Spatial Signature Estimation for Uniform Linear Arrays with Unknown Receiver Gains and Phases

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Abstract—The problem of spatial signature estimation using a uniform linear array (ULA) with unknown receiver gain and phase responses is studied. Sufficient conditions for identifying the spatial signatures are derived, and a closed-form ESPRIT-like estimator is proposed. The performance of the method is investigated by means of simulations and on experimental data collected with an antenna array in a suburban environment. The results show that the absence of receiver calibration is not critical for uplink signal waveform estimation using a plane wave model.

Index Terms—Array signal processing, calibration, parameter estimation.

I. INTRODUCTION

The use of antenna arrays at base stations in wireless communication systems has recently received substantial interest [8], [13]. The diversity offered by multiple antennas may be used for increased range resulting in increased coverage or for lowering transmit power in the mobile units. In addition, cochannel interference from other mobiles, either in the same cell or in other cells, may be efficiently suppressed, leading to increased capacity.

This work considers a problem arising in uplink (mobile to base) signal waveform estimation in environments where a plane wave model is suitable. Examples of scenarios where this may be reasonable include suburban and rural environments with the antenna array placed above roof tops so that near-field scattering is limited. The approach taken herein is to first estimate the spatial signatures or channels associated with the mobile sources and then use them in turn to estimate the transmitted signals.

Conventional narrowband source localization using antenna arrays relies on the fact that the spatial signature is determined by the direction of arrival (DOA) of the plane wave incident on the array. However, the gain and phase response of each individual antenna must also be considered together with the geometry of the array in arriving at the overall spatial signature for a given DOA. In this work, we assume a uniform linear array (ULA) geometry with elements whose gains and phases are nonuniform, angle-independent, and unknown.

The unknown calibration may be viewed as a model error, and the effects of model errors on the accuracy of DOA and signal waveform estimation have been studied in, e.g., [4], [21], [25], and [28]. These results show that substantial performance degradation may be expected due to such errors, especially in scenarios with large power differences between signals. Due to the time-varying random nature of the radio channel, situations where signals have large power differences occur frequently.

This provides motivation for examining techniques that estimate the DOA’s together with the unknown parameters of the array response: in this case, the receiver responses. Simultaneous estimation of the calibration and location parameters is known as auto calibration, and examples of such methods include [11], [24], [26], and [27]. However, in many cases, the source location and array response parameters are not independently identifiable. As is well known, if the phase characteristics of the receivers are different for each sensor and unknown, the spatial frequencies corresponding to the DOA’s may only be determined up to an unknown rotational ambiguity [11]. This loss of identifiability is of course critical for DOA estimation. However, we show that except for a set of scenarios with zero measure, the differential spatial frequencies may be determined, and this is sufficient for determining the spatial signatures.

Estimation of sensor gain and phase has been studied in [5], [12], and [16] for the case where the DOA’s are known. Iterative approaches to the problem of estimating the DOA’s and the sensors’ gains and phases have been studied in, e.g., [16] and [26]. If the receiver characteristics are approximately known, i.e., for small calibration errors, another alternative is the MAP approach presented in [19]. Except for [2] and [11], geometries other than ULA’s are studied due to the inherent ambiguity between the sensors’ phases and the DOA’s [16], [26]. The identifiability issue is nontrivial. Various necessary conditions are presented in [16] and [26], and in [7], examples are given, showing that the conditions in [26] are not sufficient.

The ULA geometry is studied in this paper as it is one of the most common array geometries and because the redundant structure may be exploited to obtain a computationally efficient solution to the spatial signature estimation problem. The identifiability of the model parameters is addressed, but the treatment is more general than in [11] as no assumptions are made on the signals’ correlation and different from [11] and [26] as identifiability from the signal subspace is studied.

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Sufficient conditions are formulated for identifiability of the differential spatial frequencies and the receivers’ responses. Under the given conditions, the rotational DOA ambiguity of the model has no impact on the identifiability of the spatial signatures. However, it turns out that it is possible to find a set of scenarios with measure zero in which the spatial signatures are not identifiable from second-order statistics when the given conditions are not satisfied. The recently proposed almost blind (AB) methods of [23] and [27] will also fail in these cases. Note that there is no contradiction between the results presented and the ones given in [11], [16], and [26], as identifiability of the spatial signatures is considered and not identifiability of the absolute DOA’s and receiver responses.

The contribution of this work is the study of identifiability of the spatial signatures (and not the DOA’s) and the sufficient conditions for identifiability from the signal subspace. In addition, two subspace-based algorithms are presented; one of them is a computational efficient ESPRIT-like estimator. The performance of the proposed estimator is investigated on experimental data. The proposed data model is compared with the model used in [27].

The paper is organized as follows. In Section II, the data model used is introduced. Identifiability of the spatial signatures is discussed, and sufficient conditions are given in Section III. The two subspace-based estimators are derived in Section IV. Numerical examples may be found in Sections V and VI involving both simulations and examples with real data.

II. A MEASUREMENT MODEL

Consider a uniformly spaced linear array whose sensors and receivers have unknown angle-independent responses. Such an array is only partially calibrated since the array response vector is a function of both the array’s geometry, which is known, and the sensors’ responses, which are unknown. The incident waves are assumed to be approximately planar so that the spatial propagation may be parameterized by the DOA of the wave only. The scenario is depicted in Fig. 1.

The signal received by an array of \(d\) such elements from \(m\) transmitters emitting narrowband signals is assumed to obey the low-rank model

\[
\mathbf{x}(t) = \mathbf{\Gamma}(\gamma) \sum_{i=1}^{d} \mathbf{a}(\omega_i) s_i(t) + \mathbf{n}(t)
\]

where \(s_i(t)\) corresponds to the signal from the \(i\)th transmitter

\[
\mathbf{s}(t) = [s_1(t), s_2(t), \ldots, s_d(t)]^T
\]

\((\cdot)^T\) denotes transpose, \(\mathbf{n}(t)\) models the noise, and

\[
\mathbf{A}(\omega) = [\mathbf{a}(\omega_1), \mathbf{a}(\omega_2), \ldots, \mathbf{a}(\omega_d)].
\]

For a uniformly spaced array with elements separated by \(\Delta\) wavelengths, the array response vector due to a plane wave with DOA \(\theta\) is

\[
\mathbf{a}(\omega) = [1, e^{j\omega}, \ldots, e^{j(m-1)\omega}]^T
\]

where

\[
\omega = 2\pi \Delta \sin \theta.
\]

In this paper, the plane waves are parameterized by their spatial frequencies, rather than the DOA’s, with the mapping defined in (3).

