Degrees of Freedom Region of Three-User MIMO Interference Channels

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***Abstract*—While the outer-bound of sum degrees of freedom (DoF) of 3-user interference channel is known, the entire DoF region is still unknown in terms of both outer-bound and achievable region. In this paper, we first give the outer-bound of DoF region. Then, we present a linear beamforming scheme based on interference alignment chain whose achievable DoF region is the same as the outer bound, with the consideration of integer DoF only.**

I. Introduction

The degrees of freedom (DoF) characterization has been recently studied for a variety of wireless networks, among which, the interference channel has drawn intensive research interest. The DoF of 2-user interference channels was fully characterized in [1]. Beyond the 2-user case, however, the only scenario in which the optimal DoF is known for K-user interference channels is when $M_T = M_R$ [2], [3], where M_T and M_R denote the number of antennas on each transmitter and receiver, respectively. When $M_T \neq M_R$, the DoF issue is not completely settled even for three-user interference channels.

The DoF of K-user $M_T \times M_R$ MIMO interference channel was studied in [4]–[6]. Specifically, [4] showed that if $\eta = \frac{\max(M_T, M_R)}{\min(M_T, M_R)}$ is an integer, each user can achieve DoF of $\min(M_T, M_R)\frac{\eta}{\eta+1}$ when $K > \eta$. The result of [4], established originally over time-varying channels, was extended to constant channels without the need for channel extensions in [5], [6]. The optimal sum DoF was solved in [7] for three-user case only, where the idea of subspace alignment chain was introduced and the outer-bound of sum DoF was derived. According to [7], the outer-bound DoF of each link equals DoF^*, where

$$DoF^* = \min\{\frac{\kappa}{2\kappa-1}M \ , \ \frac{\kappa}{2\kappa+1}N\} \tag{1}$$

where $N = \max\{M_T \ , \ M_R\}$, $M = \min\{M_T \ , \ M_R\}$ and $\kappa = \lceil \frac{M}{N-M} \rceil$. Hence, the outer-bound of the sum DoF of the network is equal to $3DoF^*$.

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As can be seen, there are still some problems left unsettled for 3-user interference channels. If the DoF of each user represents one coordinate axis, a 3-dimensional coordinates can be formed for a 3-user channel case. Let d_i denote the DoF of user i. The outer-bound of sum DoF, $d_1+d_2+d_3 \leq 3DoF^*$, can be seen as a plane in the coordinates, which is not enough to characterize the entire DoF region. For example, when $M_T = 5, \ M_R = 3$, we have $d_1 + d_2 + d_3 \leq 6$ according to (1), but it is also obvious that $d_i \leq 3$.

To obtain the exact DoF region, we first give the outer-bound DoF region based on the derivation of the outer-bound of sum DoF in [7]. Then, we present a linear beamforming scheme whose achievable DoF region is the same as the outer bound.

The paper is organized as follows. In Section II, the system model is introduced, and the outer bound of DoF region is given. In Section III, a beamforming scheme is presented based on the concept of interference alignment chain. Section IV investigates and summarizes the constraints of the parameters that are involved in the scheme. In Section V, the achievable DoF region is derived based on the constraints and shown to be the same as the outer bound. Section VI concludes the paper.

II. System Model

We consider a fully connected 3-user MIMO interference channel with M_T and M_R antennas at each transmitter and each receiver, respectively. Transmitter i transmits messages intended to receiver i ($i = 1, \ 2, \ 3$), and hence causes interference to other two receivers. Let $\mathbf{H}_{ji} \in \mathbb{C}^{M_R \times M_T}$ denote the channel from transmitter i to receiver j, the received signals on receiver j can be expressed as

$$\mathbf{y}_j = \sum_{i=1}^{3} \mathbf{H}_{ji}\mathbf{B}_i\mathbf{m}_i + \mathbf{z}_j \tag{2}$$

where $\mathbf{y}_j \in \mathbb{C}^{M_R \times 1}$ denotes the received signal; $\mathbf{B}_i \in \mathbb{C}^{M_T \times d_i}$ denotes the beamforming matrix of transmitter i; $\mathbf{m}_i \in \mathbb{C}^{d_i \times 1}$ denotes the original message vector from transmitter i; $\mathbf{z}_j \in \mathbb{C}^{M_R \times 1}$ denotes the white Gaussian noise at receiver j. Let d_i denote the DoF of user i, the sum DoF of the network is $D = \sum_{i=1}^{3} d_i$.

