporate mutual coupling effects. To see how a more general model may be obtained, suppose that \( g_k \) and \( e^{j\phi_k} \) are respectively the nominal gain and phase response of the \( k \)th sensor to a signal from direction \( \theta \); i.e., \( n_k(\theta) = g_k e^{j\phi_k} \).

The perturbed response \( \tilde{n}_k(\theta) = n_k(\theta) + \hat{n}_k(\theta) \) may be written as

\[
\begin{align*}
\tilde{n}_k(\theta) &= (g_k + \hat{g}_k)e^{j(\phi_k + \hat{\phi}_k)} \\
&= g_k e^{j\phi_k} + \hat{g}_k e^{j\phi_k} + j(g_k + \hat{g}_k)\hat{\phi}_k \approx g_k e^{j\phi_k} + j(g_k + \hat{g}_k)\hat{\phi}_k, \\
&= n_k(\theta) + p_k(\theta)\hat{n}_k(\theta),
\end{align*}
\]

(6)

where \( p_k(\theta) \triangleq \frac{[g_k + j(g_k + \hat{g}_k)\hat{\phi}_k]}{g_k} \). The \( m \times d \) matrix \( \hat{A} \) is thus described by the equation

\[
\hat{A} = [p_1(\theta_1), p_2(\theta_2), \ldots, p_d(\theta_d)],
\]

(7)

where \( P_i = \text{diag}(p_1(\theta_i), \ldots, p_d(\theta_i)) \). Though for simplicity we have written \( P_i \) as a diagonal matrix, the off-diagonal elements may be non-zero in cases involving mutual coupling between sensors [9].

The model of equation (7) is general in the sense that it applies to angle-dependent sensor errors, i.e., the perturbations \( P_1, \ldots, P_d \) are different for each DOA. If the deviations from the nominal response are due to bulk delay and gain errors in the antenna receiver electronics, or if the sources are grouped closely in angle, the errors may be assumed to be angle-independent. In such cases, the approximation \( \hat{A} \approx PA \) is used. However, in situations where the perturbations are due to imprecise knowledge of sensor locations, or where the sensor gain and phase patterns do not distort uniformly in \( \theta \), the more general model of (7) may be preferred.

In practice, the response of a given sensor is typically known to within some tolerance in gain and phase that accounts for variations in the construction of the sensor and the conditions under which it is to operate. This tolerance may be specified as limits above and below some nominal response, or as an expected deviation around the nominal. Consequently, a useful approach for modeling array errors is to assume that \( \hat{A} \) is specified in probabilistic terms (e.g., the mean and variance of the elements of \( \hat{A} \) are assumed known), and obtain statistical measures of the errors in the DOAs (e.g., means and variances). In this framework, one may think of the sensor array as one realization from the probability space of arrays specified by \( A \) and the distribution of \( \hat{A} \). As such, when simulations are performed to validate the expressions obtained in the next section, Monte Carlo trials are conducted over a large number of arrays "drawn" from the distribution specified by \( A \) and \( \hat{A} \).

IV. Error Expressions

To find expressions for the mean and variance of the estimation error \( \hat{\theta}_i - \theta_i \), an approach similar to [3] will be used. That is, an expression for \( \hat{\theta}_i - \theta_i \) will be obtained via a first order expansion of \( \hat{f} = \hat{f}(\theta_i) \), and the statistics of the estimation error will be related to the statistics of \( \tilde{E}_n \).

Expanding \( \hat{f} \) about the estimate \( \hat{\theta}_i \), for small enough errors we may write

\[
0 = \hat{f}(\hat{\theta}_i) \approx \hat{f}'(\hat{\theta}_i)(\theta_i - \hat{\theta}_i) + \frac{\hat{f}''(\hat{\theta}_i)}{2!}(\theta_i - \hat{\theta}_i)^2.
\]

(8)

With the definition

\[
\delta \theta(\theta_i) = \frac{\partial \hat{f}(\theta)}{\partial \theta|_{\theta = \theta_i}},
\]

the first derivative may be approximated as

\[
\hat{f}'(\theta_i) = 2\text{Re}[\bar{d}(\theta_i)E_n^* a(\theta_i)]
\]

\[
= 2\text{Re}[\bar{d}(\theta_i)E_n^* a(\theta_i)] + O(||\tilde{E}_n||^2),
\]

(9)