The matrix \(\mathbf{\Gamma}(\gamma)\) is a diagonal matrix containing the receivers’ unknown gain and phase responses

\[
\mathbf{\Gamma}(\gamma) = \text{diag}\{\gamma_1, \ldots, \gamma_m\} = \begin{bmatrix} \gamma_1 & 0 \\ \vdots & \ddots \\ 0 & \gamma_m \end{bmatrix}
\]

where \(\gamma\) is the complex \(m \times 1\) vector containing the diagonal elements of \(\mathbf{\Gamma}(\gamma)\)

\[
\gamma = [\gamma_1, \gamma_2, \ldots, \gamma_m]^T.
\]

Let \((\cdot)^*\) denote Hermitian transpose. The noise \(\mathbf{n}(t)\) is modeled as spatially white

\[
\mathbb{E}[\mathbf{n}(t)\mathbf{n}^*(t)] = \sigma^2 \mathbf{I}
\]

since the system is assumed to be internally noise limited as in [12], [16], [23], [26], and [27]. This means that the noise level in each receiver is not affected by the receiver gain. If the system is externally noise limited so that the receiver gain affects the noise level as much as the signal, the model in [11] applies.

Equation (1) may also be written as

\[
\mathbf{x}(t) = \mathbf{V}\mathbf{s}(t) + \mathbf{n}(t)
\]

Fig. 1. Plane waves incident on a ULA with unknown receiver responses.
where the spatial signature matrix $\mathbf{V}$ is defined as

$$\mathbf{V} = \mathbf{\Gamma}(\gamma)\mathbf{A}(\boldsymbol{\omega}).$$

(4)

Assuming that the noise and the signals are zero-mean and independent, the covariance matrix of the observations is given by

$$\mathbf{R} = E\{\mathbf{x}(t)\mathbf{x}^*(t)\} = \mathbf{V}\mathbf{S}\mathbf{V}^* + \sigma^2\mathbf{I}$$

(5)

where the covariance matrix of the emitter signals $\mathbf{S}$, which is defined as

$$\mathbf{S} = E\{\mathbf{s}(t)\mathbf{s}^*(t)\}$$

is assumed to have full rank; thus, no coherent signals are present. An eigenvalue decomposition of $\mathbf{R}$ in (5) may be written as

$$\mathbf{R} = \sum_{i=1}^{m} \lambda_i \mathbf{e}_i \mathbf{e}_i^* = \mathbf{E}_s \mathbf{\Lambda}_\omega \mathbf{E}_s^* + \sigma^2 \mathbf{E}_n \mathbf{E}_n^*$$

where the eigenvalues are ordered so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > \lambda_{m+1} = \cdots = \lambda_d = \sigma^2$, and $\mathbf{E}_s = [\mathbf{e}_1, \ldots, \mathbf{e}_d]$, $\mathbf{E}_n = [\mathbf{e}_{d+1}, \ldots, \mathbf{e}_m]$. The $d$ vectors in $\mathbf{E}_s$ span the signal subspace, and the columns of $\mathbf{E}_n$ span its orthogonal complement: the so-called noise subspace.

The problem studied herein is the estimation of the spatial signature matrix $\mathbf{V}$ using the special structure in (4) and $\mathcal{N}$ noisy observations of the array output.

### III. IDENTIFIABILITY

Inherent in the problem is the unknown scaling between the spatial signature matrix $\mathbf{V} = \mathbf{\Gamma}(\gamma)\mathbf{A}(\boldsymbol{\omega})$ and the signals $\mathbf{s}(t)$. Without any information about the signals, each spatial signature may only be determined up to an unknown scaling. In addition, the ordering of the spatial signatures is arbitrary. This means that the spatial signature matrix may only be determined up to a permutation of its columns. The scaling ambiguity may be handled by imposing a constraint on either the signals or $\mathbf{\Gamma}(\gamma)$. An example of such a constraint is to let one of the elements of $\gamma$ be equal to one, e.g., $\gamma_1 = 1$. In addition, there is a progressive phase factor ambiguity between $\mathbf{\Gamma}(\gamma)$ and $\mathbf{A}(\boldsymbol{\omega})$. This was shown in [11] for the special case of uncorrelated source signals.

Since we consider subspace-based identification methods, the relevant question is whether or not the spatial signatures can be identified from the column span of the spatial signature matrix. For this purpose, consider the relation

$$\mathbf{\Gamma}(\gamma)\mathbf{A}(\boldsymbol{\omega}) \mathbf{T} = \mathbf{\Gamma}(\gamma)\mathbf{A}(\boldsymbol{\omega})$$

(6)

where $\mathbf{T}$ is some full-rank matrix, and the first elements of $\gamma$ and $\boldsymbol{\omega}$ are set to one. If the spatial frequencies and the receiver responses are uniquely identifiable from the subspace, this means that the only solution to (6) is $\hat{\gamma} = \gamma$, and $\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}$ (subject to ordering). Due to the progressive phase ambiguity, this is not the case. The Vandermonde structure of $\mathbf{A}(\boldsymbol{\omega})$ yields

$$\mathbf{\Gamma}(\gamma)\mathbf{A}(\omega_1, \ldots, \omega_d) = \mathbf{\Gamma}(\gamma \odot \mathbf{a}(\omega))\mathbf{A}(\omega_1 - \omega_0, \ldots, \omega_d - \omega_0)$$

where $\odot$ denotes element-by-element multiplication. Thus, the parameters $\boldsymbol{\omega}$ and $\gamma$ cannot be uniquely determined simultaneously. However, as is shown next, the spatial signatures may still be determined.

