Aligned Interference Neutralization and the Degrees of Freedom of the $2 \times 2 \times 2$ Interference Channel with Interfering Relays

(Invited Paper)

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Abstract—Previous work showed that the $2 \times 2 \times 2$ interference channel, i.e., the multihop interference network formed by concatenation of two 2-user interference channels, achieves the min-cut outer bound value of 2 DoF. This work studies the $2 \times 2 \times 2$ interference channel with one additional assumption that two relays interfere with each other. It is shown that even in the presence of the interfering links between two relays, the min-cut outer bound of 2 DoF can still be achieved for almost all values of channel coefficients, for both fixed or time-varying channel coefficients. The achievable scheme relies on the idea of aligned interference neutralization as well as exploiting memory at source and relay nodes.

I. INTRODUCTION

Characterizing the capacity of wireless networks is one of the most important problems of network information theory. Rapid progress has been made recently, especially in the settings of (a) multihop multicast and (b) single hop interference networks. The study of multihop multicast networks gave rise to the deterministic channel model, leading to the result that the capacity (within a constant gap) in this setting is given by the network min-cut [1]. The study of single hop interference networks on the other hand produced many outer bounds. Moreover, this setting turns out to be the only setting where interference alignment is essential. This result is generalized by allowing all possible unicast between 2 sources and 2 destinations, i.e., the 2-sink 2-source layered multihop wireless X networks where each source has an independent message for each destination [5]. In this case, it is shown that with arbitrary number of nodes in each layer and with arbitrary connectivity between adjacent layers, the DoF belong to the set of {1, 4/3, 3/2, 5/3, 2}. In this setting, aligned interference neutralization plays more important role to achieve the DoF outer bounds.

As we can see, much progress in DoF characterizations has been made for the 2-sink 2-source layered multihop multiple unicast networks. The layered structure simplifies the problem since the lengths of all paths from sources to destinations are the same, implying that signals generated at different time do not mix. However, if we do not restrict to the layered structure, signals arriving along paths with different lengths will mix. This causes inter-user inter-symbol interference with which additional effort should be made to deal. Due to this difficulty, with a few notable exceptions [8], the problem of multiple unicasts over 2-source 2-sink arbitrary multihop
networks remains wide open. In this work our goal is to make some progress on this problem.

Inspired by the importance of the $2 \times 2 \times 2$ setting to the 2-unicast layered networks as well as its simplicity, we study the $2 \times 2 \times 2$ interference network with relays interfering with each other as shown in Fig. 1. The min-cut DoF outer bound value of this network is still 2. And we will show in this paper that the outer bound can be achieved for almost all channel coefficients. The achievable scheme is based on aligned interference neutralization designed for the $2 \times 2 \times 2$ IC in [7] as well as exploiting memory at sources and relays which helps to cancel interference between relays.

II. SYSTEM MODEL

The $2 \times 2 \times 2$ IC with interfering relays as shown in Fig. 1 is comprised of two sources, two relays and two destinations. Each source node has a message for its respective destination. The received signal at relay $R_k$, $k \in \{1, 2\}$ in time slot $t$ is

$$
Y_{R_k}(t) = F_{k1}(t)X_{S_1}(t) + F_{k2}(t)X_{S_2}(t) + \sum_{j=1}^{2} H_{kj}(t)X_{R_j}(t) + Z_k(t)
$$

where $\bar{k} = 1$ if $k = 2$ and $\bar{k} = 2$ if $k = 1$. $F_{kj}(t)$, $\forall k,j \in \{1,2\}$, is the complex channel coefficient from source $S_j$ to relay $R_k$, $H_{kj}$ is the channel coefficient from relay $R_j$ to relay $R_k$, $X_j(t)$ is the input signal from $S_j$, $X_{R_j}(t)$ is the input signal from relay $R_j$, $Y_{R_k}(t)$ is the received signal at relay $R_k$ and $Z_k(t)$ is the independent identically distributed (i.i.d) zero mean unit variance circularly symmetric complex Gaussian noise. The received signal at destination $D_k$ in time slot $t$ is given by

$$
Y_k(t) = G_{k1}(t)X_{R_1}(t) + G_{k2}(t)X_{R_2}(t) + N_k(t)
$$

where $G_{kj}(t)$, $\forall k,j \in \{1,2\}$, is the complex channel coefficient from relay $R_j$ to destination $D_k$, $Y_k(t)$ is the received signal at $D_k$ and $N_k(t)$ is the i.i.d. zero mean unit variance circularly symmetric complex Gaussian noise. We assume every node in the network has an average power constraint $P$. The relays are full-duplex. In addition, the relays are causal, i.e., the transmitted signals at relays only depend on the past received signals but not the current received signals. We assume that sources and relays know all channels and destination nodes only know channels to relays. To avoid degenerate conditions, we assume the absolute values of all channel coefficients are bounded between a non-zero minimum value and a finite maximum value. We will consider two settings where channel coefficients are time-varying or constant.

1) Channels $F_{kj}(t)$, $G_{kj}(t)$ and $H_{kj}(t)$ are time varying, i.e., the channel coefficients change and are drawn i.i.d. from a continuous distribution for every channel use.

