Transmitter Optimization for Per-Antenna Power Constrained Multi-Antenna Downlinks: An SLNR Maximization Methodology

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Abstract—In this paper, we consider signal-to-leakage-plus-noise ratio (SLNR) maximized precoding for multi-antenna downlinks under per-antenna power constraints (PAPCs). In contrast to the conventional sum power constraint, the per-antenna power constrained SLNR precoder design problem no longer admits a simple closed-form solution and acquiring its optimal solution turns out to be much more challenging. To conquer the difficulty, we first propose to perform the Charnes–Cooper transformation on the problem of interest, which yields a solvable simpler second-order cone program (SOCP). Moving beyond this approach, we are then able to obtain a semi-closed form solution to the optimal SLNR precoder. The power allocation among users is also optimized for further performance enhancement, for which we devise two numerical methods based on gradient projection and a low-complexity approximation. Finally, focusing on the imperfect channel state information (CSI) scenario, we develop a robust SLNR-based precoder to combat the performance degradation caused by random CSI errors. The superiority of all proposed precoder designs in this work is confirmed by numerical simulations.

Index Terms—Multi-antenna downlinks, signal-to-leakage-plus-noise ratio (SLNR), per-antenna power constraint (PAPC), semidefinite program (SDP), second-order cone program (SOCP), robust precoder design.

I. INTRODUCTION

MULTIUSER multiple-input multiple-output (MU-MIMO) techniques have been widely acknowledged as a powerful means to boost the spectral efficiency of modern wireless communications. Under the MU-MIMO mode, distributed users simultaneously communicate with the transmitter on the same frequency band, thus enhancing data throughput via spatial multiplexing. However, multiuser interference inevitably exists due to the non-orthogonal signaling, which constitutes a major issue for system designs.

Concerning the downlink transmission, researchers have devoted a number of efforts to devising transmit precoders in order to mitigate multiuser interference and exploit the potential gain of MU-MIMO [1]–[12]. Generally, the precoding strategies in the literature can be categorized into nonlinear and linear types in terms of the relationship between the precoded and source signals. Dirty paper coding (DPC) is a famous nonlinear scheme with capacity-achieving capability [1], [2]. Despite its prominent performance, the computational burden of DPC is rather high. Other nonlinear precoders requiring lower complexity include zero-forcing DPC (ZF-DPC) [1], Tomlinson-Harashima precoding (THP) [3], etc. In contrast to nonlinear precoders, linear precoding methods are simpler to implement. Although suboptimal in general, they can effectively suppress multiuser interference and provide satisfactory performance. A classical linear preprocessing technique is zero-forcing beamforming (ZFBF) [4], which applies channel inversion to perfectly null out multiuser interference. However, ZFBF does not aim to maximize signal power, thus leading to performance loss in noise-limited regions. Moreover, channel inversion may not be possible for arbitrary channel dimensions. One approach to overcome both shortcomings is minimum mean-square error (MMSE) beamforming [5], [6], which regularizes the channel inversion applied in ZFBF. Alternatively, the authors in [7] developed a linear precoder by maximizing the so-called signal-to-leakage-plus-noise ratio (SLNR). The SLNR based precoder, which is obtained by using the generalized Rayleigh quotient [13], achieves better signal-to-interference-plus-noise ratio (SINR) and bit error rate (BER) performance than ZFBF since it takes both signal amplification and interference suppression into consideration.

The above-mentioned works all adopt a sum power constraint (SPC) in which the total transmit power does not exceed a given value. However, in practical systems, one power amplifier is usually equipped for each transmit antenna. To accurately characterize this scenario, it is necessary to impose per-antenna power constraints (PAPCs) [14]–[23]. Compared to the precoder designs under SPC, PAPCs usually lead to much more intricate problems which need to be solved via sophisticated
optimization techniques. In particular, the conventional ZFBF under SPC was generalized to the PAPC case in [16], where the authors revealed that the optimal ZF precoding does not have a trivial pseudo-inverse form. Moreover, a recent work [17] studied the optimization of the ZF-DPC based precoder under PAPCs. In addition to the PAPCs, the authors in [18], [19] imposed a constraint on the leakage power for precoder optimization, which effectively controls the inter-user interference. Considering that the SLNR based precoder outperforms ZFBF under SPC, it is natural to investigate whether this conclusion holds for the more complicated PAPC case. Unfortunately, answering this question is non-trivial since, unlike the SPC case, the per-antenna power constrained SLNR precoding design problem cannot be solved in a straightforward manner, and to the best of our knowledge, how to determine its optimal solution is still open.

In this study, we aim to address the SLNR precoder optimization problem with practical PAPCs. Our main contributions are summarized as follows:

• We formulate the SLNR precoder design under PAPCs as a non-convex optimization problem, which cannot be directly solved via a generalized Rayleigh quotient. By employing the Charnes-Cooper transformation [24], we transform the problem into an equivalent second-order cone program (SOCP) that can be optimally solved.

• Inspired by the SOCP based method, we carefully analyze the problem under consideration and successfully derive a semi-closed form solution. The solution reveals an explicit structure of the optimal SLNR precoder under PAPCs, which generalizes the known closed-form solution under SPC [7].

• We further consider enhancing the system performance by optimizing the power allocation for multiple users. A gradient projection (GP) based algorithm is presented to achieve a locally optimal solution to the non-convex power allocation problem. In addition, we also develop an approximation scheme to approach the performance of GP with low complexity.

• We extend the above precoding design to the scenario where channel state information at the transmitter (CSIT) is imperfectly known. Assuming that the first-order and second-order statistics of the CSI errors are available, we devise an expectation based statistical robust SLNR precoder which effectively guarantees system performance in the presence of CSI mismatch.

Notation: Vectors and matrices are denoted by boldface lower-case, and boldface upper-case letters, respectively. The Euclidean norm, maximum norm, transpose, conjugate transpose of vector $a$ are denoted by $|a|$, $|a|_{\infty}$, $a^t$ and $a^{H}$, respectively. $e_i$ denotes a unit vector with its $i$-th element being 1. $\text{tr}(A)$, rank($A$), $A^{-1}$ and $A^t$ represent the trace, rank, inverse, Moore-Penrose pseudo-inverse of matrix $A$, respectively. $(A)_{ij}$ denotes the $(i,j)$-th element of matrix $A$. By $A \succeq 0$ or $A \succ 0$, we mean that the matrix $A$ is positive semidefinite or definite. $\text{max}_\text{geig}(A,B)$ returns the normalized dominant generalized eigenvector of matrix pair $(A,B)$. $\text{E}[\cdot]$ represents the expectation operation and $\mathbb{R}(\cdot)$ returns the real part of the input.