Suppose that the spatial frequencies $\{\omega_i\}_{i=1}^d$ are distinct and that the coefficients of the polynomial $b(z)$, which are defined as

$$b(z) = b_0 \prod_{i=1}^{d} (z - e^{j\omega_i})$$

(7)

are all nonzero. In addition, assume that all the coefficients of $\mathbf{\Gamma}(\gamma)$ are nonzero and that

$$d \leq m - 2.$$ 

Then, using the theorem in the Appendix, it follows that the solutions to (6) are given by

$$\hat{\gamma} = \gamma \odot \mathbf{a}(\omega^e), \quad \hat{\boldsymbol{\omega}} = \mathcal{P}\boldsymbol{\omega} - \omega^e \mathbf{e}$$

(8)

where $\mathbf{e}$ is a column vector with all elements equal to one, the subtraction in the second equation is modulo $2\pi$, and $\omega^e$ is an arbitrary real-valued rotation. The matrix $\mathcal{P}$ is a permutation matrix with one element equal to one in each row and column and the other elements equal to zero.

This means that the absolute spatial frequencies and the receiver responses may not be uniquely determined from the subspace spanned by the columns of $\mathbf{V}$. However, it is possible to determine the differential frequencies from the span as the rotation does not change them. Note that it is not sufficient to know or fix, the DOA of one of the sources in order to resolve this rotational ambiguity. Due to the nature of the rotation, there is no natural ordering of the corresponding spatial frequencies. It is not possible to determine which of the frequencies corresponds to the known DOA.

Finally, consider the set of spatial signatures formed from the set of solutions in (8) as

$$\hat{\mathbf{V}} = \mathbf{\Gamma}(\hat{\gamma})\mathbf{A}(\hat{\boldsymbol{\omega}}) = \mathbf{\Gamma}(\gamma \odot \mathbf{a}(\omega^e))\mathbf{A}(\mathcal{P}\boldsymbol{\omega} - \omega^e \mathbf{e}) = \mathbf{\Gamma}(\gamma)\mathbf{A}(\omega)\mathcal{P} = \mathbf{V}\mathcal{P}.$$ 

This means that $\mathbf{T}$ in (6) is a permutation matrix. Thus, all solutions to (6) will give the correct spatial signature matrix up to some permutation. The phase ambiguity between the spatial frequencies and the phase of the diagonal matrix $\gamma$ will only lead to a permutation determined by the rotation $\omega^e$. The result may be summarized in the following theorem.

**Theorem:** If all elements of $\gamma$ are nonzero, the spatial frequencies are such that all coefficients of the polynomial in (7) are nonzero and $d \leq m - 2$, then it is possible to determine the spatial signatures to within a scaling from the subspace spanned by the columns of the spatial signature matrix.

For the case where the array is calibrated so that $\mathbf{\Gamma}(\gamma)$ is known, up to $m - 1$ spatial frequencies may be determined. Thus, the number of signals that can be identified is only reduced by one when $\gamma$ is unknown.
A. Examples of Unidentifiability

For $d \leq m - 2$, the sufficient condition for identifiability of the spatial signatures is that the coefficients of the polynomial in (7) are nonzero. Now, assume that $d$ sources are present and that the spatial frequencies are

$$\omega_0, \omega_0 + \frac{2\pi}{d}, \ldots, \omega_0 + (d - 1) \frac{2\pi}{d}.$$ 

The polynomial in (7) is then

$$b(z) = e^{-j\omega_0/2}z^d - e^{j\omega_0/2}.$$ 

For $d \geq 2$, this polynomial has $d - 1$ zero coefficients. Assume that the source covariance matrix is the identity $\mathbf{S} = \mathbf{I}$. For $d = 2$, it is straightforward to verify that the spatial frequencies are $\omega_0$ and $\omega_0 + \pi$ and, from (3), that the DOA’s $\theta_1$ and $\theta_2$ satisfy

$$\sin \theta_1 = \sin \theta_2 = \frac{1}{2\Delta}$$ (9)

where $\Delta$ is the interelement spacing. In addition, alternating diagonals of the signal part of the covariance matrix are zero

$$(\mathbf{A}(\omega)\mathbf{A}^*(\omega))_{k,l} = 0 \quad \text{for} \quad |k - l| = 1, 3, \ldots.$$ 

As $\mathbf{\Gamma}(\gamma)\mathbf{A}(\omega)\mathbf{A}^*(\omega)\mathbf{\Gamma}^*(\gamma) = (\mathbf{A}(\omega)\mathbf{A}^*(\omega)) \odot (\gamma^*)$, it may be concluded that for $\gamma$ of the form

$$\gamma_0 = [1 \quad e^{j\phi} \quad 1 \quad e^{j\phi} \quad \ldots]^T$$

the equation

$$\mathbf{\Gamma}(\gamma_0)\mathbf{A}(\omega)\mathbf{A}^*(\omega)\mathbf{\Gamma}^*(\gamma_0) = \mathbf{A}(\omega)\mathbf{A}^*(\omega)$$

holds for all values of $\phi$. Similar examples may be constructed for $d > 2$. In these cases, it is not possible to uniquely determine the receiver responses from the covariance matrix. Other methods for almost-blind identification of the spatial signatures [27] as well as subspace methods for sensor calibration using known DOA’s [12], [16] will fail in these cases.

Finally, note that the set for which the spatial frequencies do not satisfy the sufficient conditions is a set of measure zero. However, performance may degrade near the unidentifiable points, and a numerical example is included in Section V to illustrate this. One way to handle such scenarios and overcome the identifiability problem may be modify the signal model. Examples include exploiting temporal properties of the transmitted signals and using additional knowledge of the array response.

IV. ESTIMATORS

In this section, subspace methods for estimating the parameters of the spatial signatures are proposed. A noise subspace fitting (NSF) approach similar to [1], [3], [17], and [22] and a simple ESPRIT-like algorithm are presented. The algorithms use estimates of the signal and noise subspace bases $\hat{\mathbf{E}}_n$ and $\hat{\mathbf{E}}_s$ determined from an eigenvalue decomposition of the sample covariance matrix of the $N$ observations.