Let $N = \max\{M_R\ ,\ M_T\}$, $M = \min\{M_R\ ,\ M_T\}$. The outer-bound of DoF region of three-user interference channels is

$$\begin{cases} 2d_i + 2d_j + d_k \leq 2N \\ d_i + d_j \leq N \\ d_i + d_j + d_k \leq 2M \\ d_i \leq M \end{cases} \tag{3}$$

when $\frac{M}{N} \in [\frac{1}{2}\ ,\ \frac{2}{3}]$,

$$\begin{cases} 2td_i + 2td_j + (2t-1)d_k \leq 3tM \\ (2t+1)d_i + 2td_j + 2td_k \leq 3tN \\ d_i + d_j \leq N \end{cases} \tag{4}$$

when $\frac{M}{N} \in [\frac{3t-1}{3t}\ ,\ \frac{3t}{3t+1}]$, (where $t = 1,\ 2,\ \cdots\infty$) and

$$\begin{cases} (2t+1)d_i + 2td_j + 2td_k \leq (3t+1)M \\ (2t+1)d_i + (2t+1)d_j + (2t+1)d_k \leq (3t+1)N \\ d_i + d_j \leq N \end{cases} \tag{5}$$

when $\frac{M}{N} \in [\frac{3t}{3t+1}\ ,\ \frac{3t+1}{3t+2}]$

$$\begin{cases} (2t+2)d_i + (2t+2)d_j + (2t+1)d_k \leq (3t+2)N \\ (2t+1)d_i + 2td_j + 2td_k \leq (3t+1)M \\ (2t+1)d_i + (2t+1)d_j + (2t+1)d_k \leq (3t+2)M \\ d_i + d_j \leq N \end{cases} \tag{6}$$

when $\frac{M}{N} \in [\frac{3t+1}{3t+2}\ ,\ \frac{3t+2}{3t+3}]$. (where $i,\ j,\ k = 1,\ 2,\ 3$ and $i \neq j \neq k$.)

The outer bound DoF region is derived based on the existing derivations of the sum DoF in [7].

III. A Beamforming Scheme

In this section, we present a beamforming scheme that can achieve all the combination of integer DoF in the outer-bound. We first explain the concept of alignment chain. Then, the design of beamforming matrices will be elaborated. At last, as part of the beamforming design, we discuss how to ensure the signals are linearly decoded at each receiver. We assume $M_T \geq M_R$. Then, $N = M_T,\ M = M_R$.

A. Subspace Alignment Chain

We let $\mathbf{V}^t_{i(s)} \in \mathbb{C}^{N\times Q_t}$ denote the sth Q_t-dimensional subspace transmitted by transmitter i which participates in the chain that originates from transmitter t. Now, we consider one alignment chain originating from transmitter 1, where $\mathbf{V}^1_{1(1)}$ is nulled at receiver 2 but causes an interference dimension at receiver 3. The second signal, $\mathbf{V}^1_{2(1)}$ from transmitter 2, should be aligned with $\mathbf{V}^1_{1(1)}$ on receiver 3 so that no more interference dimension is generated on receiver 3. Then, if $\mathbf{V}^1_{2(1)}$ can be zero-forced at receiver 1, the chain is finished; Otherwise, transmitter 3 should send a vector, $\mathbf{V}^1_{3(1)}$, which is aligned with $\mathbf{V}^1_{2(1)}$ on receiver 1. The chain will keep going until zero-forcing can be achieved. Mathematically, it can be expressed as follows,