2) Channels $F_{kj}(t)$, $G_{kj}(t)$ and $H_{kj}(t)$ are constant, i.e., the channel coefficients are drawn i.i.d. from a continuous distribution before the transmissions. Once they are drawn, they remain unchanged during the entire transmission. In this case, we will omit the time index for simplicity.

As shown in Fig. 1, there are two messages in the network. Source $S_k$, $k \in \{1, 2\}$ has a message $W_k$ for destination $D_k$. We denote the size of message $W_k$ as $|W_k|$. For the codewords spanning $n$ channel uses, the rates $R_k = \frac{\log(|W_k|)}{n}$ are achievable if the probability of error for both messages can be simultaneously made arbitrarily small by choosing an appropriately large $n$. The sum-capacity $C_S(P)$ is the maximum achievable sum rate. The number of degrees of freedom is defined as

$$
d = \lim_{P \to \infty} \frac{C_S(P)}{\log P}
$$

III. MAIN RESULTS

The main result of this paper is presented in the following theorem.

Theorem 1: For the $2 \times 2 \times 2$ IC with interfering relays defined in Section II, the total number of DoF is equal to 2 for both constant or time-varying channel coefficients, almost surely.

Since the min-cut DoF outer bound is 2, we only need to provide an achievable scheme to achieve 2 DoF for this channel. Recall that for the $2 \times 2 \times 2$ IC where two relays do not interfere with each other, 2 DoF can be achieved using aligned interference neutralization [7]. As we can see, the only difference between the $2 \times 2 \times 2$ IC studied in [7] and the channel studied in this paper is the additional assumption that two relays are interfering with each other. Now suppose if we can cancel the interfering signals between two relays, then the achievable scheme of the $2 \times 2 \times 2$ IC can be applied here as well. In fact, as we will show later, the achievable scheme is designed on top of the achievable scheme for $2 \times 2 \times 2$ IC with additional effort to cancel interference caused by relays. Thus, before presenting the scheme for this channel, it is helpful to first review the achievable scheme for the $2 \times 2 \times 2$ IC.

A. Preliminary - achievable scheme for $2 \times 2 \times 2$ IC

Signalling is performed over $M$ dimensions regardless of whether it is time or rational dimensions. Over these $M$ dimensions, $S_1$ sends $M$ signals and $S_2$ sends $M - 1$ signals so that a total of \(\frac{2M}{M-1}\) DoF is achieved. Since $M$ can be chosen arbitrarily large, we can achieve arbitrarily close to the cut set bound of 2. Here we illustrate the scheme for the constant channel over rational dimensions. With a scaling factor that is needed to satisfy the power constraints, the transmitted signals are

$$
X_1 = \sum_{k_1=1}^{M} v_{1,k_1} a_{k_1}
$$

$$
X_2 = \sum_{k_2=1}^{M-1} v_{2,k_2} b_{k_2}
$$

where $v_{i,m}$ are rational “beamforming” directions. $a_{k_1}$ and $b_{k_2}$ are lattice symbols that will be specified later. Essentially each symbol carries $\frac{1}{M}$ DoF. Then we choose $v_{1,k_1}$ and $v_{2,k_2}$ shown
in Table I and II, so that

\[ F_{11}v_{1,i+1} = F_{12}v_{2,i} \]
\[ F_{21}v_{1,i} = F_{22}v_{2,i}, \quad i \in \{1, \ldots, M-1\}. \]

Note that once \( v_{1,1} \) is determined, then the remaining lattice symbols can be calculated through above equations. We choose \( v_{1,1} = (F_{11}F_{22})^{-1} \). Thus, we have

\[ v_{1,i+1} = (F_{12}F_{21})^{-1}(F_{11}F_{22})^{-1} \]
\[ v_{2,i} = F_{11}F_{22}^{-1}F_{21}^{-1}F_{12}^{-1}. \]

As a consequence, a total of \( 2M-1 \) symbols are aligned in the lattice space such that they can be resolved. In this case, each relay makes hard decisions on the aligned lattice symbols. So \( R_1 \) demodulates \( a_1+a_2+1, \ldots, a_{i+1}+b_i, \ldots, a_M+b_{M-1} \). And \( R_2 \) demodulates \( a_1+b_i, \ldots, a_i+b_i, \ldots, a_{M-1}+b_{M-1}+a_M \). After resolving these symbols, they are transmitted over the second hop again in an aligned fashion similar to the first hop but with phases reversal such that interference is canceled at each destination. Specifically, \( R_1 \) sends the demodulated \( a_1+a_2+1, \ldots, a_{i+1}+b_i, \ldots, a_M+b_{M-1} \) along rational “beamforming” directions \( v_{R_1,1}, v_{R_1,2}, \ldots, v_{R_1,i}, \ldots, v_{R_1,M} \). \( R_2 \) sends the demodulated \( a_1+b_i, \ldots, a_i+b_i, \ldots, a_{M-1}+b_{M-1}+a_M \) along “beamforming” directions \( v_{R_2,1}, v_{R_2,2}, \ldots, v_{R_2,M-1} \). The “beamforming” directions are chosen as follows

\[ G_{11}v_{R_1,i+1} = -G_{12}v_{R_2,i} \]
\[ -G_{21}v_{R_1,i} = G_{22}v_{R_2,i}, \quad \forall i \in \{1, \ldots, M-1\}. \]