To resolve the scaling ambiguity between $\gamma$ and the signals, the ESPRIT-like approach constrains the response of the first sensor $\gamma_1 = 1$. The NSF estimator uses a unit norm constraint $\gamma^*\gamma = 1$, whereas

A. Noise Subspace Fitting

By using the orthogonality of the signal and noise subspaces, an NSF approach similar to [1], [3], [17], and [22] may be taken. In this technique, the estimates are obtained as the minimizing arguments of the cost function

$$V(\omega, \gamma) = \text{trace}\left\{ \mathbf{A}(\omega)\mathbf{U}^*(\gamma)\hat{\mathbf{E}}_s\hat{\mathbf{E}}_n^*\mathbf{\Gamma}(\gamma)\mathbf{A}(\omega)\mathbf{U} \right\}$$

where $\mathbf{U} = \mathbf{U}^* > 0$ is a $d \times d$ Hermitian weighting matrix. Let $(\cdot)^\dagger$ denote the pseudo inverse. Then, from [10], $\mathbf{U}$ should be chosen as

$$\mathbf{U} = (\mathbf{\Gamma}(\gamma)\mathbf{A}(\omega))^\dagger\hat{\mathbf{E}}_s\hat{\mathbf{E}}_n^* (\mathbf{A}_s - \sigma^2\mathbf{I})^2\hat{\mathbf{E}}_s^* (\mathbf{\Gamma}(\gamma)\mathbf{A}(\omega))^\dagger$$

to obtain asymptotically minimum variance estimates. Replacing the weighting with a consistent estimate will not affect the asymptotic properties.

Since $\mathbf{\Gamma}(\gamma)$ is diagonal, this cost function may be rewritten as

$$V(\omega, \gamma) = \gamma^*\mathbf{M}(\omega)\gamma$$

where

$$\mathbf{M}(\omega) = (\hat{\mathbf{E}}_s\hat{\mathbf{E}}_n^* ) \odot (\mathbf{A}(\omega)\mathbf{U}\mathbf{A}^*(\omega))^T.$$ (10)

From this, it may be recognized that a (nontrivial) estimate of $\gamma$ using the norm constraint is the eigenvector associated with the smallest eigenvalue of the matrix $\mathbf{M}(\omega)$ in (10). This is similar to the estimate of the sensors’ gains and phases with known DOA’s proposed in [12] and [16]. The difference here is that a weighting $\mathbf{U}$ has been introduced. For closely spaced sensors and a finite number of samples, such a weighting may significantly improve performance. An algorithm may now be outlined as follows.

1) Find the frequencies $\omega$ that minimize the smallest eigenvalue of the matrix $\mathbf{M}(\omega)$ in (10). As described earlier, there is an infinite number of solutions. To obtain one solution, it is possible to fix one of the spatial frequencies, e.g., $\omega_1 = 0$. A $d - 1$ dimensional search is then needed to find the remaining relative spatial frequencies.

2) Use the estimated frequencies $\hat{\omega}$ in (10), and calculate $\hat{\gamma}$ as the eigenvector corresponding to the smallest eigenvalue of $\mathbf{M}(\hat{\omega})$.

3) Calculate an estimate of the spatial signature matrix as

$$\hat{\mathbf{V}} = \mathbf{\Gamma}(\hat{\gamma})\mathbf{A}(\hat{\omega}).$$

A simple alternative to the above algorithm is to use a standard technique to find an estimate $\hat{\omega}$ with $\mathbf{\Gamma}(\hat{\gamma})$ set to the identity and then solve for $\gamma$ prior to the signal waveform estimation by forming $\mathbf{M}(\hat{\omega})$ and computing the eigenvector associated with the smallest eigenvalue. Simulations indicate that this approach improves performance over totally ignoring $\mathbf{\Gamma}(\gamma)$ but that the performance is significantly worse than when estimating $\gamma$ and $\omega$ jointly. Finally, as in [16], a solution based on signal subspace fitting may also be formulated.

The minimization in the algorithm described above is quite complex, and finding the global minimum is difficult. Therefore, an alternative approach that avoids this multidimensional optimization is presented in the next section.
B. An ESPRIT-Like Approach

For a ULA, the array response vector given in (2) is Vandermonde, and similar to [6] and [20], an algorithm based on ESPRIT [14], [15] may be derived. Let the matrices \( A_1 \) and \( A_2 \) be constructed from the first and last \( m-1 \) rows of \( A(\omega) \), respectively, as

\[
A_1 = J_1 A(\omega), \quad A_2 = J_2 A(\omega)
\]

where \( J_1 \) and \( J_2 \) are selection matrices defined by \( J_1 = \begin{bmatrix} I_{m-1} & 0 \end{bmatrix} \) and \( J_2 = \begin{bmatrix} 0 & I_{m-1} \end{bmatrix} \). Define the diagonal matrices

\[
\Gamma_1 = J_1 \Gamma(\gamma) J_1^T, \quad \Gamma_2 = J_2 \Gamma(\gamma) J_2^T.
\]

Due to the Vandermonde structure of \( A(\omega) \), the matrices \( A_1 \) and \( A_2 \) satisfy the relationship

\[
A_2 = A_1 \Phi(\omega)
\]

where \( \Phi(\omega) \) is a \( d \times d \) diagonal matrix defined as

\[
\Phi(\omega) = \text{diag}\{e^{j\omega_1}, \ldots, e^{j\omega_d}\}.
\]

Furthermore, let \( E_{s1} \) be a basis for the signal subspace, and define \( E_{s1} \) and \( E_{s2} \) as

\[
E_{s1} = J_1 E_s, \quad E_{s2} = J_2 E_s.
\]

Now, since \( E_s = A(\omega)T \) for some full-rank matrix \( T \), two equations

\[
E_{s1} = \Gamma_1 A_1 T, \quad E_{s2} = \Gamma_2 A_2 T
\]

may be formulated. Together with (11), the above equations lead to

\[
D(d) E_{s2} = E_{s1} \Psi
\]

where

\[
\Psi = T^{-1} \Phi T
\]

and the vector \( d = [d_1, d_2, \ldots, d_{m-1}]^T \) is formed from the diagonal elements of the diagonal matrix \( D(d) \) defined as

\[
D(d) = \Gamma_1 \Gamma_2^{-1}.
\]

The frequencies \( \omega \) may be determined from the arguments of the eigenvalues of the operator \( \Psi \). Except for the diagonal scaling \( D(d) \), this is the relation used by ESPRIT (see [14] and [15]).

From (12), it may noted that there is a scaling ambiguity between \( D(d) \) and \( \Psi \) when \( \Psi \) is an arbitrary matrix. One way to fix the magnitude of this scaling ambiguity is to impose a unit magnitude constraint on the product of eigenvalues, or the determinant, of \( \Psi \). This is because

\[
\det\{\Psi\} = \exp\left( j \sum_{i=1}^{d} \omega_i \right)
\]

clearly has unit magnitude. Since the spatial frequencies are unknown, the angle of the determinant is unknown. This results in the progressive phase factor ambiguity between the receivers’ phase and the spatial frequencies.