$$\underbrace{\begin{bmatrix} \mathbf{H}_{21} & 0 & \cdots & \cdots & 0 \\ \mathbf{H}_{31} & \mathbf{H}_{32} & 0 & \cdots & 0 \\ 0 & \mathbf{H}_{12} & \mathbf{H}_{13} & 0 & \cdots \\ \vdots & \ddots & \cdots & & \\ 0 & \cdots & \cdots & \mathbf{H}_{ri} & \mathbf{H}_{rt} \\ 0 & 0 & \cdots & 0 & \mathbf{H}_{it} \end{bmatrix}}_{\mathbf{H}\in\mathbb{C}^{M(S+1)\times S\cdot N}} \underbrace{\begin{bmatrix} \mathbf{V}^1_{1(1)} \\ \mathbf{V}^1_{2(1)} \\ \mathbf{V}^1_{3(1)} \\ \mathbf{V}^1_{1(2)} \\ \vdots \\ \mathbf{V}^1_{i(s)} \\ \mathbf{V}^1_{t(s)} \end{bmatrix}}_{\mathbf{V}\in\mathbb{C}^{S\cdot N\times Q_1}} = \mathbf{0} \tag{7}$$

where S denotes the total number of subspaces that involved in the chain, i.e., the length of the chain.

As can be seen, the chain can be finished once the matrix $\mathbf{H}$ turns into a "fat" matrix, i.e., $S\cdot N > (S+1)M \Rightarrow S > \frac{M}{N-M}$. Hence, the length of the shortest chain can be expressed as

$$S = \begin{cases} \kappa + 1 & \text{when } \frac{M}{N} = \frac{p}{p+1} \\ \kappa & \text{when } \frac{M}{N} \neq \frac{p}{p+1} \end{cases} \tag{8}$$

where $\kappa = \lceil \frac{M}{N-M} \rceil$ and p is an arbitrary natural number. Note that for any length that is larger than S, $\mathbf{H}$ will always be a "fat" matrix, which means for each antenna configuration, there exists multiple chains with length equals $S,\ S+1, \cdots$.

We refer to the chains with the length of S as the original alignment chains. As can be seen, there are three original chains and each originates from one transmitter, i.e., $t = 1,\ 2,\ 3$, and each chain has S Q_t-dimensional subspaces. We denote the three original chains as follows

$$\begin{aligned} &\mathbf{0} \overset{R_2}{\longleftrightarrow} \mathbf{V}^1_{1(1)} \overset{R_3}{\longleftrightarrow} \mathbf{V}^1_{2(1)} \overset{R_1}{\longleftrightarrow} \mathbf{V}^1_{3(1)} \overset{R_2}{\longleftrightarrow} \mathbf{V}^1_{1(2)} \cdots \mathbf{0} \\ &\mathbf{0} \overset{R_3}{\longleftrightarrow} \mathbf{V}^2_{2(1)} \overset{R_1}{\longleftrightarrow} \mathbf{V}^2_{3(1)} \overset{R_2}{\longleftrightarrow} \mathbf{V}^2_{1(1)} \overset{R_3}{\longleftrightarrow} \mathbf{V}^2_{2(2)} \cdots \mathbf{0} \\ &\mathbf{0} \overset{R_1}{\longleftrightarrow} \mathbf{V}^3_{3(1)} \overset{R_2}{\longleftrightarrow} \mathbf{V}^3_{1(1)} \overset{R_3}{\longleftrightarrow} \mathbf{V}^3_{2(1)} \overset{R_1}{\longleftrightarrow} \mathbf{V}^3_{3(2)} \cdots \mathbf{0} \end{aligned} \tag{9}$$

where $\mathbf{V}^1_{1(1)} \overset{R_3}{\longleftrightarrow} \mathbf{V}^1_{2(1)}$ means that the interference generated by $\mathbf{V}^1_{1(1)}$ and $\mathbf{V}^1_{2(1)}$ at receiver 3 are aligned together, i.e., $\mathbf{H}_{31}\mathbf{V}^1_{1(1)} = \mathbf{H}_{32}\mathbf{V}^1_{2(1)}$, as shown in (7).

B. A Beamforming Scheme

In [7], the outer-bound of sum DoF is obtained based on the original chains only. However, it is not enough to achieve the entire DoF region. In our proposed scheme, the beamforming matrix contains three types of subspaces that are designed according to original chains, long chains (with length $\bar{S} = S+1$), and the null space of interfering channels, respectively.