Again, once \( v_{R_1,1} \) is determined, then all other scaling factors can be calculated using above equations. We choose \( v_{R_1,1} = (G_{11}G_{22})^{-1} \). Thus, we have \( \forall i \in \{1, \ldots, M-1\} \)

\[ v_{R_1,i+1} = (G_{12}G_{21})^{-1}(G_{11}G_{22})^{-1} \]
\[ v_{R_2,i} = -G_{11}^{-1}G_{12}^{-1}(G_{21}G_{22})^{-1} \]

### B. Achievable scheme for constant channels

Now consider the 2×2×2 IC with interfering relays. Since relays interfere with each other, each relay receives not only signals from sources but also signals from the other relay. If we still want to use the achievable schemes for the 2×2×2, then while a total of \( 2M-1 \) signals from sources are aligned into \( M \) dimensions at relays, due to additional interfering symbols from relays, these aligned signals cannot be resolved. To resolve these aligned symbols, we need to cancel interference from relays. Next, we will illustrate how to achieve this by exploiting memories at transmitters and relays.

1) Intuition: We first illustrate the intuitions behind the achievable schemes. The rigorous description of the achievable scheme is deferred to Section III-B2. First consider time slot 1. \( S_1 \) will send symbol \( a_1(1), \ldots, a_M(1) \) and \( S_2 \) will send symbol \( b_1(1), \ldots, b_{M-1}(1) \) along “beamforming” directions \( v_{1,1}, v_{1,2}, \ldots, v_{1,M} \) and \( v_{2,1}, v_{2,2}, \ldots, v_{2,M-1} \) shown in (5) and (6), respectively. Note that relays are causal, i.e., their transmitted signals only depend on \( past \) received signals but not the current ones. Thus, in the first time slot relays do not transmit any signal. As a consequence, relays only receive signals from sources so that they can demodulate those aligned symbols as shown in Table I and II.

Consider time slot 2. \( S_1 \) and \( S_2 \) will transmit new symbols \( a_1(2), \ldots, a_M(2) \) and \( b_1(2), \ldots, b_{M-1}(2) \) again along \( v_{1,1}, \ldots, v_{1,M} \) and \( v_{2,1}, \ldots, v_{2,M-1} \), respectively, so that they are aligned into \( M \) dimensions. Now consider relays. Relays will transmit what they demodulated in the previous time slot using the scheme for the 2×2×2 IC as described in the Section III-A. Specifically, \( R_1 \) sends the demodulated \( a_1(1), a_2(1)+b_1(1), \ldots, a_{i+1}(1)+b_i(1), \ldots, a_M(1)+b_{M-1}(1) \) through “beamforming” directions \( v_{R_1,1}, v_{R_1,2}, \ldots, v_{R_1,i}, \ldots, v_{R_1,M} \) as shown in (9). \( R_2 \) sends the demodulated \( a_1(1)+b_1(1), \ldots, a_i(1)+b_i(1), \ldots, a_{M-1}(1)+b_{M-1}(1) \) through “beamforming” directions \( v_{R_2,1}, v_{R_2,2}, \ldots, v_{R_2,i}, \ldots, v_{R_2,M-1} \) as shown in (10). Due to interfering links between relays, these symbols are not only received at destinations where they can be decoded as in the 2×2×2 IC, but also they are received at relays. As a consequence, each relay not only receives \( 2M-1 \) new symbols from sources which are aligned into \( M \) dimensions but also receives previous demodulated symbols. Since signalling is over \( M \) dimensions and there are more than \( M \) signals that reside in each relay’s signal space, \( M \) new aligned symbols cannot be resolved. To resolve them, we need to cancel all the interference symbols from relays. Next, we will illustrate how to achieve this.

Recall that each relay has demodulated the sum of previous symbols. Thus, some of the interference can be canceled using these demodulated symbols. The interfering symbols and the demodulated symbols at \( R_1 \) and \( R_2 \) are listed in Table III and IV, respectively. Then our goal is to cancel all interfering
symbols $a_1(1), \ldots, a_M(1)$ and $b_1(1), \ldots, b_{M-1}(1)$.

First, consider canceling $a_1(1)$. At $R_1$, since $a_1(1)$ is received along $H_{12}v_{R_2,1}$ as shown in Table III, it can be canceled by adding $-H_{12}v_{R_2,1}a_1(1)$ to its received signal. Similarly, $R_2$ can cancel $a_1(1)$ by adding $-H_{21}v_{R_1,1}(a_1(1) + b_1(1))$ to its received signal.

Second, consider canceling $b_1(1)$. First consider $R_2$. While $a_1(1)$ can be canceled using memory at $R_2$, $b_1(1)$ cannot be canceled using the memory because it is aligned with $a_1(1)$. But the interfering $a_1(1)$ is not aligned with $b_1(1)$.