II. SYSTEM MODEL AND PROBLEM FORMULATION

Consider a $K$-user downlink multiple-input single-output (MISO) system where the base station (BS) has $N_t$ antennas, as depicted in Fig. 1. Let $w_k \in \mathbb{C}^{N_t \times 1}$, $h_{kj} \in \mathbb{C}^{1 \times N_t}$ and $\sigma_{n,k}^2$ denote the transmit beamformer for the $k$-th user, the channel between the BS and $k$-th user, and the noise variance of user $k$, respectively. According to [7], the SLNR metric for user $k$ is defined by

$$\text{SLNR}_k = \frac{\mathbb{E}(h_k^H h_k h_j^H w_k)}{\sum_{j \neq k} \mathbb{E}(w_k^H h_j h_j^H w_k) + \sigma_{n,k}^2} \quad (1)$$

where the term $\sum_{j \neq k} \mathbb{E}(w_k^H h_j h_j^H w_k)$ represents the power of the leakage from user $k$ to others. Assuming that the total transmit power budget is $P$ and $\|w_k\|^2 \leq P/K$, i.e., equal power allocation among users, the optimal beamformer that maximizes SLNR is given by [7]

$$w_k^{\text{opt}} = \sqrt{\frac{P}{K} \cdot \text{max}_\text{geig} \left( h_k h_k^H \cdot \sum_{j \neq k} h_j h_j^H + \frac{K \sigma_{n,k}^2}{P} I \right)} \quad (2)$$

Although a sum power constraint is commonly used in the literature for simplifying the transmitter optimization, in practical systems, the radio frequency (RF) front-end of each antenna has its own power amplifier with limited dynamic range. Accordingly, it becomes necessary to constrain per-antenna transmit power instead of total power. Concerning the MISO system illustrated above, the transmit power at the $m$-th antenna is given by $\sum_{k=1}^{K} |w_{k,m}|^2 = \sum_{k=1}^{K} e_m^T w_k w_k^H e_m$, where $w_{k,m}$ is the $m$-th entry of $w_k$. Denoting the power budget of the $m$-th antenna with $P_m$, we obtain per-antenna power constraints as $\sum_{k=1}^{K} e_m^T w_k w_k^H e_m \leq P_m$, $m = 1, \ldots, N_t$. For ease of exposition, here we temporarily assume that the maximum powers allocated to each user at each antenna are identical\(^1\), which is expressed by $e_m^T w_k w_k^H e_m \leq \frac{P_m}{K}$, $m = 1, \ldots, N_t$, $k = 1, \ldots, K$. Now we are able to formulate the per-antenna power

\(^1\)We will include the more general unequal power allocation into our design framework in Section III.B so as to gain further performance improvement.
constrained beamforming design problem via SLNR maximiza-

tion as follows:

\[
\begin{align*}
\text{maximize} & \quad w_k^H h_k h_k^H w_k + \frac{P_m}{\sigma_n^2} \\
\text{subject to} & \quad e_m^T w_k w_k^H e_m \leq \frac{P_m}{K}, \quad m = 1, \ldots, N_t
\end{align*}
\]  

which is specified for each user \(k\).

We need to emphasize that this problem is highly non-convex since the objective function has a complicated quadratic fractional form. Furthermore, it is quite different from the well-known generalized Rayleigh quotient problem due to the multiple quadratic constraints that characterize the power limit for each antenna. Hence, it turns out to be a non-trivial task to achieve the optimal solution of problem (3).

III. OPTIMIZING SLNR PRECODER UNDER PER-ANTENNA POWER CONSTRAINTS

Since it is hard to resolve the per-antenna power constrained SLNR precoder optimization in a straightforward manner, we need to leverage some sophisticated techniques to obtain a tractable form for the original problem. To be more specific, in this section we first apply the Charnes-Cooper transformation to convert problem (3) to an SOCP, from which the globally optimal solution can be achieved. Motivated by this method, we further establish a semi-closed form solution that explicitly reveals the structure of the optimal SLNR precoder under PAPCs.

A. Globally Optimal SLNR Precoder Solution

The major difficulty of addressing problem (3) lies in the objective function, which is clearly non-concave with respect to the optimization variable. To handle this, we first transform problem (3) into a more tractable form by making use of the so-called Charnes-Cooper transformation \[24\]. Specifically, we define

\[ w_k = z_k / \sqrt{\xi_k} \]

where \(\xi_k > 0\) and recast problem (3) in the following form:

\[
\begin{align*}
\text{maximize} & \quad |h_k^T z_k|^2 \\
\text{subject to} & \quad \sum_{j \neq k} |h_j^T z_k|^2 + \xi_k \sigma_n^2 = 1 \\
& \quad |e_m^T z_k|^2 \leq \frac{\xi_k P_m}{K}, \quad m = 1, \ldots, N_t \\
& \quad \xi_k \geq 0.
\end{align*}
\]  

Note that the constraint \(\xi_k \geq 0\) is equivalent to \(\xi_k > 0\) since \(\xi_k = 0\) is not feasible for problem (4), which can be verified by contradiction. Specifically, if \(\xi_k = 0\) is feasible, then according to the first equality constraint, \(\sum_{j \neq k} h_j^T z_k^2 = 1\) must hold. Meanwhile, all the entries of \(z_k\) should be zero due to the inequality constraints, which contradicts the equality \(\sum_{j \neq k} h_j^T z_k^2 = 1\). Although problem (4) is a non-convex quadratically constrained quadratic program (QCQP), we will show that it can be transformed into an SOCP which allows a globally optimal solution.

**Theorem 1:** Problem (4) can be cast as the following SOCP problem:

\[
\begin{align*}
\text{maximize} & \quad \Re \langle h_k^T z_k \rangle \\
\text{subject to} & \quad \Im \{h_k^T z_k\} = 0 \\
& \quad z_k^T \left( e_m e_m^T + \frac{P_m}{K \sigma_n^2} \sum_{j \neq k} h_j h_j^T \right) z_k \leq \frac{P_m}{K \sigma_n^2}, \quad m = 1, \ldots, N_t.
\end{align*}
\]

**Proof:** First, we eliminate the variable \(\xi_k\) by exploiting the equality constraint. To this end, we substitute

\[\xi_k = 1 - \sum_{j \neq k} |h_j^T z_k|^2\]

into the constraints \(|e_m^T z_k|^2 \leq \xi_k P_m / K\), which gives rise to an equivalent form of problem (4) as follows:

\[
\begin{align*}
\text{maximize} & \quad z_k^T h_k^T h_k^H z_k \\
\text{subject to} & \quad z_k^T \left( e_m e_m^T + \frac{P_m}{K \sigma_n^2} \sum_{j \neq k} h_j h_j^T \right) z_k \leq \frac{P_m}{K \sigma_n^2}, \quad m = 1, \ldots, N_t.
\end{align*}
\]

Note that the objective function and constraints in (6) will not be changed by any phase rotation on \(z_k\), thus \(h_k^T z_k\) can be a non-negative real number at the optimal point. This observation implies that, without loss of optimality, we can replace the objective function of problem (6) with \(\Re \langle h_k^T z_k \rangle\), which results in the SOCP problem in (5).

In Algorithm 1, we summarize the SOCP based method to achieve a globally optimal SLNR beamformer for each user \(k\) under PAPCs.

**Algorithm 1:** The SOCP method for solving problem (3)

1: Solve the SOCP problem in (5) using solvers such as SeDuMi [25] or SDPT3 [26] to obtain \(z_k^*\).
2: Calculate the optimal \(\xi_k^* = \frac{1}{\sum_{\frac{1}{2}} |h_j^T z_k^*|^2} \).
3: Calculate the -th user’s optimal SLNR beamforming vector as \(w_k^* = z_k^* / \sqrt{\xi_k^*}\).