The beamforming matrix of transmitter i can be expressed as $\mathbf{B}_i = \begin{bmatrix} \mathbf{V}_i & \bar{\mathbf{V}}_i & \mathbf{U}_i \end{bmatrix}$, where $\mathbf{V}_i$ is composed of all the subspaces from transmitter i that participate in the original chains (as shown in (9)), i.e., $\mathbf{V}_i = \begin{bmatrix} \mathbf{V}^1_{i(1)} & \cdots & \mathbf{V}^2_{i(1)} & \cdots & \mathbf{V}^3_{i(2)} & \cdots \end{bmatrix}$, $\bar{\mathbf{V}}_i$ is composed of all the subspaces from transmitter i that participate in the longer chains (which is similar to (9) except with one more subspace at the end of each chain), and $\mathbf{U}_i = \begin{bmatrix} \mathbf{U}_i^1 & \mathbf{U}_i^2 \end{bmatrix} \in \mathbb{C}^{M \times q_i}$, which is designed as follows

$$\begin{aligned} \mathbf{H}_{21}\mathbf{U}_1^1 = \mathbf{0},\ \mathbf{H}_{31}\mathbf{U}_1^2 = \mathbf{0} \\ \mathbf{H}_{12}\mathbf{U}_2^1 = \mathbf{0},\ \mathbf{H}_{32}\mathbf{U}_2^2 = \mathbf{0} \\ \mathbf{H}_{13}\mathbf{U}_3^1 = \mathbf{0},\ \mathbf{H}_{23}\mathbf{U}_3^2 = \mathbf{0} \end{aligned} \tag{10}$$

To ensure the desired signals on each receiver can be linearly decoded, two conditions must be satisfied, i.e.,

1) The beamforming matrix, $\mathbf{B}_i$, has full column rank.
2) On each receiver, the desired signal space does not overlap the interference space.

• **Condition 1**

First, we need to guarantee that each single subspace, ($\mathbf{V}_{i(s)}^t$ and $\bar{\mathbf{V}}_{i(\bar{s})}^t$), has full column rank. From (7) we can see that each chain is in the null space of $\mathbf{H}$, which means the column rank of $\mathbf{V}$ cannot be larger than the nullity of $\mathbf{H}$, i.e.,

$$Q_t \leq S \cdot N - (S+1)M \tag{11}$$

$$Q_s = \sum_{t=1}^{3} Q_t \leq 3(S \cdot N - (S+1)M) \tag{12}$$

Similarly, for long chains we have

$$\bar{Q}_t \leq (S+1) \cdot N - (S+2)M \tag{13}$$

$$\bar{Q}_s = \sum_{t=1}^{3} \bar{Q}_t \leq 3((S+1) \cdot N - (S+2)M) \tag{14}$$

where $\bar{Q}_t$ is the number of dimensions of $\bar{\mathbf{V}}_{i(\bar{s})}^t$.

Next, we discuss the full rank of $\mathbf{B}_i$. Take $\mathbf{B}_1$ for example, among all the subspaces in $\mathbf{B}_1$, three of them, $\begin{bmatrix} \mathbf{V}_{1(1)}^1 & \bar{\mathbf{V}}_{1(1)}^1 & \mathbf{U}_1^1 \end{bmatrix}$, are all in the null space of $\mathbf{H}_{21}$. Hence, we should guarantee that

$$Q_1 + \bar{Q}_1 + q_{31} \leq N - M \tag{15}$$

where q_{ji} denotes the number of interference dimensions generated on receiver j by $\mathbf{U}_i$. Specifically, we have $\mathbf{U}_1^1 \in \mathbb{C}^{N \times q_{31}}$, $\mathbf{U}_1^2 \in \mathbb{C}^{N \times q_{21}}$, $\mathbf{U}_2^1 \in \mathbb{C}^{N \times q_{32}}$, $\mathbf{U}_2^2 \in \mathbb{C}^{N \times q_{12}}$, $\mathbf{U}_3^1 \in \mathbb{C}^{N \times q_{23}}$, $\mathbf{U}_3^2 \in \mathbb{C}^{N \times q_{13}}$.

Similarly, for $\mathbf{B}_2$ and $\mathbf{B}_3$ we can get

$$Q_2 + \bar{Q}_2 + q_{12} \leq N - M \tag{16}$$

$$Q_3 + \bar{Q}_3 + q_{23} \leq N - M \tag{17}$$

Then, it can be proved that under the constraints (11)-(17), $\mathbf{B}_i$ will have full rank for sure. (The proof is omitted.)