In the presence of noise, the basis of the signal subspace is approximated by \( \hat{E}_{s1} \), and \( D(d) \) and \( \Psi \) are estimated as the solutions to the least squares problem

\[
\left\{ \hat{d}, \hat{\Psi} \right\} = \arg \min_{d, \Psi} \| D(d) \hat{E}_{s2} - \hat{E}_{s1} \Psi \|_F^2
\]

(15)

where the elements of \( d \) are to be nonzero, the eigenvalues of \( \Psi \) are to lie on the unit circle, and \( \|\cdot\|_F \) is the Frobenius norm. The constraint on the eigenvalues of \( \Psi \) is replaced by a necessary constraint, namely, that the determinant of \( \Psi \) is one

\[
\det\{\Psi\} = 1.
\]

This is much easier than imposing a constraint on the individual eigenvalues. The constraint will however rotate the spatial frequencies so that \( \sum_{i=1}^{d} \omega_i \) is a multiple of \( 2\pi \). Solving for \( \Psi \) in terms of \( D(d) \) gives

\[
\hat{\Psi} = \hat{E}_{s1} D(d) \hat{E}_{s2}
\]

where \( \hat{d} \) is scaled so that (16) is satisfied. Substituting this solution into (15) and using the diagonal structure of \( D(d) \) gives

\[
\hat{d} = \arg \min_{d} \| P_{\hat{E}_{s1}} \circ \left( \hat{E}_{s2} \right)^* \|^2 \| D(d) \|_F \]

subject to

\[
\det\{\hat{E}_{s1} D(d) \hat{E}_{s2}\} = 1
\]

where \( P_{\hat{E}_{s1}} = I - E_{s1} E_{s1}^T \). Note that the function above is minimized when \( \hat{d} \) is parallel with the eigenvector associated with smallest eigenvalue of the matrix

\[
P_{\hat{E}_{s1}} \circ \left( \hat{E}_{s2} \right)^* T.
\]

(17)

Let \( d_{\min} \) be the eigenvector associated with the smallest eigenvalue of the matrix in (17). For any scalar \( \alpha \)

\[
\det\{\hat{E}_{s1} D(\alpha d_{\min}) \hat{E}_{s2}\} = \alpha^d \det\{\hat{E}_{s1} D(d_{\min}) \hat{E}_{s2}\}.
\]

An estimate of \( d \) may thus be taken as

\[
\hat{d} = \frac{1}{\left( \det\{\hat{E}_{s1} D(d_{\min}) \hat{E}_{s2}\} \right)^{1/d}} d_{\min}.
\]

The estimated frequencies are then obtained by constructing \( D(\hat{d}) \) and calculating the arguments of the eigenvalues of

\[
\hat{\Psi} = \hat{E}_{s1} D(\hat{d}) \hat{E}_{s2}.
\]

(19)

Note that (16) will rotate the estimated spatial frequencies. However, for the purpose of determining the spatial signatures to within some scaling, this ambiguity may be ignored since all solutions will give the same estimates of the spatial signatures.
With (14) and the constraint \( \gamma_1 = 1 \), it is straightforward to establish that

\[
\gamma_1 = 1, \quad \gamma_i = \frac{1}{\prod_{j=1}^{i-1} d_j}, \quad 2 \leq i \leq m. \tag{20}
\]

By using estimates of \( d_j \), estimates of \( \gamma_i \) may be calculated. An alternative is to estimate the differential spatial frequencies with the ESPRIT-like approach and then construct \( \mathbf{M}(\hat{\omega}) \) in (10). An estimate of \( \gamma \) may then be calculated as the eigenvector associated with the smallest eigenvalue of \( \mathbf{M}(\hat{\omega}) \). Simulations that are not presented here indicate that this more computationally complex approach may perform slightly better.

An algorithm may now be outlined as follows.

1) Determine an estimate \( \hat{\mathbf{d}} \) according to (18), where \( \hat{\mathbf{d}}_{\text{train}} \) may be determined as the eigenvector associated with the smallest eigenvalue of the matrix in (17).

2) Determine an estimate \( \hat{\mathbf{V}} \) according to (19), and calculate the estimates of the spatial frequencies \( \hat{\omega} \) as the arguments of the eigenvalues of \( \hat{\mathbf{V}} \).

3) Use \( \hat{\mathbf{V}} \) in (20) to calculate an estimate of the receivers’ responses \( \hat{\mathbf{r}} \).

4) Calculate an estimate of the spatial signature matrix as

\[
\hat{\mathbf{V}} = \Gamma(\hat{\gamma})\mathbf{A}(\hat{\omega}).
\]

V. SIMULATION EXAMPLES

To examine the performance of the derived estimators, simulations were conducted with data generated according to the model. The results of using the AB method proposed in [27] are also included. This approach assumes that the array response vectors have an arbitrary phase but an angle independent gain response

\[
\mathbf{V} = \begin{bmatrix}
g_{f_1} & 0 & \cdots & \cdots & \cdots & g_{f_{m}} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
g_{m_1} & 0 & \cdots & \cdots & \cdots & g_{m_{m}}
\end{bmatrix}
\begin{bmatrix}
ej\phi_{f_1} & \cdots & \cdots & \cdots & \cdots & ej\phi_{f_{m}} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
ej\phi_{m_1} & \cdots & \cdots & \cdots & \cdots & ej\phi_{m_{m}}
\end{bmatrix}.
\]

The gain parameters \( g_{f_1}, \ldots, g_{m_{m}} \) are assumed to be real and positive and the phases \( \phi_{f_{ij}} \) arbitrary. The method is based on iterative decompositions of the covariance matrix, and the source signals are assumed to be uncorrelated. The algorithm terminated either when a maximum number of 1000 iterations was reached or when the relative change in the estimated signatures was less than \( 10^{-4} \)

\[
\left\| \hat{\mathbf{V}}^{(k)} - \hat{\mathbf{V}}^{(k-1)} \right\|_F < 10^{-4} \cdot \left\| \hat{\mathbf{V}}^{(k-1)} \right\|_F
\]

where \( \hat{\mathbf{V}}^{(k)} \) is the estimated spatial signature matrix after the \( k \)th iteration.