Although the SOCP based approach optimally solves problem (3), it does not provide much insight into the structure of the optimal SLNR precoder under PAPCs. To gain a deeper understanding of the solution to problem (3), we need to carefully examine its equivalent QCQP form shown in (6). Despite its non-convexity, we are still able to prove that problem (6) admits strong duality.

**Proposition 1:** The QCQP problem (6) has a zero duality gap, i.e., strong duality holds.

**Proof:** See Appendix A.

The above strong duality property enables us to solve the primal problem (6) via its dual, which actually yields a semi-closed form solution as shown in the subsequent theorem.
Theorem 2: Problem (6) has a semi-closed form optimal solution given by
\[
\mathbf{z}_k^* = \max_{n_{\text{m},k}} \left( \mathbf{h}_k \mathbf{h}_k^H \sum_{m=1}^{N_t} \lambda_{m,k}^* \left( \mathbf{e}_m \mathbf{e}_m^T \frac{P_m}{K \sigma_n^2} + \sum_{j \neq k} \mathbf{h}_j \mathbf{h}_j^H \right) \right)
\]
where \( \lambda_{m,k}^* \) is the optimal solution to the following convex problem
\[
\begin{align*}
\text{minimize} & \quad \frac{1}{K \sigma_n^2} \sum_{m=1}^{N_t} \lambda_{m,k}^* P_m \\
\text{subject to} & \quad \sum_{m=1}^{N_t} \lambda_{m,k}^* \left( \mathbf{e}_m \mathbf{e}_m^T \frac{P_m}{K \sigma_n^2} + \sum_{j \neq k} \mathbf{h}_j \mathbf{h}_j^H \right) \mathbf{h}_k \mathbf{h}_k^H \succeq 0 \\
\end{align*}
\]
and \( d_k \) is given by
\[
d_k = \min \left\{ \sqrt{\frac{P_1}{K \sigma_n^2 \Phi_{1,k} \Phi_{1,k}}} \cdots \sqrt{\frac{P_{N_t}}{K \sigma_n^2 \Phi_{N_t,k} \Phi_{N_t,k}}} \right\}
\]
with
\[
\Phi_{m,k} = \max_{n_{\text{m},k}} \left( \mathbf{h}_k \mathbf{h}_k^H \sum_{m=1}^{N_t} \lambda_{m,k}^* \left( \mathbf{e}_m \mathbf{e}_m^T \frac{P_m}{K \sigma_n^2} + \sum_{j \neq k} \mathbf{h}_j \mathbf{h}_j^H \right) \right)
\]
and
\[
\Phi_{m,k} = \mathbf{e}_m \mathbf{e}_m^T \frac{P_m}{K \sigma_n^2} + \sum_{j \neq k} \mathbf{h}_j \mathbf{h}_j^H, \quad m = 1, \ldots, N_t.
\]

Proof: See Appendix B.

Remark 1: The importance of Theorem 2 lies in the fact that it sheds light on the intrinsic structure of the optimal SLNR beamformer under PAPCs, which, to the best of our knowledge, has not yet been discovered by previous works. By comparing (7) with (2), we observe that the results under both kinds of power constraints share a dominant generalized eigenvector form, which is mainly due to the use of the SLNR metric. However, the optimal beamforming vector under PAPCs differs from the one under SPC in two aspects: 1) It is relevant to \( N_t \) optimal dual variables associated with PAPCs, which can be acquired by solving the problem in (8); 2) The optimal transmit power, i.e., \( d_k^2 / \xi_k^* \), depends on the beamforming direction.

B. Sum Rate Enhancement via Power Allocation

One assumption we adopted in problem (3) is that the maximum powers allocated to all users at each antenna are equal, which is embodied in the PAPCs as
\[
\mathbf{e}_m^T \mathbf{w}_k \mathbf{w}_k^H \mathbf{e}_m \leq \frac{P_k}{K}, \quad m = 1, \ldots, N_t.
\]
Although such a restriction simplifies the beamforming optimization, it may lead to suboptimal system performance. Thereby, for further performance improvement, we can design appropriate power allocation techniques for multiple users. Focusing on the sum rate maximization objective, we formulate the power allocation problem as
\[
\begin{align*}
\text{maximize} & \quad \sum_{k=1}^{K} \log_2 \left( 1 + \frac{P_k \mathbf{h}_k^H \mathbf{w}_k^H \mathbf{e}_m \mathbf{e}_m^T \frac{P_m}{K \sigma_n^2} + \sum_{j \neq k} P_j \mathbf{h}_j^H \mathbf{w}_j^H \mathbf{e}_m \mathbf{e}_m^T \frac{P_m}{K \sigma_n^2} \right) \\
\text{subject to} & \quad \sum_{k=1}^{K} \mathbf{e}_m^T \mathbf{w}_k \mathbf{w}_k^H \mathbf{e}_m \leq P_m, \quad m = 1, \ldots, N_t
\end{align*}
\]
where \( P_k \) is the power allocated to user \( k \) and the beamforming direction is fixed with \( \mathbf{w}_k \) which is the normalized beamformer calculated under the equal power allocation assumption.

We find that it is quite hard to achieve the globally optimal solution of this problem since the objective function is rather complicated. To attain a high-quality solution, we propose to apply the gradient projection (GP) method [28] which iteratively searches for a locally optimal solution. Let \( \mathbf{p}^{(n)} \) denote the value of \( \mathbf{p} \) at the \( n \)-th iteration. Then, the update of \( \mathbf{p} \) consists of three steps:

1) Calculate \( \tilde{\mathbf{p}}^{(n)} = \mathbf{p}^{(n)} + \delta^{(n)} \mathbf{g}^{(n)} \) where \( \delta^{(n)} \) is the step size and \( \mathbf{g}^{(n)} = \begin{bmatrix} \frac{\partial f(\mathbf{p})}{\partial P_1} \cdots \frac{\partial f(\mathbf{p})}{\partial P_K} \end{bmatrix} \) represents the gradient vector with \( f(\mathbf{p}) \) representing the objective function of problem (12). By performing some basic manipulations, we have the \( k \)-th entry of \( \mathbf{g}^{(n)} \) as (9), shown at the bottom of the page.

2) Project \( \tilde{\mathbf{p}}^{(n)} \) onto the feasible solution set of problem (12). The essence of the projection operation is to find a feasible \( \hat{\mathbf{p}}^{(n)} \) that is nearest to \( \tilde{\mathbf{p}}^{(n)} \) in terms of Euclidean distance. This can be expressed by the convex problem below:
\[
\begin{align*}
\text{minimize} & \quad \| \mathbf{p}^{(n)} - \hat{\mathbf{p}}^{(n)} \|^2 \\
\text{subject to} & \quad \sum_{k=1}^{K} \mathbf{e}_m^T \mathbf{w}_k \mathbf{w}_k^H \mathbf{e}_m \leq P_m, \quad m = 1, \ldots, N_t \\
& \quad \mathbf{p}_k^{(n)} \geq 0, \quad k = 1, \ldots, K
\end{align*}
\]
\[\text{Problem (8) is the dual of the semidefinite program (SDP) problem in (31) (see Appendix A). According to a recent finding in [27], the former can be solved more efficiently by a parser, especially for a large number of transmit antennas.} \]
where $\mathbf{\bar{p}}_k^{(n)}$ is the $k$-th entry of $\mathbf{\bar{p}}^{(n)}$. We can efficiently solve this quadratic program using off-the-shelf solvers such as QPC [29].