• **Condition 2**

Since the direct channel matrices, $\mathbf{H}_{11}$, $\mathbf{H}_{22}$ and $\mathbf{H}_{33}$, are not used in the design of beamforming subspaces, **Condition 2** can be satisfied as long as the sum number of dimensions of the desired signals and interference does not exceed the number of dimensions on each receiver.

Take receiver 1 for example, the number of desired signals and interference dimensions is equal to d_1 and $P_1 + q_{12} + q_{13}$, respectively, where P_1 denotes the number of interference dimensions that are generated by the six alignment chains. Hence, we have

$$d_1 + P_1 + q_{12} + q_{13} \leq M \tag{18}$$

Similarly, for receivers 2 and 3, we have

$$d_2 + P_2 + q_{21} + q_{23} \leq M \tag{19}$$

$$d_3 + P_3 + q_{31} + q_{32} \leq M \tag{20}$$

Therefore, **Condition 2** can be satisfied by (18)-(20).

IV. Constraints

Based on the beamforming scheme, the DoF of each user is determined by the value of Q_t, $\bar{Q}_t$ and q_t, (where $q_1 = q_{21} + q_{31},\ q_2 = q_{12} + q_{32},\ q_3 = q_{13} + q_{23}$). Hence, before exploring the bounds of DoF, we need to find out all the constraints of these parameters. Note that some constraints have been given in (11)-(20).

We first transform (15)-(17), and (18)-(20) into expressions which are related to the DoF of each user. Since $q_{31} = q_1 - q_{21}$, (15) can be expressed as

$$N - M - Q_1 - \bar{Q}_1 \geq q_1 - q_{21} \tag{21}$$

and (19) can be expressed as

$$M - d_2 - P_2 \geq q_{21} + q_{23} \tag{22}$$

By adding up (21) and (22), we can get

$$N - Q_1 - \bar{Q}_1 - d_2 - P_2 \geq q_1 + q_{23} \tag{23}$$

Since $q_1 + q_{23} \leq q_s$, (23) can be guaranteed by

$$N - Q_1 - \bar{Q}_1 - d_2 - P_2 \geq q_s \tag{24}$$

We can see that satisfying (24) is not equivalent to satisfying both (21) and (22). However, note that q_1, q_{21} and q_{23} are integers that can be as small as zero. Hence, as long as $N - M - Q_1 - \bar{Q}_1 \geq 0$ and $M - d_2 - P_2 \geq 0$, we can always find suitable values of q_1, q_{21} and q_{23} that satisfy (21) and (22) under the constraint of (24).

Since $N - M - Q_1 - \bar{Q}_1 \geq 0$ which is guaranteed by (15), we let (24) and the following inequality to be the constraints instead of (15) and (19).

$$M - d_2 - P_2 \geq 0 \tag{25}$$

Similarly, based on (16), (17), (18) and (20), we have

$$M - d_1 - P_1 \geq 0 \tag{26}$$

$$M - d_3 - P_3 \geq 0 \tag{27}$$

$$N - Q_3 - \bar{Q}_3 - d_1 - P_1 \geq q_s \tag{28}$$

$$N - Q_2 - \bar{Q}_2 - d_3 - P_3 \geq q_s \tag{29}$$

The constraints (15)-(20) are converted into (24)-(29).

Next, note that (11)-(14) give the upper bounds of Q_t, Q_s, $\bar{Q}_t$ and $\bar{Q}_s$. We also need to find the lower bounds of these parameters.

First, the total DoF of the network can be calculated as

$$D = d_1 + d_2 + d_3 = S \cdot Q_s + (S+1)\bar{Q}_s + q_s \tag{30}$$

where $q_s = q_1 + q_2 + q_3$.

Then, since each original chain and long chain occupy $2S-1$ and $2S+1$ dimensions, respectively, and there are totally $3M$ dimension on the receivers' side, we have

$$(2S-1)Q_s + (2S+1)\bar{Q}_s + 2q_s \leq 3M \tag{31}$$

By taking (30) into (31), we can get

$$Q_s + \bar{Q}_s \geq 2D - 3M \tag{32}$$

(32) indicates the lower bound of $Q_s + \bar{Q}_s$.