In the simulations, the quality of the spatial signature estimates obtained with the different estimators is compared by constructing signal waveform estimators and comparing the signal-to-interference-plus-noise ratio (SINR) of the estimated signals. The structured stochastic estimator (SSE) described in [9] was used.

A. Different Angular Separation

In the first example, a ULA with six elements separated by half a wavelength was considered. Two signals with equal power were incident on the array, and 100 snapshots were collected. The SNR was 10 dB. The DOA of one source was held fixed at \(-45^\circ\), and the DOA of the second source was varied. The average SINR in the estimated signals is plotted versus DOA of the second signal in Fig. 2. The AB method is iterative and typically requires a large number of iterations, each iteration having roughly the same complexity as the ESPRIT-like approach.

As the first DOA is \(-45^\circ\), it so happens that the polynomial in (7) has a zero coefficient for

\[
\sin^{-1}(1 + \sin(-45^\circ)) = 17^\circ
\]

where (9) is satisfied. At this point, the sensor gain and phase responses are not uniquely identifiable. In the vicinity of this point, performance degrades.

B. Different Source Correlation

In the second example, the same scenario as in the first example was considered, but the DOA of the second source was held fixed at \(0^\circ\). The source covariance matrix \( \mathbf{S} \) was modeled as

\[
\mathbf{S} = \mathbb{E}\{\mathbf{s}(t)\mathbf{s}^*(t)\} = \rho \begin{bmatrix} 1 \\ \rho \end{bmatrix}
\]

for some real scalars \( \rho \) and \( \rho \). The average SINR in the estimated signals is plotted versus the parameter determining the correlation between the transmitted signals \( \rho \) in Fig. 3. The phase reference was chosen to coincide with the first element of the array. The subspace-based methods perform well for moderately correlated sources, whereas performance of the AB method, which relies on the fact that the transmitted signals are uncorrelated, degrades significantly with increasing
correlation. The ESPRIT-like approach was found in most cases to have about the same performance as the NSF approach for relatively well-separated sources; see Figs. 2 and 3. In what follows, only the ESPRIT-like approach will be considered.

C. Different Number of Snapshots

In the third example, the number of snapshots was varied. Three signals with DOA’s 0°, 20°, and −20° and SNR’s 30, 20, and 30 dB were present. The array was a ULA with six elements, but the sensors had a zero-mean Gaussian distributed phase error, independent from sensor to sensor with a standard deviation of 1°. For comparison, results are included for the conventional ESPRIT algorithm, which assumes the sensors to be calibrated. The average SINR in the estimate of the weaker signal is plotted in Fig. 4. As can be seen, for a small number of snapshots, it is advantageous to use the standard plane wave model together with the conventional ESPRIT algorithm. This is because the number of parameters is small, and finite sample errors dominate the calibration error. For this particular scenario, the proposed ESPRIT approach needs about 150 snapshots to obtain the best possible performance. The AB method has many more parameters, and therefore, a large number of snapshots are needed to obtain the best possible performance.

VI. REAL DATA EXPERIMENTS

To further test the above technique as well as to study its applicability to real world scenarios, data was used from a number of experiments with an array in a suburban environment. A nominal ULA consisting of 12 dipoles placed 18 cm apart was mounted on the top of a 15-m high tower on a hillside overlooking a relatively flat residential area. The combined height of the antenna (hill plus tower) was several hundred feet. Two mobile transmitters in vehicles were present, transmitting analog FM 1-kHz sine waves with center frequencies $f_c = 825.27$ MHz. Both mobiles traveled at speeds between 0–50 km/h and were located at a range of 2–3 km from the array. The azimuth angles of the sources varied in the intervals [−5°, 5°] and [−35°, −25°] relative to array broadside. In the experiments, the inner eight elements of the array were used, and data from the array’s RF front end were sampled at 71.4 kHz and stored in blocks of 512 samples. An on-site receiver calibration was performed by measuring the broadside (0°) response of the array with (nominally) no other sources present.

Since the center frequencies of the two signals were separated by 15 kHz and since their bandwidths were only a few kilohertz, they were spectrally disjoint. The SNR in the experiments was estimated to be in excess of 27 dB, and since this is quite high, the interference rejection was measured by simply taking the highest peaks in the frequency bands above and below the carrier, comparing their levels before (single sensor) and after signal separation.

The algorithms studied were applied directly to the raw data, and in Fig. 5, the average interference rejection of the algorithms is plotted versus the maximum number of iterations for the AB method. Conventional ESPRIT refers to the conventional ESPRIT algorithm using the measured calibration. The estimates of the spatial signatures are then simply $\mathbf{T}(\hat{\Psi})\mathbf{A}(\hat{\Theta})$ with $\hat{\Psi}$ measured in the offline calibration. The conventional ESPRIT method and the proposed ESPRIT algorithm are noniterative and require roughly the same order of computations as one iteration of the AB algorithm. The signal estimates were obtained using the least squares (LS) beamformer $\hat{s}(t) = (\hat{\mathbf{V}}^\dagger \hat{\mathbf{V}})^{-1}\hat{\mathbf{V}}x(t)$, and the results shown were averaged over both of the sources and a total of 2000 data sets.

As can be seen from the figure, there is very little difference (less than 1 dB) between measuring the calibration offline and using the online technique presented herein. This is significant since it may not always be possible to make
Fig. 5. Average interference rejection. Conventional ESPRIT uses calibration data measured off line.

such offline measurements with no other sources on the air. Experiments with the conventional ESPRIT algorithm were also done ignoring the measured receiver calibration. The average interference rejection combining was then of the order of 9 dB. These experiments clearly demonstrate the need for calibration, either online or offline.

For a large number of iterations, the AB method offers about 4 dB of additional improvement as compared to the ESPRIT algorithms. This is due to the more general phase response model for the array where the phase errors are angle dependent. However, it should be noted that the computational burden is significantly higher: roughly a factor 100 for the 4-dB improvement. In addition, more than 50 iterations are needed for the AB method to perform better than the ESPRIT algorithms. This and other examples illustrate a tradeoff between computational complexity and performance as well as the number of model parameters and finite sample effects. In addition, note that the proposed method could be used to initialize an iterative scheme such as the AB method in order to reduce its computational burden.