To cancel $b_1(1)$, we rely on $S_2$. Note that after adding $-H_{21}v_{R_1,1}(a_1(1) + b_1(1))$ to the received signal at $R_2$ to cancel $a_1(1)$, the effective direction of $b_1(1)$ becomes $H_{21}(v_{R_2,2} - v_{R_1,1})$. Then $S_2$ can resend $b_1(1)$ along $u_{2,1} = -F_{22}^{-1}H_{21}(v_{R_2,2} - v_{R_1,1})$ such that $b_1(1)$ can be canceled at $R_2$. Now consider $R_1$. Note that the effective direction of $b_1(1)$ becomes the sum of the direction of interfering $b_1(1)$ from $R_2$ and that of the resent $b_1(1)$ from $S_2$, which is $H_{12}v_{R_2,1} + F_{12}u_{2,1}$. Therefore, $R_1$ will add $-(H_{12}v_{R_2,1} + F_{12}u_{2,1})(a_2(1) + b_1(1))$ to its received signal to cancel $b_1(1)$.

Third, consider canceling $a_2(1)$. First consider $R_1$. Again, since $b_1(1)$ is not aligned with $a_2(1)$ for the interfering signals at $R_1$, we cannot cancel $a_2(1)$ using memory after canceling $b_1(1)$. Instead, $S_1$ will resend $a_1(1)$ along a direction such that it is canceled at $R_1$. Note that the effective direction of $a_1(1)$ at $R_1$ is the sum of the direction of interfering $a_1(1)$ from $R_2$ and that of added $a_2(1) + b_1(1)$, which is $H_{12}(v_{R_2,2} - v_{R_1,1}) - F_{12}u_{2,1}$. Therefore, the “beamforming” direction of $a_1(1)$ is chosen as $u_{1,1} = -F_{11}^{-1}(H_{12}(v_{R_2,2} - v_{R_1,1}) - F_{12}u_{2,1})$. Now consider $R_2$. The effective direction of $a_2(1)$ is the sum of the direction of the interfering symbol from $R_1$ and that of the resent $a_2(1)$ from $S_1$, which is $H_{21}v_{R_1,2} + F_{21}u_{1,1}$. Then we will add $-(H_{21}v_{R_1,2} + F_{21}u_{1,1})(a_2(1) + b_1(1))$ to the received signal at $R_2$ to cancel $a_2(1)$.

Continuing this procedure, the remaining interference symbols can be canceled. Then after we cancel $a_1(1)$, we have following general way to cancel $a_2(1), \ldots, a_M(1)$ and $b_1(1), \ldots, b_{M-1}(1)$. Let $u_{1,0} = 0$ and $b_{M}(1) = 0$. For all $i = 1, \ldots, M - 1$,

- Cancel $b_1(1)$. First consider $R_2$. The effective direction of $b_1(1)$ is $H_{21}(v_{R_1,i+1} - v_{R_2,i}) - F_{21}u_{1,i+1}$. Thus, $S_2$ will resend $b_i(1)$ along $u_{2,i} = -F_{22}^{-1}(H_{21}(v_{R_1,i+1} - v_{R_2,i}) - F_{21}u_{1,i+1})$ to cancel $b_i(1)$. Then consider $R_1$. The effective direction of $b_1(1)$ is $H_{12}v_{R_2,i} + F_{12}u_{2,i}$. Therefore, it will add $-(H_{12}v_{R_2,i} + F_{12}u_{2,i})(a_{i+1}(1) + b_i(1))$ to its received signal to cancel $b_i(1)$.

- Cancel $a_{i+1}(1)$. First consider $R_1$. The effective direction of $a_{i+1}(1)$ is $H_{12}(v_{R_1,i+1} - v_{R_2,i}) - F_{12}u_{2,i}$. Therefore, $S_1$ will resend $a_{i+1}(1)$ along the direction $u_{1,i} = -F_{11}^{-1}(H_{12}(v_{R_1,i+1} - v_{R_2,i}) - F_{12}u_{2,i})$. Now consider $R_2$. The effective direction of $a_{i+1}(1)$ is $H_{21}v_{R_1,i+1} + F_{21}u_{1,i}$. Therefore, $R_2$ will add $-(H_{21}v_{R_1,i+1} + F_{21}u_{1,i})(a_{i+1}(1) + b_{i+1}(1))$ to cancel $a_{i+1}(1)$.

2) Detailed description of the achievable scheme: Now we have the following achievable schemes.

Sources: At source node $S_1$, message $W_1$ is split into $M$ sub-messages. Sub-message $W_{1,k_1}, k_1 \in \{1, \ldots, M\}$, is encoded using a codebook with the codeword of length $\ell$ denoted as $a_{k_1}(1), \ldots, a_{k_1}(\ell)$. Similarly, at source node $S_2$, message $W_2$ is split into $M - 1$ sub-messages. Sub-message $W_{2,k_2}, k_2 \in \{1, \ldots, M - 1\}$, is encoded using a codebook with the codeword of length $\ell$ denoted as $b_{k_2}(1), \ldots, b_{k_2}(\ell)$. For any $\epsilon > 0$ and a constant $\gamma$, let $C$ denote all integers in the interval $\left[-\gamma P^{\frac{1-\epsilon}{M+\epsilon}}, \gamma P^{\frac{1-\epsilon}{M+\epsilon}}\right]$, i.e.,

$$ C = \left\{ x : x \in \mathbb{Z} \cap \left[-\gamma P^{\frac{1-\epsilon}{M+\epsilon}}, \gamma P^{\frac{1-\epsilon}{M+\epsilon}}\right] \right\}. $$  