Then, we find the lower bound of Q_s. Note that the number of signals participate in the original chains equals $S \cdot Q_s$. Assuming the rest $D - S \cdot Q_s$ signals all participate in the longer chain, each of them takes at least $\frac{2S+1}{S+1}$ dimensions. Hence, we have

$$\frac{2S+1}{S+1}(D - S \cdot Q_s) + (2S-1)Q_s \leq 3M \tag{33}$$

which leads to

$$Q_s \geq [(2S+1)D - 3(S+1)M]^+ \tag{34}$$

where $[A]^+ = \max\{A,\ 0\}$.

Moreover, since $Q_s = \sum_{i=1}^{3} Q_i$ and $Q_i \leq S \cdot N - (S+1)M$ (according to (11)), we can get

$$Q_i \geq [(2S+1)D - (S+1)M - 2SN]^+ \tag{35}$$

Next, since $\bar{Q}_s \geq 2D - 3M - Q_s$ and $Q_s \leq 3(S \cdot N - (S+1)M)$, we have

$$\bar{Q}_s \geq [2D - 3SN + 3SM]^+ \tag{36}$$

Since $\bar{Q}_i \leq (S+1)N - (S+2)M$, we can get

$$\bar{Q}_i \geq [2D - (5S+2)N + (5S+4)M]^+ = 0 \tag{37}$$

As a result, the constraints can be summarized as (11)-(14), (24)-(29), and (32)-(37).

V. Achievable Degrees of Freedom Region

In this section, we characterize the achievable DoF region based on the obtained constraints. The network region we are interested in is $\frac{1}{2} < \frac{M}{N} < 1$, which can be expressed as $[\frac{S-1}{S}\ ,\ \frac{S}{S+1})$, (where $S = 2, 3, \cdots \infty$). It can be divided into four cases, i.e., $S = 2$, $S = 3t$, $S = 3t+1$ and $S = 3t+2$, where $t = 1, 2, \cdots \infty$. The achievable DoF region will be studied for different cases.

Note that for 3-user interference networks, the DoF region is the combination of the bounds of D, $d_i + d_j$, and d_i. We first investigate the bounds of D which have general forms for all four cases. Then, the bounds of $d_i + d_j$ and d_i will be developed according to different cases.

Note that (32) and (34) is the lower bound of $Q_s + \bar{Q}_s$ and Q_s, respectively. Since $\bar{Q}_s \geq 0$, the lower bound of $Q_s + \bar{Q}_s$ cannot be less than that of Q_s, i.e.,

$$\begin{aligned} 2D - 3M &\geq [(2S+1)D - 3(S+1)M]^+ \\ &\Rightarrow 2D - 3M \geq 0 \text{ and } (2S-1)D \leq 3SM \end{aligned}$$

Since $2D - 3M \geq 0$ is trivial, it can be written as

$$D = d_1 + d_2 + d_3 \leq \frac{3SM}{2S-1} \tag{38}$$

Next, since $S \cdot N - (S+1)M \geq Q_i$ and $Q_i \geq [(2S+1)D - (S+1)M - 2SN]^+$ (from (35)), we can get

$$\begin{aligned} S \cdot N - (S+1)M &\geq [(2S+1)D - (S+1)M - 2SN]^+ \\ \Rightarrow D &\leq \frac{3SN}{2S+1} \end{aligned} \tag{39}$$

(38) and (39) are the two bounds of D that are applied to all cases. As we can see, these two bounds are in fact the M-bound and N-bound of sum DoF, respectively.

Next, we investigate the bounds of $d_i + d_j$ and d_i in different cases.