Some additional studies using the real data were also conducted. Since the sources were spectrally disjoint, their individual spatial signatures could be estimated by means of temporal filtering. The estimated signatures were normalized to have length $\sqrt{m}$ and used to synthesize simulation data. In this way, performance was easy to measure, and it was possible to vary different parameters. In order to study the effect of calibration errors, the receiver calibration was perturbed by a normally distributed phase error independent from sensor to sensor

$$\mathbf{\Gamma}(\gamma) = \text{diag}\{e^{j\phi_1}, \ldots, e^{j\phi_m}\}$$

where the $\phi_i$'s are independent zero-mean with variance $\sigma^2_p$. The estimated spatial signatures were used to synthesize data with 100 snapshots. The average SNR of both mobiles was 20 dB. The SSE signal waveform estimator of [9] was used with the different estimates of the spatial signatures. In Fig. 6, the average SINR in the estimated signals are shown.

Fig. 6. Average SINR in the estimated signals, Signal 1 with DOA $[\pm 35^\circ, -25^\circ]$ and Signal 2 with DOA $[-5^\circ, 5^\circ]$.

The performance degradation for estimating the receiver responses as opposed to directly measuring them is in this case only about 0.5 dB in average SINR. As the phase perturbation in the receiver calibration is increased, the performance of the proposed approach remains unchanged, whereas the standard approach that neglects the calibration errors suffers from a significant performance degradation.

If the true spatial signatures are used, an average SINR of about 29 dB results. The loss of 3–5 dB is due to deviations from the ideal array response model, as assumed in (2). Examples of error sources are multipath propagation, near field scattering, and mutual coupling. However, by examining the time variations from burst to burst, coherent multipath propagation appears to be the most likely explanation.

VII. CONCLUSIONS

The problem of spatial signature estimation for a partially calibrated ULA was studied. Sufficient conditions for identifiability of the spatial signatures were derived, and examples where the receivers’ responses are not identifiable from the covariance matrix were given. The results may also be used when calibrating an antenna array using calibration sources at known locations as the theorem gives conditions for when the parameters are identifiable.

In addition to a weighted subspace fitting approach, an ESPRIT-like method was proposed for estimating the parameters of the spatial signatures. The ESPRIT-like method utilizes the special structure of a ULA to obtain a closed-form solution for the parameters of the spatial signatures. The algorithms were applied to simulated data, as well as to “semi-real” and real data collected in a suburban environment.

The experiments indicate that accurate receiver calibration is preferable, although satisfactory performance may be obtained even without knowledge of the receivers’ gain and phase responses. From the experiments, it is clear that there is a tradeoff between model parameterization and finite sample effects as well as computational burden and performance.
The major advantage of the proposed ESPRIT method is its low computational burden as compared with the other methods studied. In addition, the proposed method can be used to initialize other iterative methods, such as the AB method, in order to reduce the large number of iterations needed.

From the results, it is seen that the real data deviates from the assumed ULA model used, as well as from the model used by the AB method. This indicates that other approaches using no model for the spatial signatures, but temporal properties of the signals, may improve performance.

**APPENDIX**

Consider the equality

\[ \Gamma(\gamma_1)\mathbf{A}(\omega_1, \cdots, \omega_d)\mathbf{T}_1 = \Gamma(\gamma_2)\mathbf{A}(\phi_1, \cdots, \phi_d)\mathbf{T}_2 \]  
(21)

where \( \mathbf{T}_1 \) and \( \mathbf{T}_2 \) are \( d \times d \) full-rank matrices. The spatial frequencies are distinct, \( \omega_i \neq \omega_j, \ i \neq j \), and all elements of the diagonal matrix \( \Gamma(\gamma_1) \) are nonzero. In addition, let the first element of \( \Gamma(\gamma_1) \) and \( \Gamma(\gamma_2) \) be one. Define the two polynomials

\[ b_\omega(z) = b_{\omega,0} + b_{\omega,1}z + \cdots + b_{\omega,d}z^d \]  
(22)

\[ b_\phi(z) = b_{\phi,0} + b_{\phi,1}z + \cdots + b_{\phi,d}z^d \]  
(23)

with \( b_{\omega,0} \) and \( b_{\phi,0} \) having unit magnitude and phase chosen so that the polynomials are conjugate symmetric

\[ b_{\omega,l} = \overline{b_{\phi,l}}, \quad l = 0, \cdots, d \]  
(24)

\[ b_{\phi,l} = \overline{b_{\omega,l}}, \quad l = 0, \cdots, d \]  
(25)

If all the coefficients of \( b_\omega(z) \) are nonzero, then (21) holds if and only if

\[ \gamma_1 = \gamma_2 \odot \mathbf{a}(\omega^\phi) \]  
(26)

\[ b_\omega(z) = \pm e^{-(d/2)\omega^2}b_{\omega,0}(z e^{i\omega^\phi}) \]  
(27)

**Proof:** Let \( \mathbf{\omega} = [\omega_1, \omega_2, \cdots, \omega_d] \) and \( \mathbf{\phi} = [\phi_1, \phi_2, \cdots, \phi_d] \). Form two vectors from the coefficients of the two polynomials defined in (22) and (23) as

\[ \mathbf{b}_\omega = [b_{\omega,0}, \cdots, b_{\omega,d}]^T \]  
\[ \mathbf{b}_\phi = [b_{\phi,0}, \cdots, b_{\phi,d}]^T \]  
(28)

As is well known, it is possible to parameterize the nullspace of \( \mathbf{A}(\omega^\phi) \) in terms of coefficients of the polynomial in (22) (see, e.g., [18]), and this approach will be pursued. Define the \( m \times m - d \) matrix \( \mathbf{B}(\mathbf{b}_\omega) \) as

\[ \mathbf{B}(\mathbf{b}_\omega) = \begin{bmatrix} b_{\omega,0} & b_{\omega,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{\omega,d} & \cdots & b_{\omega,0} \end{bmatrix} \]  
(29)

The matrix \( \mathbf{B}(\mathbf{b}_\phi) \) is constructed in the same way. By construction, the matrices will satisfy

\[ \mathbf{B}^*(\mathbf{b}_\omega)\mathbf{A}(\omega) = \mathbf{0}, \quad \mathbf{B}^*(\mathbf{b}_\phi)\mathbf{A}(\phi) = \mathbf{0} \]  
(30)