$a_{k_1}(t)$ and $b_{k_2}(t)$ are obtained by uniform i.i.d. sampling on $C$. Essentially, each sub-message carries $\frac{1-\epsilon}{M+\epsilon}$ DoF. Then at time slot $1$, the transmitted signals are

$$ X_1(1) = A \sum_{k_1=1}^{M} v_{1,k_1} a_{k_1}(t) $$  

$$ X_2(1) = A \sum_{k_2=1}^{M-1} v_{2,k_2} b_{k_2}(t) $$

where $v_{1,k_1}$ and $v_{2,k_2}$ are chosen as in (5) and (6), respectively. To satisfy the power constraints, the constant scaling factor $A$ is chosen to be $\xi P^{\frac{1-\epsilon}{M+\epsilon}}$ where $\xi = \min\left(\frac{1}{\sum_{k_1=1}^{M} v_{1,k_1}^2}, \frac{1}{\sum_{k_2=1}^{M-1} v_{2,k_2}^2}\right)$ as in [7].

### Table III

<table>
<thead>
<tr>
<th>Dimension of interfering symbols</th>
<th>$H_{12}v_{R_2,1}$</th>
<th>$H_{12}v_{R_2,2}$</th>
<th>$H_{12}v_{R_2,3}$</th>
<th>$H_{12}v_{R_2,M-1}$</th>
<th>$H_{12}v_{R_2,M}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interfering symbols</td>
<td>$a_1(1) + b_1(1)$</td>
<td>$a_2(1) + b_2(1)$</td>
<td>$a_3(1) + b_3(1)$</td>
<td>$a_{M-1}(1) + b_{M-1}(1)$</td>
<td>$a_M(1) + b_M(1)$</td>
</tr>
<tr>
<td>Demodulated symbols</td>
<td>$a_1(1)$</td>
<td>$a_2(1) + b_1(1)$</td>
<td>$a_3(1) + b_1(1)$</td>
<td>$a_{M-1}(1) + b_{M-2}(1)$</td>
<td>$a_M(1) + b_{M-1}(1)$</td>
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### Table IV

<table>
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<tr>
<th>Dimension of interfering symbols</th>
<th>$H_{21}v_{R_1,1}$</th>
<th>$H_{21}v_{R_1,2}$</th>
<th>$H_{21}v_{R_1,3}$</th>
<th>$H_{21}v_{R_1,M-1}$</th>
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Then at time $t = 2, \ldots, n$, the transmitted signals are

$$X_1(t) = A' \left( \sum_{k_1=1}^{M} v_{1,k_1} a_{k_1}(t) + \sum_{i=1}^{M-1} u_{1,i} a_{i+1}(t-1) \right)$$

$$X_2(t) = A' \left( \sum_{k_2=1}^{M} v_{2,k_2} b_{k_2}(t) + \sum_{i=1}^{M-1} u_{2,i} b_{i}(t-1) \right)$$

where

$$u_{2,i} = -F_{22}^{-1}(H_{21}(v_{1,i+1} - v_{R_i})) - F_{21} u_{1,i}(t-1)$$

$$u_{1,i} = -F_{11}^{-1}(H_{21}(v_{R_{i+1}} - v_{R_i}) - F_{12} u_{2,i}(t)).$$

Note that $u_{1,0}$ is defined to be 0. And $A'$ should be chosen to satisfy the power constraints, i.e.,

$$E[X_j^2(t)] \leq \gamma^2 A'^2 \left( \sum_{k_j=1}^{M_j} v_{j,k_j}^2 + \sum_{i=1}^{M-1} u_{j,i}^2 \right) P^{\frac{1}{\xi_j}} \leq P$$

where $M_1 = M$ and $M_2 = M - 1$. To satisfy power constraints at both transmitters, we choose $A' = \min(C, B)$ where $C$ and $B$ are scaling factors that can ensure sources and relays satisfy the power constraints, respectively. Specifically, $C = \frac{\xi_1 P^{\frac{M+1}{M-\gamma}}}{\gamma}$ and $\xi_1$ is given in (16), and $B = \frac{\xi_2 P^{\frac{M+2}{M-\gamma}}}{\gamma}$ where $\xi_2 = \min(\frac{1}{\xi_1}, \frac{1}{\xi_2})$ and $\xi_2 = \sum_{m,n=1}^{M_j} |v_{R_m,n}|$ as given in (7).

Relays: Let us consider relays. First consider the first time slot. As mentioned before, since relays do not transmit in the first time slot, the received signals are the same as the $2 \times 2 \times 2$ IC [7], which is omitted here. The relays will demodulate the aligned symbols as described in [7]. Specifically, $R_1$ will demodulate $a_1(1), a_1(1) + b_1(1), \ldots, a_M(1) + b_{M-1}(1)$ as $\hat{x}_{R_1,1}(1), \hat{x}_{R_1,2}(1), \ldots, \hat{x}_{R_1,M}(1)$, respectively. $R_2$ will demodulate $a_1(1) + b_1(1), \ldots, a_1(1) + b_1(1), \ldots, a_{M-1}(1) + b_{M-1}(1), a_M(1)$ as $\hat{x}_{R_2,1}(1), \hat{x}_{R_2,2}(1), \ldots, \hat{x}_{R_2,M-1}(1), \hat{x}_{R_2,M}(1)$, respectively.