- $\frac{1}{2} < \frac{N}{M} < \frac{2}{3}$ ($S = 2$)

According to (30), we can get

$$d_1 + d_2 = D - (Q_s + \bar{Q}_s) + Q_1 - q_3 \tag{40}$$

Since $Q_s + \bar{Q}_s \geq 2D - 3M$ (from (32)), $Q_1 \leq 2N - 3M$ (from (11)) and $q_3 \geq 0$, we can get

$$d_1 + d_2 \leq 2N - D \tag{41}$$

which is equivalent to

$$2d_1 + 2d_2 + d_3 \leq 2N \tag{42}$$

Since the links are interchangeable, we have

$$2d_i + 2d_j + d_k \leq 2N \tag{43}$$

Next, since $d_2 = Q_1 + Q_2 + \bar{Q}_s + q_2$ and $P_2 = Q_3 + \bar{Q}_2 + \bar{Q}_3$, (24) can be written as

$$D - (Q_s + \bar{Q}_s) + Q_1 \quad \leq \quad N \tag{44}$$

By taking (44) into (40), we have

$$d_i + d_j \leq D - (Q_s + \bar{Q}_s) + Q_1 \leq N \tag{45}$$

In addition, (25) can be expressed as

$$d_2 \leq M - (Q_s + \bar{Q}_s - (Q_1 + Q_2 + \bar{Q}_1)) \tag{46}$$

which can lead to

$$d_i \leq \min\{7N - 6M - 2D,\ M,\ 4N + 4M - 5D\} \tag{47}$$

Finally, by combining all the bounds, the DoF region is proved to be the same as (3).

- $\frac{3t-1}{3t} \leq \frac{N}{M} < \frac{3t}{3t+1}$ ($S = 3t$)

In this case, we have $d_i = t \cdot Q_s + t \cdot \bar{Q}_s + \bar{Q}_i + q_i$. Accordingly,

$$d_1 + d_2 = D - t(Q_s + \bar{Q}_s) - \bar{Q}_3 - q_3 \tag{48}$$

Since $\bar{Q}_3 \geq 0$ (according to (37)), $q_3 \geq 0$ and $Q_s + \bar{Q}_s \geq 2D - 3M$, we have

$$d_1 + d_2 \leq D - t(Q_s + \bar{Q}_s) \leq 3tM - (2t-1)D \tag{49}$$

which leads to

$$2td_i + 2td_j + (2t-1)d_k \leq 3tM \tag{50}$$

Then, since $d_2 = t(Q_s + \bar{Q}_s) + \bar{Q}_2 + q_2$ and $P_2 = t \cdot (Q_s + \bar{Q}_s) - Q_1$, (24) can be written as

$$2tQ_s + (2t+1)\bar{Q}_s + q_s - \bar{Q}_3 + q_2 \leq N \tag{51}$$

which can lead to

$$D - t(Q_s + \bar{Q}_S) - \bar{Q}_3 \leq N - q_2 \leq N \tag{52}$$

By taking (52) into (48), we can get

$$d_1 + d_2 \leq N \;\Rightarrow\; d_i + d_j \leq N \tag{53}$$

Next, (25) can be expressed as

$$d_2 \leq M - t(Q_s + \bar{Q}_s) + Q_1 \tag{54}$$

Since $Q_s + \bar{Q}_s \geq 2D - 3M$ and $Q_1 \leq 3tN - (3t+1)M$, we have $d_2 \leq 3tN - 2tD$, which leads to

$$(2t+1)d_i + 2td_j + 2td_k \leq 3tN \tag{55}$$

Finally, the achievable DoF region for $\frac{3t-1}{3t} \leq \frac{N}{M} < \frac{3t}{3t+1}$ can be determined the same as (4).

- $\frac{3t}{3t+1} \leq \frac{N}{M} < \frac{3t+1}{3t+2}$ ($S = 3t+1$)

First, we have

$$\begin{aligned} d_1 + d_2 &= 2t \cdot Q_s + (2t+1)\bar{Q}_s + Q_1 + Q_2 + \bar{Q}_1 + q_s - q_3 \\ &\leq D - t(Q_s + \bar{Q}_s) - ((Q_s + \bar{Q}_s) - (Q_1 + Q_2 + \bar{Q}_1)) \end{aligned} \tag{56}$$

Based on (32), (11), (13) and (34), we can get

$$d_i + d_j \leq \begin{cases} (9t+4)N - (6t+4)M - (2t+1)D \\ 3tM - (2t-1)D \\ (6t+2)(N+M) - (8t+2)D \end{cases} \tag{57}$$