3) Compute $\mathbf{p}^{(n+1)} = \mathbf{p}^{(n)} + \delta^{(n)}(\mathbf{p}^{(n)} - \mathbf{p}^{(n)})$ where $\delta^{(n)}$ is the step size.

To determine appropriate values for $\delta^{(n)}$ and $\delta^{(n)}$ in Steps 1) and 3), we adopt the well-known Armijo step size rule in [28]. Accordingly, we set $\delta^{(n)} = \delta$ and $\delta^{(n)} = \beta m_n$, where $\beta \in \{0, 1, 0.5\}$ and $m_n$ is chosen as the smallest nonnegative integer such that

\[
I\left(\mathbf{p}^{(n+1)}\right) - I\left(\mathbf{p}^{(n)}\right) \geq \alpha \delta^{(n)} \mathbf{g}^{(n)} \mathbf{T} \left(\mathbf{p}^{(n)} - \mathbf{p}^{(n)}\right)
\]

where $\alpha \in [10^{-5}, 10^{-1}]$. The Armijo rule guarantees the convergence of vector $\mathbf{p}$, i.e., $\|\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)}\| < \epsilon$ where $\epsilon$ is the pre-defined convergence accuracy.

The detailed procedure of the GP method is summarized in Algorithm 2, where the notation $C$ in the fourth line represents the feasible set of problem (12). It is necessary to note that the GP algorithm always obtains a locally optimal solution to the non-convex problem (12). However, it is quite computationally inefficient since it involves an iterative procedure and a convex problem needs to be solved in each iteration. To address this issue, we now develop an alternative low complexity power allocation scheme which only requires solving one convex problem while also providing satisfactory sum rate performance. Specifically, we propose an approximation method that omits the interference term $\sum_{j=1}^{K} p_j k^H \mathbf{w}_j^H$ in the objective function of problem (12), which is intuitively effective in the noise-limited region where the interference power can be much weaker than the noise power. The resulting power allocation problem becomes

\[
\begin{aligned}
\maximize_{\mathbf{p}_k} & \sum_{k=1}^{K} \log_2 \left(1 + \mathbf{p}_k \mathbf{h}_k^H \mathbf{w}_k \mathbf{w}_k \mathbf{H}_m \mathbf{e}_m \right) \\
\text{subject to} & \sum_{k=1}^{K} \mathbf{p}_k \mathbf{e}_m \mathbf{w}_k \mathbf{w}_k \mathbf{H}_m \mathbf{e}_m \leq P_m, m = 1, \ldots, N_t.
\end{aligned}
\]

Algorithm 2: The GP method for solving the power allocation problem in (12)

1: Select initial point, step size and convergence accuracy; Set $n = 1$.
2: Repeat
3: $\mathbf{\tilde{p}}^{(n)} = \mathbf{p}^{(n)} + \delta^{(n)}(\mathbf{p}^{(n)} - \mathbf{p}^{(n)})$.
4: $\mathbf{\tilde{p}}^{(n)} = \text{project}_{C}(\mathbf{\tilde{p}}^{(n)})$.
5: $\mathbf{p}^{(n+1)} = \mathbf{p}^{(n)} + \delta^{(n)}(\mathbf{p}^{(n)} - \mathbf{p}^{(n)})$.
6: $n = n + 1$.
7: Until $\|\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)}\| < \epsilon$.

Clearly, this is a convex problem and thus can be optimally solved. It is interesting to note that, by applying similar techniques in [30], we can further transform problem (16) into an SOCP given by (17), where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ respectively represent the ceiling and floor operations, and $Q = 2^{\log_2 K}$. The detailed procedure is shown in Appendix C. It should be noted that the solution to the simplified problem in (16) or (17) is not optimal for the original one, but it still achieves a remarkably higher sum rate than the equal power allocation scheme over a wide range of signal-to-noise ratios (SNRs), as will be verified by numerical tests in Section V. Moreover, it is also possible to solve the power allocation problem in (12) via other numerical algorithms such as the one in [30].

IV. SLNR PRECODER OPTIMIZATION UNDER PAPCS AND IMPERFECT CSIT

When optimizing the SLNR precoder under PAPCs in the previous section, we assumed that the transmitter has perfect knowledge about the CSI of all users. Unfortunately, due to factors such as channel estimation errors, this assumption is not satisfied in some practical applications. The CSI mismatch sometimes severely degrades system performance. Therefore, it becomes necessary to further study robust SLNR precoder optimization that takes CSI imperfections into account, which is the primary goal of this section.

We consider the following channel error model

\[
\mathbf{h}_k = \mathbf{\bar{h}}_k + \mathbf{e}_k
\]

where $\mathbf{\bar{h}}_k$ denotes the channel estimate of user $k$, and the mean and covariance of the channel error vector $\mathbf{e}_k$ are $\mathbf{0}$ and $\Sigma_k$, respectively. Note that compared to the stochastic CSI mismatch models used in prior works [31]–[34], the above model is more general since we do not restrict the probability density function of $\mathbf{e}_k$. To incorporate the CSI uncertainties, we consider the modified SLNR metric below:

\[
\text{SLNR}_k^* = \frac{\mathbf{w}_k^H \mathbf{P}^\dagger \{\mathbf{h}_k \mathbf{h}_k^H\} \mathbf{w}_k}{\sum_{j \neq k} \mathbf{w}_k^H \mathbf{P}^\dagger \{\mathbf{h}_j \mathbf{h}_j^H\} \mathbf{w}_k + \sigma_{n,k}^2}
\]

\[
= \frac{\mathbf{w}_k^H \mathbf{\bar{h}}_k \mathbf{\bar{h}}_k^H + \Sigma_k \mathbf{w}_k}{\sum_{j \neq k} \mathbf{w}_k^H \mathbf{\bar{h}}_j \mathbf{\bar{h}}_j^H + \Sigma_k \mathbf{w}_k + \sigma_{n,k}^2}
\]

(19)
which averages over CSI error components. Accordingly, the robust SLNR maximization problem under PAPCs takes the form

\[
\text{maximize } \mathbf{w}^H_k \left( \mathbf{h}_k \mathbf{h}_k^H + \Sigma_k \right) \mathbf{w}_k
\]

\[
\text{subject to } e_m^T \mathbf{w}_k \mathbf{w}_k^H e_m \leq P_m/K, \quad m = 1, \ldots, N_t.
\]

Compared to the non-robust design in problem (3), the original rank one matrices \( \mathbf{h}_k \mathbf{h}_k^H \), \( k = 1, \ldots, K \) in the objective function are replaced here with \( \mathbf{h}_k \mathbf{h}_k^H + \Sigma_k \) whose rank will be higher than one in most cases. As we will see later, we can still employ techniques similar to those in Section III to solve problem (20), but some conclusions drawn therein will no longer be valid due to this subtle change.