Consider time slot $t = 2$. Let us first consider the transmitted signal. Relays will send the demodulated signals at time slot 1. Specifically, the transmitted signals are

$$X_{R_1}(2) = A' \sum_{k_1=1}^{M} v_{R_1,k_1} \hat{x}_{R_1,k_1}(1)$$

$$X_{R_2}(2) = A' \sum_{k_2=1}^{M} v_{R_2,k_2} \hat{x}_{R_2,k_2}(1)$$

where $A'$ is chosen to satisfy the power constraint and $v_{R_1,k_1}$ and $v_{R_2,k_2}$ are given in (9) and (10), respectively.

Now consider the received signals:

$$Y_{R_1}(2) = F_{11}X_1(2) + F_{12}X_2(2) + H_{12}X_{R_2}(2) + Z_1(2)$$

$$Y_{R_2}(2) = F_{22}X_2(2) + F_{21}X_1(2) + H_{21}X_{R_1}(2) + Z_2(2)$$

Relays will locally generate some signals and add them to their received signals before demodulating the aligned symbols. Specifically, $R_1$ will generate the following signal and add it to its received signal:

$$\hat{Y}_{R_1}(2) = A'(-H_{12}v_{R_2,1} \hat{x}_{R_1,1}(1)$$

$$- \sum_{i=1}^{M-1} (H_{12}v_{R_2,i} + F_{12}u_{2,i}) \hat{x}_{R_1,i+1}(1)).$$

Next, $R_1$ will demodulate the aligned symbols using the signal $Y_{R_1}(2) + \hat{Y}_{R_1}(2)$ as follows. Define the following constellation set

$$C_{R_1} = \{ A' (F_{11}v_{1,1}x_{R_1,1} + \cdots + F_{11}v_{1,M}x_{R_1,M}) \}.$$

where $x_{R_1,1}$ is an integer in the interval $\left[ -\gamma P^{\frac{1}{\xi_1}}, \gamma P^{\frac{1}{\xi_1}} \right]$ and $x_{R_1,M}$ is also an integer but in the interval $\left[ -2\gamma P^{\frac{1}{\xi_2}}, 2\gamma P^{\frac{1}{\xi_2}} \right]$. Notice that $v_{1,1}, \ldots, v_{1,M}$ are distinct monomial functions of channel coefficients and thus rationally independent almost surely. Thus, there is a one-to-one mapping from $C_{R_1}$ to $x_{R_1,k_1}$, $k_1 \in \{1, \ldots, M\}$. Then Relay $R_1$ will find the point in $C_{R_1}$, which has the minimal distance between $Y_{R_1}(2) + \hat{Y}_{R_1}(2)$, and then make a hard decision on $x_{R_1,k_1}$ by mapping the point to $\hat{x}_{R_1,k_1}(2)$.

Similarly, $R_2$ will generate and then add the following signal to its received signal:

$$\hat{Y}_{R_2}(2) = A'(-H_{21}v_{R_1,1} \hat{x}_{R_2,1}(1)$$

$$- \sum_{i=1}^{M-1} (H_{21}v_{R_1,i+1} + F_{21}u_{1,i}) \hat{x}_{R_2,i+1}(1))$$

Then, $R_2$ will demodulate the aligned symbols from $\hat{Y}_{R_2}(2) + \hat{Y}_{R_2}(2)$ as $\hat{x}_{R_2,1}(2), \ldots, \hat{x}_{R_2,M}(2)$ in the same manner.

After relays demodulate the aligned symbols, they will transmit them in the next time slot. Then at time slot $t = 3, \ldots, n + 1$, relays will follow the same procedure. Simply replacing the time indices 2 and 1 in the above analysis with $t$ and $t - 1$, respectively, we can obtain the equations for the transmitted and received signals at relays.

**Destinations:** The destinations will use the same decoding method as $2 \times 2 \times 2$ IC in [7]. The probability of error of all the demodulations will go to zero as SNR approaches infinity. Then, the coded rate for each symbol approaches 1 bit according to the rational dimension framework [3]. Therefore, a total of $\frac{2M}{M-1}$ DoF can be achieved. As $M$ goes to infinity, $\frac{2M}{M-1} \to 2$.

**Remark:** Note that the scheme presented above works for all $H_{12}$ and $H_{21}$. Thus, 2 DoF can be achieved for some special cases of $H_{12}$ and $H_{21}$ as well. For example, if $H_{12}$ or $H_{21}$ is equal to zero, i.e., interference is unidirectional, then 2 DoF can still be achieved using the above scheme. Another special case is when $H_{12} = H_{21}$, for which again 2 DoF can be achieved. Also note that while the scheme is designed for real channels where all signals, channel coefficients and noises are real values, it can be easily generalized to the complex case using Theorem 7 in [6].
C. Achievable scheme for time-varying channels

So far, we have considered the case when the channels are constant. For the time-varying channels, the scheme designed before can be applied as well. In this case, $M$ rational dimensions become $M$ time dimensions which are obtained through symbol extensions, and rational “beamforming” directions are replaced by linear beamforming vectors. Unlike the constant channel case, the sources will generate codewords using Gaussian codebook and relays will separate aligned symbols in each dimension through linear operations instead of demodulating them. Next, we will provide the achievable scheme.