Then, since $d_2 = t \cdot Q_s + Q_2 + (t+1)\bar{Q}_s - \bar{Q}_3 + q_2$ and $P_2 = t \cdot (Q_s + \bar{Q}_s) + \bar{Q}_3$, (24) can be written as

$$2tQ_s + (2t+1)\bar{Q}_s + q_s + Q_1 + Q_2 + \bar{Q}_1 + q_2 \leq N \tag{58}$$

which leads to

$$D - (t+1)(Q_s + \bar{Q}_S) + Q_1 + Q_2 + \bar{Q}_1 \leq N - q_2 \leq N \tag{59}$$

By taking (59) into (56), we can get

$$d_i + d_j \leq N \tag{60}$$

Next, (25) can be expressed as

$$d_2 \leq M - t(Q_s + \bar{Q}_s) - \bar{Q}_3 \tag{61}$$

Since $Q_s + \bar{Q}_s \geq 2D - 3M$ and $\bar{Q}_3 \geq 0$, we can get

$$d_2 \leq (3t+1)M - 2tD \tag{62}$$

which leads to

$$(2t+1)d_i + 2td_j + 2td_k \leq (3t+1)M \tag{63}$$

As a result, the achievable DoF region for $\frac{3t-1}{3t} \leq \frac{N}{M} < \frac{3t}{3t+1}$ can be determined to be the same as (5).

- $\frac{3t+1}{3t+2} \leq \frac{N}{M} < \frac{3t+2}{3t+3}$ ($S = 3t+2$)

In this case, we have

$$\begin{aligned} d_1 + d_2 &= (2t+1) \cdot Q_s + (2t+2)\bar{Q}_s + Q_1 + q_s - q_3 \\ &= D - t(Q_s + \bar{Q}_s) - (Q_s + \bar{Q}_s - Q_1) - q_3 \end{aligned} \tag{64}$$

Since $Q_s + \bar{Q}_s \geq 2D - 3M$, $Q_1 \leq (3t+2)N - (3t+3)M$ and $q_3 \geq 0$, (64) can be written as

$$d_1 + d_2 \leq \min\{3tM - (2t-1)D \;,\; (3t+2)N - (2t+1)D\}$$

which leads to

$$\begin{aligned} 2td_i + 2td_j + (2t-1)d_k &\leq 3tM & (65) \\ (2t+2)d_i + (2t+2)d_j + (2t+1)d_k &\leq (3t+2)N & (66) \end{aligned}$$

Then, since $d_2 = (t+1)Q_s - Q_3 + (t+1)\bar{Q}_s + q_2$ and $P_2 = t \cdot Q_s + Q_3 + (t+1)\bar{Q}_s - \bar{Q}_1$, (24) can be written as

$$(2t+1)Q_s + (2t+2)\bar{Q}_s + q_s + Q_1 + q_2 \leq N \tag{67}$$

which implies that

$$D - (t+1)(Q_s + \bar{Q}_S) + Q_1 \leq N - q_2 \leq N \tag{68}$$

By taking (68) into (64), we can get

$$d_i + d_j \leq N \tag{69}$$

Next, (25) can be expressed as

$$d_2 \leq M - t(Q_s + \bar{Q}_s) - (Q_s + \bar{Q}_s - (Q_1 + Q_2 + \bar{Q}_1))$$

which can lead to

$$d_i \leq \begin{cases} (9t+7)N - (6t+6)M - (2t+2)D \\ (3t+1)M - 2tD \\ (6t+4)(N+M) - (8t+5)D \end{cases} \tag{70}$$

Hence, the achievable DoF region for $\frac{3t-1}{3t} \leq \frac{N}{M} < \frac{3t}{3t+1}$ can be determined to be the same as (6).

VI. Conclusion

The outer-bound of DoF region of 3-user MIMO interference channels is given in this paper. Then, a linear beamforming scheme based on alignment chain is proposed, whose achievable DoF region is the same as the outer bound. This result implies that all the combination of $(d_1,\ d_2,\ d_3)$ inside the region is achievable (d_i is integer), yet all the ones that outside the region cannot be achieved for sure. Hence, the region can be seen as the necessary and sufficient condition for the feasibility of linear interference alignment in 3-user interference networks.

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