Remark 2: We would like to highlight that the rationality of adopting metric \( \text{SLNR}_k^* \) in (18) for the imperfect CSI case lies in the following facts: 1) \( \text{SLNR}_k^* \) has an explicit expression and hence simplifies the robust precoder optimization; 2) This metric has also been employed in Sadek’s early paper [7] and some recent works like [35], [36], and has been shown to be effective in providing robustness against CSI errors under SPC; 3) For PAPCs, adopting (19) as the design objective can achieve a noticeable gain over the non-robust method under imperfect CSI (see Section V.B), which is consistent with the phenomenon observed under SPC [7], [35], [36].

A. Solving Problem (20) via SDP

First, similar to the perfect CSIT case, we apply the Charnes-Cooper transformation on problem (20) which yields the QCQP problem:

\[
\text{maximize } \mathbf{z}_k^H \left( \mathbf{h}_k \mathbf{h}_k^H + \Sigma_k \right) \mathbf{z}_k
\]

\[
\text{subject to } \mathbf{z}_k^H \sum_{j \neq k} \left( \hat{\mathbf{h}}_j \hat{\mathbf{h}}_j^H + \Sigma_j \right) \mathbf{z}_k + \xi_k \sigma_{n,k}^2 - 1 = 0
\]

\[
e_m^T \mathbf{z}_k e_m \leq \xi_k P_m/K, \quad m = 1, \ldots, N_t
\]

\[
\xi_k \geq 0.
\]

We eliminate the variable \( \xi_k \) by substituting \( \xi_k = \frac{1}{1 \text{tr} \left( \sum_{j \neq k} \left( \mathbf{h}_j \mathbf{h}_j^H + \Sigma_j \right) \mathbf{Z}_k \right)} \) into problem (21), which now becomes

\[
\text{maximize } \mathbf{z}_k \quad \text{tr} \left( \mathbf{h}_k \mathbf{h}_k^H + \Sigma_k \right) \mathbf{Z}_k
\]

\[
\text{subject to } \text{tr} \left( \sum_{j \neq k} \left( \hat{\mathbf{h}}_j \hat{\mathbf{h}}_j^H + \Sigma_j \right) \mathbf{Z}_k \right) + \xi_k \sigma_{n,k}^2 = 1
\]

\[
e_m^T \mathbf{Z}_k e_m \leq \xi_k P_m/K, \quad m = 1, \ldots, N_t
\]

\[
\mathbf{Z}_k \succeq 0, \quad \xi_k \geq 0.
\]

We find it quite hard to prove that the above problem always has a solution with unit rank. Despite this difficulty, we prove in the subsequent theorem that this conclusion still holds under certain conditions.

Theorem 3: Problem (23) must admit a rank-1 solution when either of the following conditions is satisfied: 1) \( N_t \leq 3 \); 2) \( \sum_{m=1}^{N_t} \lambda_{m,k} \left( e_m e_m^T + \frac{P_m}{K \sigma_{n,k}^2} \sum_{j \neq k} \left( \hat{\mathbf{h}}_j \hat{\mathbf{h}}_j^H + \Sigma_j \right) \right) \) – \( \Sigma_k > 0 \), where \( \lambda_{m,k}^* \) is the solution of the convex problem

\[
\text{minimize } \lambda_{m,k}^* \geq 0
\]

\[
\text{subject to } \sum_{m=1}^{N_t} \lambda_{m,k} \left( e_m e_m^T + \frac{P_m}{K \sigma_{n,k}^2} \sum_{j \neq k} \left( \hat{\mathbf{h}}_j \hat{\mathbf{h}}_j^H + \Sigma_j \right) \right) \geq 0.
\]

Proof: We start with the first condition. By utilizing the rank bound result shown in ([38], Theorem 5.1), the SDP problem in (23) always admits an optimal solution \( \mathbf{Z}_k^* \) satisfying \( \text{rank} \mathbf{Z}_k^* \leq N_t \). Thereby, if \( N_t \leq 3 \), there must exist a \( \mathbf{Z}_k^* \) whose rank is either 0 or 1. Note that \( \mathbf{Z}_k^* = 0 \) is impossible because we can always construct a feasible \( \mathbf{Z}_k = \alpha \mathbf{I} \), \( \alpha > 0 \) such that the objective value is positive. Hence, a rank-1 solution always exists. In fact, when \( N_t \leq 3 \), even if SDP solvers such as SeDuMi [25] return a higher rank solution \( \mathbf{Z}_k \), one can apply rank reduction techniques [38] to generate an optimal rank-1 solution from \( \mathbf{Z}_k^* \).

To validate the second sufficient condition, we need to investigate the following Karush-Kuhn-Tucker (KKT) conditions of problem (23):

\[
\sum_{m=1}^{N_t} \lambda_{m,k} \left( e_m e_m^T + \frac{P_m}{K \sigma_{n,k}^2} \sum_{j \neq k} \left( \hat{\mathbf{h}}_j \hat{\mathbf{h}}_j^H + \Sigma_j \right) \right) - \Sigma_k - \mathbf{h}_k \mathbf{h}_k^H = \Phi_k^*
\]

\[
\Phi_k^* \mathbf{Z}_k^* = 0.
\]

where \( \Phi_k^* \) is the optimal dual variable associated with the constraint \( \mathbf{Z}_k \succeq 0 \). When the condition \( \sum_{m=1}^{N_t} \lambda_{m,k} \left( e_m e_m^T + \frac{P_m}{K \sigma_{n,k}^2} \sum_{j \neq k} \left( \hat{\mathbf{h}}_j \hat{\mathbf{h}}_j^H + \Sigma_j \right) \right) - \Sigma_k > 0 \) is met, we must have \( \text{rank} \left( \Phi_k^* \right) > N_t - 1 \). Moreover, it can be inferred from the equality \( \Phi_k^* \mathbf{Z}_k^* = 0 \) that \( \mathbf{Z}_k^* \) lies in the null space of \( \Phi_k^* \). Therefore, \( \text{rank} \left( \mathbf{Z}_k^* \right) \leq 1 \). Recalling that \( \mathbf{Z}_k^* \neq 0 \), \( \text{rank} \left( \mathbf{Z}_k^* \right) = 1 \) must hold. This conclusion
indicates that the rank of any optimal solution equals one when the second condition holds, which is different from the first condition that only guarantees the existence of a rank-1 solution.

After solving problem (23), we need to acquire the optimal beamformer $\mathbf{w}^*_k$ from $\mathbf{Z}^*_k$. Specifically, if $\text{rank}(\mathbf{Z}^*_k) = 1$, we perform a rank-one decomposition of $\mathbf{Z}^*_k$ to obtain $\mathbf{x}^*_k$ and calculate $\mathbf{w}^*_k = \frac{\mathbf{x}^*_k}{\sqrt{\xi^*_k}}$ where $\xi^*_k = 1 - \sum_{j \neq k} (\mathbf{s}^*_j)^H (\mathbf{h}_j \mathbf{h}_j^H + \Sigma_j) \mathbf{s}^*_j$. Otherwise, we first apply Gaussian randomization [37] to generate an approximate rank-1 solution. It is also interesting to note that, although we cannot prove that the SDP problem in (23) always admits a rank-one optimal solution, we seldom observe solutions with higher rank during numerical simulations.