Communication is performed over $n+1$ blocks, each block consisting of $M$ channel uses, which is referred to as $M$ symbol-extended channel. Over the extended channel, the input-output relations are

$$ Y_{R_k}(t) = F_{k1}(t)X_1(t) + F_{k2}(t)X_2(t) + H_{kk}X_{R_k}(t) + Z_k(t) $$

$$ Y_k(t) = G_{k1}(t)X_{R1}(t) + G_{k2}(t)X_{R2}(t) + N_k(t) $$

where $k = 1, 2$ if $k = 2, 1$, respectively and $t = 1, \ldots, n+1$.

\[ F_{kj}(t) = \begin{bmatrix} F_{kj}(M(t-1)+1) & 0 & \cdots & 0 \\ 0 & F_{kj}(M(t-1)+2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_{kj}(M(t)) \end{bmatrix} \]

\[ G_{kj}(t) = \begin{bmatrix} G_{kj}(M(t-1)+1) & 0 & \cdots & 0 \\ 0 & G_{kj}(M(t-1)+2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_{kj}(M(t)) \end{bmatrix} \]

and $X, Y, Z$ and $N$ are $M \times 1$ vectors representing $M$ symbol extensions of $X, Y, Z$ and $N$, respectively.

Sources: At source node $S_1$, message $W_1$ is split into $M$ sub-messages. Sub-message $W_{1,k_1}$, $k_1 \in \{1, \ldots, M\}$, is encoded using a Gaussian codebook with the codeword of length $n$ denoted as $a_{k_1}(1), \ldots, a_{k_1}(n)$. Similarly, at source node $S_2$, message $W_2$ is split into $M$ sub-messages. Sub-message $W_{2,k_2}$, $k_2 \in \{1, \ldots, M-1\}$, is encoded using a Gaussian codebook with the codeword of length $n$ denoted as $b_{k_2}(1), \ldots, b_{k_2}(n)$. Each sub-message carries one DoF. Over the $t$th, $t = 1, \ldots, n$ block (the sources do not transmit in the $(n+1)$th block), the transmitted signals are

$$ X_1(t) = \sum_{k_1=1}^{M} v_{1,k_1}(t)a_{k_1}(t) + \sum_{i=1}^{M-1} u_{1,i}(t)a_{i+1}(t-1) $$

$$ X_2(t) = \sum_{k_2=1}^{M-1} v_{2,k_2}(t)b_{k_2}(t) + \sum_{i=1}^{M-1} u_{2,i}(t)b_{i}(t-1) $$

where $v_{1,k_1}$ and $v_{2,k_2}$ are chosen in the same manner as the $2 \times 2 \times 2$ IC in [7] so that $2M-1$ symbols are aligned into $M$ dimensions at relays as shown in Table I and II. Specifically, $v_{1,1}$ is an $M \times 1$ vector with all elements equal to one, i.e., $v_{1,1}(t) = [1 1 \cdots 1]^T$ and $\forall i = 1, \ldots, M-1$

$$ v_{1,i+1}(t) = (F_{11}^{-1}(t)F_{12}(t)F_{21}^{-1}(t))^{i}v_{1,1}(t) $$

$$ v_{2,i}(t) = (F_{22}^{-1}(t)F_{21}(t)F_{12}^{-1}(t))^{i-1}F_{22}^{-1}(t)F_{21}(t)v_{1,1}(t). $$

Note that when $t = 1$, $a_{i+1}(0)$ and $b_{i}(0)$ are set to be zero. In addition, $u_{k,i}$ are chosen essentially in the same manner as the constant case to cancel interfering symbols between two relays. Specifically, by replacing channel coefficients with channel matrices and rational “beamforming” numbers with linear beamforming vectors in (14) and (15), we have

$$ u_{2,i}(t) = -F_{21}^{-1}(t)(H_{21}(t)v_{R_1,i+1}(t) - v_{R_1,i}(t)) $$

$$ v_{1,i}(t) = -F_{11}^{-1}(t)(H_{12}(t)v_{R_2,i+1}(t) - v_{R_2,i}(t)) $$

(20)

(21)

Note that when $i = 1$, $u_{i,i-1}(t) = u_{i,0}(t)$ which is set to be zero.