Using similar arguments, the second term of (8) is approximately given by

\[
\hat{f}''(\theta_i)(\theta_i - \hat{\theta}_i) = 2\text{Re}[\bar{d}(\theta_i)E_n^* \tilde{E}_n a(\theta_i)] + \text{Re}[\bar{d}(\theta_i)E_n^* \tilde{E}_n a(\theta_i)](\theta_i - \hat{\theta}_i)
\]

\[
\approx 2\text{Re}[\bar{d}(\theta_i)E_n^* \tilde{E}_n a(\theta_i)](\theta_i - \hat{\theta}_i).
\]

(10)

In the previous section, it was shown that \( \tilde{E}_n a(\theta_i) \approx E_n^* \tilde{a}(\theta_i) \), so the expression for the estimation error becomes

\[
\hat{\theta}_i - \theta_i \approx -\frac{\text{Re}[\bar{d}(\theta_i)E_n^* \tilde{a}(\theta_i)]}{\text{Re}[\bar{d}(\theta_i)E_n^* \tilde{a}(\theta_i)]}.
\]

(11)

Rao and Hari showed [5] that the perturbation in \( \theta_i \) for cost-MUSIC can be written as

\[
\hat{\theta}_i - \theta_i \approx -\frac{\hat{f}(\theta_i)}{2\text{Re}[\bar{d}(\theta_i)E_n^* \tilde{a}(\theta_i)]}.
\]

(12)

Although the goal of [5] was an asymptotic analysis of cost-MUSIC for finite sample approximations to \( R \), the general perturbation equations are general and apply to any source of error.

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It is clear from the arguments used in deriving (11) that (11) and (12) are equivalent, so for small perturbations, the same expressions can be used to analyze both MUSIC and root-MUSIC.

For cases where \( \tilde{\theta}(\theta) \) is zero-mean, it is clear that \( \mathcal{E}(\tilde{\theta}_i - \theta_i) = 0 \). This condition will be assumed for all simulations in the next section, so the bias term will not be considered further. As for the second moment of the estimation error, it is easily shown that

\[
\mathcal{E}[(\tilde{\theta}_i - \theta_i)^2] = \frac{\Re(\mathbf{h}_i^H \mathcal{E}(\tilde{\theta}_i \tilde{\theta}_i^*) \mathbf{h}_i + \mathbf{h}_i^H \mathcal{E}(\tilde{\theta}_i^* \tilde{\theta}_i) \mathbf{h}_i)}{2(\mathbf{h}_i^H \mathbf{h}_i)^2},
\]

(13)

where \( \tilde{\theta}_i \) denotes conjugation and the definitions \( \tilde{\theta}_i(\theta) = \tilde{\theta}_i \) and \( \mathbf{h}_i = E_{\theta_{\mathbf{h}}, \theta} \mathbf{d}(\theta) \) have been used for simplicity. While (13) may be evaluated in its present form using any statistical model for \( \tilde{\theta}_i \), it is useful to examine some special cases. In the examples which follow, the nominal array is assumed to be composed of identical sensors with unity gain in the directions of interest, and the sensor errors are assumed to be independent and identically distributed from element to element.

1. **Model 1**: \( \tilde{\theta}_i \sim N(0, \gamma^2) \), \( \mathcal{E}(\tilde{\theta}_i \tilde{\theta}_i^*) = 0 \), where \( \mathbf{I} \) is the identity matrix with one diagonal element set to zero,

\[
\mathcal{E}[(\tilde{\theta}_i - \theta_i)^2] = \frac{\gamma^2}{2h_i^2}.
\]

2. **Model 2**: \( \tilde{\theta}_i = \mathbf{P}_i \tilde{\theta}_i \), as in (7), where \( \mathbf{P}_i \) is diagonal, \( \mathbf{P}_i(1,1) = 0 \), and \( \phi \) and \( \tilde{\theta}_i \) are independent, zero-mean random variables with variance \( \sigma_\phi^2 \) and \( \sigma_\theta^2 \) respectively,

\[
\mathcal{E}[(\tilde{\theta}_i - \theta_i)^2] = \frac{\gamma_1}{2h_i^2} + \frac{\gamma_2 \Re(h_i^H \tilde{\theta}_i \tilde{\theta}_i^* \mathbf{d}(\theta_i) \mathbf{h}_i)}{2(h_i^H h_i)^2},
\]

where \( \gamma_1 = \sigma_\phi^2 + (1 + \sigma_\phi^2)\sigma_\theta^2 \), \( \gamma_2 = \sigma_\phi^2 - (1 + \sigma_\phi^2)\sigma_\theta^2 \), and \( \mathbf{d}(\theta_i) \) is the diagonal matrix with \( \tilde{\theta}_i \) as its diagonal elements.