B. Semi-Closed Form Robust Solution

Although CSI mismatch induces some differences in the SLNR maximized precoder optimization, we show here that a semi-closed form solution is still available for the imperfect CSI case. Let us first consider the problem as follows:

$$
\max_{\mathbf{x}_k} \quad \mathbf{x}_k^H \left( \mathbf{h}_k \mathbf{h}_k^H + \Sigma_k \right) \mathbf{x}_k
$$

subject to

$$
\mathbf{x}_k^H \left( e_m e_m^T + \frac{P_m}{K \sigma_{n,k}^2} \sum_{j \neq k} (\mathbf{h}_j \mathbf{h}_j^H + \Sigma_j) \right) \mathbf{x}_k \
\leq \frac{P_m}{K \sigma_{n,k}^2}, \quad m = 1, \ldots, N_t.
$$

(26)

According to the previous subsection, it can be deduced that this QCQP problem is equivalent to problem (20) and also enjoys the nice property of strong duality, as long as the SDP problem in (23) admits an optimal solution with unit rank, e.g., either condition in Theorem 3 is fulfilled. Therefore, under such conditions, we are able to address problem (20) by solving the Lagrangian dual of problem (26) and eventually obtain a semi-closed form solution to problem (20), which is given in the proposition below.

Proposition 2: Under either condition in Theorem 3, problem (20) has a semi-closed form solution as in (27) where the optimal dual variables $\lambda^*_{m,k}$, $m = 1, \ldots, N_t$ can be achieved via solving problem (24), and the coefficient $d_k$ is given by (28)

$$
\mathbf{z}^*_k = d_k \cdot \max_{\mathbf{h}_k, \mathbf{h}_k^H} \left( \mathbf{h}_k \mathbf{h}_k^H \
+ \Sigma_k, \sum_{m=1}^{N_t} \lambda^*_{m,k} \left( e_m e_m^T + \frac{P_m}{K \sigma_{n,k}^2} \sum_{j \neq k} (\mathbf{h}_j \mathbf{h}_j^H + \Sigma_j) \right) \right)
$$

(27)

$$
d_k = \min \left\{ \sqrt{\frac{P_1}{K \sigma_{n,1}^2 \mathbf{h}_1 \mathbf{h}_1^H \Phi_{1,k} \mathbf{z}_k}}, \ldots, \sqrt{\frac{P_{N_t}}{K \sigma_{n,N_t}^2 \mathbf{h}_N \mathbf{h}_N^H \Phi_{N_t,k} \mathbf{z}_k}} \right\}
$$

(28)

With the optimal $\mathbf{z}^*_k$ given in (27), the robust beamformer for user $k$ takes the form $\mathbf{w}^*_k = \frac{\mathbf{x}^*_k}{\sqrt{\xi^*_k}}$ where $\xi^*_k = 1 - \sum_{j \neq k} (\mathbf{s}^*_j)^H (\mathbf{h}_j \mathbf{h}_j^H + \Sigma_j) \mathbf{s}^*_j$.

Proof: The proof is analogous to that of Theorem 2 and hence is omitted for brevity.

V. SIMULATION RESULTS AND COMPLEXITY ANALYSIS

In this section, we first apply Monte Carlo simulations to investigate the performance of the proposed SLNR precoder designs under PACPs and then analyze the complexity of the considered methods. During simulations, we set $P_m = P/N_t$ and normalize the noise variance $\sigma_{n,k}^2 = 1$ for simplicity. The elements of the channel matrix $\mathbf{H} = [\mathbf{h}_1 \mathbf{h}_1^H, \ldots, \mathbf{h}_N \mathbf{h}_N^H]$ are generated with independent and identically distributed (i.i.d.) complex Gaussian variables with zero mean and unit variance unless otherwise specified.

A. Perfect CSIT Case

Under perfect CSIT, we first consider a homogeneous user scenario where the average channel gains of all users are equal to 1. The sum rate performance of various transmit precoding schemes is depicted in Fig. 2, where the notations “LC” and “GP” represent the low complexity power allocation scheme in (16) and the gradient projection based method in Algorithm 2, respectively, and “Scaled SPC” refers to scaling the optimal SLNR beamformer under SPC such that PACPs are fulfilled.

3Following the proof of Theorem 2, we can readily determine the SLNR maximized scaling factor of user $k$ as $\eta_k^* = \min \left\{ \sqrt{r_1}, \ldots, \sqrt{r_{N_t}} \right\}$, where $w_k \Phi_{m,k}$ is the $k$-th user's SLNR beamformer under SPC.
power allocation for the SLNR precoding scheme provides a performance enhancement. In particular, the proposed low complexity power allocation scheme achieves almost the same sum rate as the gradient projection based algorithm.

We depict the sum rate performance of different precoders versus the number of transmit antennas in Fig. 3, where \( N_t = K \) and the transmit power is fixed to 15 dB. The results demonstrate that the proposed SLNR based precoding schemes outperform the ZF precoder and scaled SPC solution by remarkable sum rate gaps, especially when the antenna number \( N_t \) is large. It should be pointed out that both Figs. 2 and 3 show that the rate maximization algorithm developed in [30] achieves a higher sum rate than the proposed SLNR based precoders. This is due to the fact that we did not adopt sum rate maximization as the objective of the beamforming design. Nevertheless, as will be analyzed in Section V.C, the proposed schemes “SLNR PAPC” and “SLNR PAPC LC” are more computationally efficient than the iterative algorithm in [30]. Moreover, unlike the method in [30], the optimal SLNR beamformers of all users can be calculated in a parallel fashion for both “SLNR PAPC” and “SLNR PAPC LC”, which is an appealing feature.

In Fig. 4, we compare the performance of the two proposed power allocation methods, and we find that the sum rate of the low complexity scheme closely approaches that of the gradient projection based method under different system configurations, which verifies the effectiveness of the approximation utilized by the former scheme.

In the sequel, we focus on a heterogenous users scenario where the average channel gain varies for different users. We set average channel gains to \([5; 0; -6; -10]\) dB and keep other simulation parameters in Fig. 2 unchanged. The corresponding sum rate performance is shown in Fig. 5. Compared to Fig. 2, it turns out that the power allocation schemes provide larger sum rate gains owing to the variation in user channel gains.

Fairness is also an important issue of multiuser communications, especially for the scenario with heterogenous users. Accordingly, we have simulated the SINR cumulative distribution function (CDF) for each precoding method in Fig. 6,
where the average channel gains for users 1~4 are 5, 0, -6, and -10 dB, respectively. As shown in Fig. 6, although the methods “SLNR PAPC GP” and “SLNR PAPC LC” provide larger SINR than “SLNR PAPC” for users with higher channel gains (see Figs. 6(a) and (b)), the opposite happens for users with low channel gains (see Fig. 6(d)). This phenomenon indicates that “SLNR PAPC” achieves better fairness than “SLNR PAPC GP” and “SLNR PAPC LC”, which is due to the fact that the latter two approaches aim to maximize sum rate when devising the power allocation. It can also be observed that the ZF scheme [16] does not favor the users with low channel gains, since it adopts sum rate maximization as the design objective.