It is straightforward to verify the “if” part of the theorem. Equation (27) implies that the spatial frequencies of \( \phi \) and \( \omega^\phi \) are related by a rotation \( \omega^\phi \) around the unit circle. With (26)

\[ \Gamma(\gamma_1)\mathbf{A}(\omega) = \Gamma(\gamma_2)\mathbf{A}(\omega_1 + \omega^\phi, \cdots, \omega_d + \omega^\phi) \]  
(31)

\[ = \Gamma(\gamma_2)\mathbf{A}(\phi_1, \cdots, \phi_d)T_2 \]  
(32)

where \( T_2 \) is a \( d \times d \) matrix with one element equal to one in each column and each row. The matrix \( T_2 \) is then a permutation matrix since the ordering in \( \phi \) is not fixed. Thus, (21) holds.

For the second part of the proof, it is assumed that (21) holds. Rewriting (21) as

\[ \mathbf{A}(\omega) = \Gamma^{-1}(\gamma_1)\Gamma(\gamma_2)\mathbf{A}(\phi)T_2T_1^{-1} \]  
(33)

shows that the elements of \( \Gamma(\gamma_2) \) must also be nonzero, as all the elements of \( \mathbf{A}(\omega) \) have unit magnitude. From (29), it is seen that the columns of \( \Gamma^{-1}(\gamma_1)\mathbf{B}(\mathbf{b}_\omega) \) will span the orthogonal complement of the space spanned by the columns of \( \Gamma(\gamma_1)\mathbf{A}(\omega) \). In addition, \( \Gamma(\gamma_2)\mathbf{A}(\phi) \) and \( \Gamma^{-1}(\gamma_2)\mathbf{B}(\mathbf{b}_\phi) \) are related in the same way. Equation (21) states that \( \Gamma(\gamma_1)\mathbf{A}(\omega) \) and \( \Gamma(\gamma_2)\mathbf{A}(\phi) \) have the same column span. They must then also have the same nullspace. Equation (21) is thus equivalent to

\[ \mathbf{B}(\mathbf{b}_\omega)P = G^*\mathbf{B}(\mathbf{b}_\phi) \]  
(34)

where \( P \) is some \( (m - d) \times (m - d) \) full-rank matrix, and \( G = \Gamma(\gamma_1)\Gamma^{-1}(\gamma_2) \) is a diagonal matrix. As the first elements of \( \Gamma(\gamma_1) \) and \( \Gamma(\gamma_2) \) are both fixed to one, the top diagonal element of \( G \) is also one

\[ G = \text{diag} \{ g_1, g_2, \cdots, g_m \}, \quad g_1 = 1 \]  
(35)

As the matrices \( \mathbf{B}(\mathbf{b}_\omega) \) and \( \mathbf{B}(\mathbf{b}_\phi) \) have \( (m - d)^2 - (m - d) \) zero entries, and \( b_{\omega,0} \) and \( b_{\phi,0} \) are nonzero, it may be established using (30) that \( P \) is a diagonal matrix

\[ P = \text{diag} \{ \alpha_1, \cdots, \alpha_{m-d} \} \]  
(36)

with \( \alpha_i \neq 0 \) for \( i = 1, \cdots, m - d \).

Writing out the equations in (30) and using the structure of \( P \) gives

\[ \left\{ \begin{array}{l} b_{\omega,0} = G_{00}^*b_{\phi,0} \\ b_{\omega,1} = G_{01}^*b_{\phi,1} \\ \vdots \\ b_{\omega,m-d} = G_{0(m-d)}^*b_{\phi,m-d} \end{array} \right. \]  
(37)

Here, \( G_i \) is a \( (d+1) \times (d+1) \) matrix defined as

\[ G_i = \text{diag} \{ g_i, g_{i+1}, \cdots, g_{i+d} \} \]  
(38)

As all the coefficients of \( \mathbf{b}_\omega \) and \( \mathbf{b}_\phi \) are assumed to be nonzero and as it was shown that this implied that the \( \alpha_i \)'s are also
nonzero, the coefficients of $b_\omega$ are all nonzero. Moreover, for $m - d \geq 2$, the only solution to the system is
\[
\frac{g_{l+1}}{g_l} = \lambda, \quad \frac{\alpha_{l+1}}{\alpha_l} = \lambda^c \quad (34)
\]
for some $\lambda$. Next, it is shown that the conjugate symmetry of the coefficients implies that the magnitude of $\lambda$ is one. The conjugate symmetry constraint may be formulated as

\[
b_\omega = b_{\omega}^c, \quad \tilde{\mathbf{I}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

Conjugate the first equation in (33) as

\[
b_{\omega}^c = G_1 b_\phi.
\]

With conjugate symmetry,

\[
b_\omega \alpha_\omega^c = \tilde{G}_1 b_\phi
\]

and combining this with the first equation in (33) gives

\[
G_1 b_{\omega} = \tilde{G}_1 b_\phi
\]

This equation together with (34) and the fact that $g_1 = 1$ proves that $\lambda$ has unit magnitude and, hence, that

\[
\lambda = \exp(j\omega^c)
\]

for some $\omega^c$. This then implies that

\[
G = \text{diag} \{a(\omega^c)\}
\]

and as $G = \Gamma(\gamma_1)\Gamma^{-1}(\gamma_2)$, (26) is established, i.e.,

\[
\gamma_1 = \gamma_2 \circ a(\omega^c)
\]

Since the polynomials were scaled so that the zeroth coefficients have unit magnitude, (33) then shows that $\alpha_1$ has unit magnitude. This, together with (37), gives

\[
\alpha_1 = \pm \exp(-j\omega^c/2).
\]

Using the results of (38) and (40) in (33) relates the coefficients of the polynomials $b_\omega(z)$ and $b_\phi(z)$. From the definitions of the polynomials, it is clear that

\[
b_\omega(z) = \pm e^{-j(\omega^c/2)\omega^c} b_{\omega} \left( z^{e^{j\omega^c}} \right)
\]

which is (27). This establishes the theorem.

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References


[24] M. Viberg and A. Swindlehurst, “A Bayesian approach to auto-

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