3. **Model 3**: \( \mathbf{P}_i \) as before, except each row has \( \mathbf{m} \) off-diagonal elements (e.g., due to mutual coupling); all off-diagonal elements are independent, zero-mean random variables with variance \( \sigma_m^2 \),

\[
\mathcal{E}[(\tilde{\theta}_i - \theta_i)^2] = \frac{\gamma_1 + k_m \sigma_m^2}{2h_i^2} + \frac{\gamma_2 \Re(h_i^H \tilde{\theta}_i^* \tilde{\theta}_i^* \mathbf{d}(\theta_i) \mathbf{h}_i)}{2(h_i^H h_i)^2}.
\]

V. SIMULATION RESULTS

To validate the analysis of the previous section, four separate simulation studies were performed. In each case, 500 trials were conducted assuming a nominal 12 element ULA of identical sensors with \( \lambda/2 \) interelement spacing. For each trial a random perturbation drawn from a fixed distribution was made to the nominal array response. For simplicity, the perturbation was assumed to be angle-independent. The standard deviation of the MUSIC and root-MUSIC DOA estimates were calculated and compared with the corresponding theoretical expressions. Excellent agreement between predicted and measured standard deviations was obtained in all cases.

The emitter signals were generated as constant amplitude plane waves with random phase, uniformly distributed on \([0, 2\pi]\), and the nominal gain of all sensor array elements was assumed to be unity in the direction of the impinging signals. In all cases, fifty noise-free snapshots were used to form the perturbed covariance matrix, and it was assumed that the detection problem was correctly solved; i.e., a signal subspace of correct dimension was always used.

**Case 1: Model 1**

For this set of simulations, the sensors errors were generated according to model 1 as described at the end of the previous section, and the variable \( \gamma \) was allowed to range from 0.0014 to 0.4243. Two uncorrelated signals were simulated, with DOAs of 10° and 15°. Figure 1 shows the standard deviation of the MUSIC and root-MUSIC estimates for each value of \( \gamma \), as well as the corresponding predicted values (there are actually two curves plotted, one for each of the signals, but they lie essentially on top of one another). Note that, for the last case where \( \gamma = 0.4243 \), the estimate error dips slightly below the predicted curve. This is because the error calculation did not include the estimate(s) obtained when MUSIC failed to resolve two sources (50% of the cases), or when root-MUSIC failed to pick the appropriate polynomial roots (20% of the cases). Although the theoretical expressions tended to slightly underestimate the DOA errors, agreement between predicted and measured standard deviations is very good.

**Case 2: Model 2, Gain Errors Only**

The only difference between this case and the previous one is that model 2 was used to generate the sensor errors. No phase errors were simulated (\( \sigma_\phi = 0 \)), while \( \sigma_\theta \) was varied from 0.001 to 0.5. Figure 2 shows a plot of the predicted and measured standard deviations for this case. The failure percentages for the last two values of \( \sigma_\phi \) were 16 and 27% for MUSIC, and 4 and 18% for root-MUSIC. As before, the measured estimate errors fall below the predicted curve for these values of \( \sigma_\phi \) since they were not included in the error calculation.

**Case 3: Model 2, Phase Errors Only**

For this case, sensor errors were again generated using model 2, this time with \( \sigma_\phi = 0 \) and \( \sigma_\theta \) allowed to vary from 0.001 to 0.5 radians (0.6° to 29°). Figure 3 shows excellent agreement between the predicted and measured standard deviations for each value of \( \sigma_\theta \). The failure percentages for the last two values of \( \sigma_\theta \) were 10 and 20% for MUSIC, and 2 and 18% for root-MUSIC.

**Case 4: Model 3, Mutual Coupling**
Model 3 was used to generate the sensor errors for these trials, with \( \sigma_g = 0.01 \), \( \sigma_\phi = 0.01 \) (0.57'), \( k_\phi = 2 \), and where \( \sigma_m \) was varied from 0.001 to 0.1. The plot of predicted and measured estimate errors in Figure 4 shows that for this case, mutual coupling errors deteriorate performance only when their magnitude exceeds that of the gain and phase errors. The theoretical expressions accurately track the effects of \( \sigma_m \).

![Figure 1: Standard Deviation of DOA Estimates vs. \( \gamma \) – Case 1](image1)

![Figure 2: Standard Deviation of DOA Estimates vs. \( \sigma_g \) – Case 2](image2)

![Figure 4: Standard Deviation of DOA Estimates vs. \( \sigma_m \) – Case 4](image4)

**REFERENCES**