Different from previous simulations using Rayleigh fading channels, we now test the proposed schemes under a uniform linear array (ULA) channel model [39], [40]. Specifically, the $k$-th user’s channel is characterized as

$$h_k^H = \frac{1}{N_p} \sum_{l=1}^{N_p} \gamma_{k,l} \alpha_l^H \angle \theta_{k,l}$$

(30)

where $N_p$ represents the number of propagation paths, $\gamma_{k,l}$ is the complex gain of the $l$-th path, and $\alpha_l = (1 + e^{j\pi \sin(\theta_{k,l})}, \ldots, e^{j\pi (N_p-1) \sin(\theta_{k,l})})^T$ denotes the array response vector of a half-wavelength spaced ULA at an azimuth angle of $\theta_{k,l}$. During simulations, we set $N_p = 3$, and assume that $\gamma_{k,l}$ is a zero-mean complex Gaussian variable with unit variance and $\theta_{k,l}$ is uniformly distributed over $[0, 2\pi]$. The corresponding simulated results are shown in Fig. 7, from which we observe that the proposed precoding approaches still exhibit excellent performance.

B. Imperfect CSIT Case

We now investigate the case with imperfectly known CSI at the transmitter. For this scenario, we randomly generate the entries of the channel estimate $\hat{h}_k$ and channel error $\sigma_k$ according to the distributions $CN(0, \sigma_{\hat{h},k}^2 I)$ and $CN(0, \sigma_{\sigma, k}^2 I)$, respectively. Concerning the homogeneous user case, we depict the sum rate performance in Fig. 8, where the notation “RSLNR” refers to the proposed robust SLNR precoding scheme and $\sigma_{\sigma, k}^2 = 0.1$ for any $k$. We note that the
ZF scheme and non-robust SLNR methods are implemented by directly treating the estimated CSI as the true CSI. As can be seen in the figure, the robust design provides excellent performance in the presence of mismatched CSI. Compared to its non-robust counterpart, the robust precoder significantly enhances the sum rate in the interference-limited region, i.e., when the transmit power is relatively high. This makes sense since CSI errors lead to severe multiuser interference that degrades system performance, and the robust design effectively suppresses the interference by taking CSI imperfections into account.

We also test the performance of various precoding methods for different numbers of antennas in Fig. 9, where one can draw similar conclusions as Fig. 3. By varying the channel error variance $\sigma^2_{e,k}$, we compare the robust SLNR precoder design and the non-robust one in Fig. 10. Clearly, the gain achieved by incorporating CSI errors gradually grows with the increase of $\sigma^2_{e,k}$. In Fig. 11, the sum rate of the proposed two power allocation
schemes is compared. It can be observed that the approximation adopted by the low complexity scheme causes a minor performance loss for relatively large transmit power and numbers of antennas. The heterogenous user case is simulated in Fig. 12, where one can observe that both power allocations schemes provide higher rate gains than the homogeneous user case shown in Fig. 8.

C. Complexity Analysis

Before stating the analysis results, we first show the complexity required for solving standard SOCP and SDP problems. It has been pointed out in [41] that an SOCP can be solved with a worst-case complexity as \( \mathcal{O}(K^2N_t^2) \), where \( N_t \) is the number of SOC constraints, the dimension of the optimization variable, the dimension of the \( i \)-th SOC constraint and the solution accuracy. On the other hand, the worst-case complexity of solving an SDP is shown to be \( \mathcal{O}(N_t^3) \), where \( N_t \) is the number of linear constraints and \( N_t^2 \) is the dimension of the positive semidefinite matrix variable [37].

We now list the complexity of the above discussed methods in Table I, where \( n_{GPiter} \), \( n_{MaxRateIter} \) and \( n_{EPiter} \) denote the number of iterations required by the gradient projection based power allocation, the iterative algorithm in [30] and the ellipsoid method in [42]. When analyzing the complexity of the SLNR scheme under the perfect CSIT case, two power allocation schemes, and the rate maximization algorithm in [30], we refer to the expression of \( \nu_{\text{SOC}} \). Note that we assume \( K = 2^n \), \( q > 1 \) when studying the complexity of the low complexity power allocation and the scheme in [30] in order to obtain concise expressions. For the robust SLNR precoder under imperfect CSIT, we apply the result given by \( \nu_{\text{SDP}} \). The complexity of the scaled SPC solution can be readily obtained from the closed-form expression in (2). Finally, the complexity result of the ZF scheme comes from a recent work [43].

TABLE I

<table>
<thead>
<tr>
<th>Method Name</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>SLNR PA PC</td>
<td>( \mathcal{O}(K^2N_t^2) )</td>
</tr>
<tr>
<td>SLNR PA PC LC</td>
<td>( \mathcal{O}(K^2(N_t^2 + (N_t + K)^2)) )</td>
</tr>
<tr>
<td>SLNR PA PC GP</td>
<td>( \mathcal{O}(K^2(N_t^2 + n_{GPiter}(N_t + K)^2)) )</td>
</tr>
<tr>
<td>RSLNR PA PC</td>
<td>( \mathcal{O}(K(N_t^2)) )</td>
</tr>
<tr>
<td>RSLNR PA PC LC</td>
<td>( \mathcal{O}(K(N_t^2 + (N_t + K)^2)) )</td>
</tr>
<tr>
<td>RSLNR PA PC GP</td>
<td>( \mathcal{O}(K(N_t^2 + n_{GPiter}(N_t + K)^2)) )</td>
</tr>
<tr>
<td>MaxRate PA PC [30]</td>
<td>( \mathcal{O}(n_{MaxRateIter}(N_t + K)^2N_t^2) )</td>
</tr>
<tr>
<td>SLNR Scaled SPC</td>
<td>( \mathcal{O}(K^2N_t^2) )</td>
</tr>
<tr>
<td>ZF PA PC [16]</td>
<td>( \mathcal{O}(N_tK^2 + n_{EPiter}KN_t(N_t - K + 1)^2) )</td>
</tr>
</tbody>
</table>

VI. CONCLUSION

We studied SLNR maximized precoding design under practical PAPCs for the multiuser MIMO downlink. The PAPC renders the optimization problem of interest rather complicated, which cannot be straightforwardly solved with generalized Rayleigh quotient as in the case with a sum power constraint. To tackle this difficult non-convex problem, we first converted it to an easier-to-handle SOCP form with the aid of the Charnes-Cooper transformation. Then, based upon this approach, we successfully derived a semi-closed form solution which unveils the structure of the optimal SLNR precoder. Aiming at further enhancing the data rate, we applied a gradient projection algorithm and also developed a low complexity approximation method to optimize the transmit power allocation among users. The robust precoder design was finally considered by maximizing a modified SLNR metric that accommodates the statistics of the CSI errors. Simulations were carried out to verify the effectiveness of the precoder designs established in this work.