Relays: Similar to the constant channel case, each relay will locally generate proper signal and add it to its received signal so as to cancel interfering symbols from the other relay. At block $t = 1, \ldots, n+1$, $R_1$ and $R_2$ will generate the following signals and add them to their received signals at block $t$, respectively:

$$ Y_{R_1}(t) = -H_{12}(t)v_{R_2,t}(t)x_{R_1,t}(t-1) - \sum_{i=1}^{M-1} (H_{12}(t)v_{R_{2,i},t}(t) + F_{12}(t)u_{2,i}(t)) $$

$$ x_{R_1,i+1}(t-1) $$

$$ Y_{R_2}(t) = -H_{21}(t)v_{R_1,t}(t)x_{R_2,t}(t-1) - \sum_{i=1}^{M-1} (H_{21}(t)v_{R_{1,i+1},t}(t) + F_{21}(t)u_{1,i}(t)) $$

$$ x_{R_2,i+1}(t-1). $$

where $x_{R_1,t}(t-1), \ldots, x_{R_1,M}(t-1)$ and $x_{R_2,t}(t-1), \ldots, x_{R_2,M}(t-1)$ are the resolved aligned symbols with noise in the previous block, i.e., block $t-1$, which will be specified shortly. Note that when $t = 1$, i.e., in the first block, $x_{R_1,k}(0)$ and $x_{R_2,k}(0)$ are set to zero, which leads to $Y_{R_1}(0) = Y_{R_2}(0) = 0$. This means that in the first block, relays do not add any signals to their received signals since in the first block relays do not transmit, implying that no interference between two relays. After adding the locally generated signals, each relay cancels all interference from the other relay and thus can resolve the $M$ aligned symbols in its $M$ dimensional signal space by inverting the effective channels.
of the $M$ symbols. For $R_k$, $k = 1, 2$

$$
\begin{bmatrix}
  x_{R_{k,1}}(t) \\
  x_{R_{k,2}}(t) \\
  \vdots \\
  x_{R_{2,i+1}}(t) \\
  \vdots \\
  x_{R_{k,M}}(t)
\end{bmatrix} = F_{R_k}^{-1}(t)(Y_{R_k}(t) + \hat{Y}_{R_k}(t))
$$

where $F_{R_k}(t) = [F_{k,k}(t)v_{1,1}(t) F_{k,k}(t)v_{1,2}(t) \cdots F_{k,k}(t)v_{1,M}(t)]$. Since interference between relays are completely canceled, the resolved aligned symbols (ignoring noise) are just the ones shown in Table I and II.

After resolving the aligned symbols, they are sent with one block delay and in an aligned manner exactly the same as are just the ones shown in Table I and II.

Specifically, the transmitted signals in block $t$ are

$$
X_{R_1}(t) = \sum_{k_1=1}^{M} v_{R_1,k_1}(t)x_{R_1,k_1}(t-1)
$$

$$
X_{R_2}(t) = \sum_{k_2=1}^{M-1} v_{R_2,k_2}(t)x_{R_2,k_2}(t-1)
$$

where $v_{R_1,k_1}$ and $v_{R_2,k_2}$ are

$$
v_{R_1,i+1}(t) = (G_{11}^{-1}(t)G_{12}(t)G_{21}^{-1}(t)G_{21}(t))^{i} v_{R_1,1}(t)
$$

$$
v_{R_2,i}(t) = -(G_{22}^{-1}(t)G_{21}(t)G_{12}^{-1}(t)G_{12}(t))^{i-1} G_{22}^{-1}(t)G_{21}(t)v_{R_1,1}(t)
$$

where $v_{R_1,1}(t) = [1 \cdots 1]^T$.

**Destinations**: Since interference is canceled, each destination can decode its desired symbols. The decoding method is the same as $2 \times 2 \times 2$ IC in [7] and thus is omitted here.

### IV. $2 \times 2 \times 2$ INTERFERENCE NETWORKS WITH ARBITRARY CONNECTIVITY

With the achievable scheme for the $2 \times 2 \times 2$ IC with interfering relays, the result can be easily generalized to $2 \times 2 \times 2$ interference networks with arbitrary connectivity as shown in Fig. 2. The red dashed lines indicate arbitrary connectivity on top of the layered fully connected $2 \times 2 \times 2$ interference channel. Note that there may exist two way connections between any two nodes. It can be shown that using the achievable scheme for the $2 \times 2 \times 2$ IC with interfering relays derived in last section, the outer bound value of 2 DoF can be achieved with arbitrary connectivity except if there is a direct link between a source and its interfering destination. In this case, the DoF collapse to 1 as already shown in [8].

The detailed proof can be found in [9].

### V. CONCLUSION

We show that the $2 \times 2 \times 2$ interference network with interfering relays achieves the DoF outer bound of 2 for almost all channel coefficients. Compared to the layered $2 \times 2 \times 2$ interference networks where the relays do not interfere with each other, the achievable scheme exploits the memory at source and relay nodes to cancel the additional interference between relays. While the setting focused in this paper is special, we believe the idea of exploiting memory with aligned interference neutralization will be useful to deal with the interference arriving along paths with different lengths in many multihop multiple unicast networks which do not have a layered structure.

While the added intra-layer links to the $2 \times 2 \times 2$ layered interference network do not affect the DoF, the inter-layer link between a source and its interfering destination lead to a DoF collapse. To understand if the inter-layer links create a DoF bottleneck for the multihop multiple unicast network, we study a class of non-layered two unicast wireless networks in [9]. Specifically, we consider the $2^n$ interference network which is formed by concatenation of $n$ two user interference channels. On top of this network, we allow arbitrary inter-layer links but no intra-layer links. We show that the min-cut outer bound value of 2 DoF can be always achieved except if there is a direct link between a source and its interfering destination.

### REFERENCES