APPENDIX A

PROOF FOR Proposition 1

Let us first perform SDR on problem (6). Concretely, we define \( Z_k = z_kz_k^H \) and temporarily relax the constraint \( \text{rank}(Z_k) = 1 \), resulting in the SDP

\[
\begin{align*}
\text{maximize}_{Z_k} & \quad h_k^HZ_kh_k \\
\text{subject to} & \quad \text{tr}\left(\sum_{j \neq k} h_jh_j^H + \frac{P_m}{K\sigma_K^2} \sum_{j \neq k} h_jh_j^H\right) Z_k \leq \frac{P_m}{K\sigma_K^2}, \\
& \quad Z_k \succeq 0.
\end{align*}
\]

According to ([16], Lemma 1), the above problem must admit a rank-1 optimal solution and hence it is tantamount to problem (6).

The Lagrangian dual of the problem in (6) takes the form in (32) where \( \lambda_{1,k}, \ldots, \lambda_{N_t,k} \) are the dual variables associated with the \( N_t \) quadratic constraints of problem (6). Denote the optimal objective value of the inner maximization problem with \( g(\lambda_{1,k}, \ldots, \lambda_{N_t,k}) \). Then, it is easy to obtain from (32) that (33)
holds. Accordingly, the dual of the problem in (6) amounts to

\[
\text{minimize } \frac{1}{K \sigma_{n,k}^2} \sum_{m=1}^{N_t} \lambda_{m,k} P_m
\]

subject to \( \sum_{m=1}^{N_t} \lambda_{m,k} \left( e_m e_m^T + \frac{P_m}{K \sigma_{n,k}^2} \sum_{j \neq k} h_j h_j^H \right) - h_k h_k^H \geq 0 \)

(34)

which can be verified to be exactly identical to the dual of problem (31). Since problem (31) is convex and satisfies Slater’s condition, it must have a zero duality gap [44]. Moreover, by invoking the equivalence between problems (31) and (6), we conclude that strong duality also holds for the QCQP problem in (6).

APPENDIX B
PROOF FOR Theorem 2

Denote the Lagrangian dual function of problem (6) with \( L(z_k, \lambda_{1,k}, \ldots, \lambda_{N_t,k}) \) whose expression is

\[
z_k^H \left( h_k h_k^H - \sum_{m=1}^{N_t} \lambda_{m,k} \left( e_m e_m^T + \frac{P_m}{K \sigma_{n,k}^2} \sum_{j \neq k} h_j h_j^H \right) \right) z_k
\]

\[
+ \frac{1}{K \sigma_{n,k}^2} \sum_{m=1}^{N_t} \lambda_{m,k} P_m.
\]

(35)

Let \( \lambda_{1,k}^*, \ldots, \lambda_{N_t,k}^* \) denote the optimal dual variables of problem (6) which can be obtained by solving problem (8) (see Appendix A). Then, according to Proposition 1 and [44], the optimal \( z_k^* \) must be a minimizer of \( L(z_k, \lambda_{1,k}^*, \ldots, \lambda_{N_t,k}^*) \), or equivalently, \( \nabla_z L(z_k, \lambda_{1,k}^*, \ldots, \lambda_{N_t,k}^*) |_{z_k = z_k^*} = 0 \) holds. By substituting (35) into this equation, we have the equality below:

\[
\left( \sum_{m=1}^{N_t} \lambda_{m,k}^* \left( e_m e_m^T + \frac{P_m}{K \sigma_{n,k}^2} \sum_{j \neq k} h_j h_j^H \right) \right) z_k^* - h_k h_k^H z_k^* = 0.
\]

(36)

Define \( \Upsilon_k = \sum_{m=1}^{N_t} \lambda_{m,k}^* \left( e_m e_m^T + \frac{P_m}{K \sigma_{n,k}^2} \sum_{j \neq k} h_j h_j^H \right) \). Then, \( z_k^* \) should be a generalized eigenvector of matrix pair \( (h_k h_k^H, \Upsilon_k) \) with its corresponding generalized eigenvalue being 1. Moreover, it is clear from (34) that \( \Upsilon_k - h_k h_k^H \geq 0 \) or, equivalently \( z_k^* \Upsilon_k z_k^* \geq z_k^* h_k h_k^H z_k^* \), \( \forall z_k \), which means that the maximum generalized eigenvalue of matrix pair \( (h_k h_k^H, \Upsilon_k) \) cannot be larger than 1. Hence, we have

\[
z_k^* = d_k \cdot \text{max}\text{-geig} (h_k h_k^H, \Upsilon_k)
\]

(37)

with the coefficient \( d_k \) yet to be determined. Since any phase rotation on \( z_k^* \) will not alter the optimal objective value of problem (6), \( d_k \) can be a positive real number at optimality. Moreover, any optimal \( z_k^* \) at least makes one inequality constraint of problem (6) active, because otherwise one can always find another \( z_k' = \epsilon z_k^* \) with \( \epsilon > 1 \), which activates one constraint and corresponds to a larger objective value. Therefore, in light of these two facts, we calculate the value of \( d_k \) as in (9).

With \( z_k^* \) available, it follows from Theorem 1 and the equality constraint in problem (4) that

\[
\zeta_k^* = \frac{1 - \sum_{j \neq k} \langle h_j h_j^H, z_k^* \rangle^2}{\sigma_{n,k}^2}.
\]

Finally, we obtain the optimal beamformer for user \( k \) as \( w_k^* = \frac{z_k^*}{\sqrt{\zeta_k^*}} \).

APPENDIX C
THE PROCEDURE OF RECASTING PROBLEM (16) AS (17)

We first remove the log term in the objective function of problem (16) and accordingly have

\[
\text{maximize } \prod_{k=1}^{K} \left( 1 + p_k |h_k^H \hat{w}_k|^2 \right)
\]

subject to \( \sum_{k=1}^{K} p_k e_m^T \hat{w}_k \hat{w}_k^H e_m \leq P_m, \quad m = 1, \ldots, N_t. \) (38)

In accordance with [30], we can equivalently transform the above problem into an SOCP. More specifically, by employing the equivalence between \( xy \geq t^2 \) and \( [2t, x - y, y] \leq x + y \) with \( x \geq 0 \) and \( y \geq 0 \), we obtain problem (39). Applying the above procedure \( \lceil \log_2 K \rceil - 1 \) times, we eventually arrive at the SOCP problem in (17).

\[
\text{maximize } \prod_{j_1=1}^{Q/2} t_{j_1},
\]

subject to \( \left[ 2t_{j_1,1}, P_{2j_1-1} |h_{2j_1-1}^H \tilde{w}_{2j_1-1}|^2 \right]^T \leq 2 + p_{2j_1-1} |h_{2j_1-1}^H \tilde{w}_{2j_1-1}|^2 + P_{2j_1} |h_{2j_1}^H \tilde{w}_{2j_1}|^2, \quad j_1 = 1, \ldots, \lceil K/2 \rceil
\]

\( t_{j_1,1} \leq p_{2j_1-1} |h_{2j_1-1}^H \tilde{w}_{2j_1-1}|^2 + 1, \quad j_1 = 1, \ldots, \lceil K/2 \rceil \), \( j_1 \) is odd

\( t_{j_1,1} \leq 1, \quad j_1 = \lceil K/2 \rceil + 1, \ldots, Q/2 \)

\( \sum_{k=1}^{K} p_k e_m^T \hat{w}_k \hat{w}_k^H e_m \leq P_m, \quad m = 1, \ldots, N_t
\)

(39)
REFERENCES


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