On the Optimality of Treating Interference as Noise: General Message Sets

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***Abstract*—To be considered for an IEEE Jack Keil Wolf ISIT Student Paper Award. In a K-user Gaussian interference channel, it has been shown that if for each user the desired signal strength is no less than the sum of the strengths of the strongest interference from this user and the strongest interference to this user (all values in dB scale), then treating interference as noise (TIN) is optimal from the perspective of generalized degrees-of-freedom (GDoF) and achieves the entire channel capacity region to within a constant gap. In this work, we show that for such TIN-optimal interference channels, even if the message set is expanded to include an independent message from each transmitter to each receiver, operating the new channel as the original interference channel and treating interference as noise is still optimal for the *sum* capacity up to a constant gap.**

I. Introduction

Treating interference as noise (TIN) when it is sufficiently weak is an attractive interference management principle for wireless networks in practice due to its simplicity and robustness. Remarkably, TIN is also information-theoretically optimal when the interference is sufficiently weak. This is established in [1], [2], [3], [4], [5], [6] from an exact capacity perspective, and in [7], [8], [9], [10], [11], [12] from an approximate capacity perspective. Each approach has its merits – the former identifies relatively narrow regimes where TIN achieves exact capacity, whereas the latter identifies significantly broader regimes where TIN is approximately optimal. Most relevant to this work are the results by Geng et al. in [12] where it is shown that in a general K-user interference channel, if for each user the desired signal strength is no less than the sum of the strengths of the strongest interference *from* this user and the strongest interference *to* this user (all values in dB scale), then TIN is optimal for the entire channel capacity region up to a constant gap of no more than $\log_2(3K)$ bits.

In this paper we explore the sum-rate optimality of TIN when the message set is expanded to include an independent message from each transmitter to each receiver, i.e., the X channel setting [13], [14], [15]. Related prior works on the X setting in [16], [17] have primarily focused on the case with 2 transmitters and 2 receivers. In [16], Huang, Cadambe and Jafar characterize the sum-GDoF for the symmetric X channel and identify sufficient conditions for TIN to achieve exact capacity in the asymmetric case. In [17], Niesen and Maddah-Ali characterize the capacity for the general asymmetric case within a constant gap subject to an outage set.

The main contribution of this work is to show that, for the K-user TIN-optimal interference channels identified by Geng et al. in [12], even if the message set is expanded to also include an independent message from each transmitter to each receiver, operating as the original interference channel and treating interference as noise at each receiver is still optimal for the *sum* capacity up to a constant gap.

II. Preliminaries

A. Channel Model

Consider the wireless channel with K transmitters and K receivers, which can be described by the following input-output equations,

$$Y_k(t) = \sum_{i=1}^{K} \tilde{h}_{ki}\tilde{X}_i(t) + Z_k(t), \quad \forall k \in \{1, 2, ..., K\}, \quad (1)$$

where $\tilde{h}_{ki}$ is the complex channel gain value from transmitter i to receiver k. $\tilde{X}_i(t)$, $Y_k(t)$ and $Z_k(t)$ are the transmitted symbol of transmitter i, the received signal of receiver k, and the additive circularly symmetric complex Gaussian noise with zero mean and unit variance seen by receiver k, respectively, at each time index t. All the symbols are complex. Each transmitter i is subject to the power constraint $E[|\tilde{X}_i(t)|^2] \le P_i$.

Following similar approaches in [7], [12], we translate the standard channel model (1) into an equivalent normalized form that is more conducive for GDoF studies. We define

$$\alpha_{ki} \triangleq \frac{\log(\max\{1, |\tilde{h}_{ki}|^2 P_i\})}{\log P}, \quad \forall i, k \in \{1, 2, ..., K\}, \quad (2)$$

where $P > 1$ is a nominal power value.

Now according to (2), we represent the original channel model (1) in the following form,

$$\begin{aligned} Y_k(t) &= \sum_{i=1}^{K} h_{ki} X_i(t) + Z_k(t) \\ &= \sum_{i=1}^{K} \sqrt{P^{\alpha_{ki}}} e^{j\theta_{ki}} X_i(t) + Z_k(t), \ \forall k \in \{1, 2, ..., K\}, \end{aligned} \quad (3)$$

where $X_i(t) = \tilde{X}_i(t)/\sqrt{P_i}$ is the normalized transmit symbol of transmitter i, subject to the unit power constraint, i.e., $E[|X_i(t)|^2] \le 1$. $\sqrt{P^{\alpha_{ki}}}$ and θ_{ki} are the magnitude and the phase, respectively, of the channel between transmitter i and receiver k. The exponent α_{ki} is called the channel strength level of the link between transmitter i and receiver k. As in [7], [12], for the GDoF metric, we preserve the ratios α_{ki} as all SNRs approach infinity. In the rest of the paper, we only consider the equivalent channel model in (3).

In the K-user interference channel, each transmitter intends to send one independent message to its corresponding receiver. Because we wish to prove the negative result that additional messages do not add to the sum-GDoF in a TIN-optimal network, the strongest result corresponds to the case where we include messages from every transmitter to every receiver. Therefore, we will consider the X channel setting. In the $K \times K$ X channel, transmitter i has message W_{ki} intended for receiver k, and the messages $\{W_{ki}\}$ are independent, $\forall i, k \in \{1, 2, ..., K\}$. The size of the message set $\{W_{ki}\}$ is denoted by $|W_{ki}|$. For codewords spanning n channel uses, the rates $R_{ki} = \frac{\log |W_{ki}|}{n}$ are achievable if the probability of error of all messages can be made arbitrarily small simultaneously by choosing an appropriately large n. The channel capacity region $\mathcal{C}$ is the closure of the set of all achievable rate tuples. Collecting the channel strength levels and phases in the sets $\alpha \triangleq \{\alpha_{ki}\}$, $\theta \triangleq \{\theta_{ki}\}$, $\forall i, k \in \{1, 2, ..., K\}$, the capacity region is denoted as $\mathcal{C}(P, \alpha, \theta)$, which is a function of α, θ, and P. The sum channel capacity is defined as

$$C_{\Sigma,X} = \max_{\mathcal{C}(P,\alpha,\theta)} \sum_{i=1}^{K} \sum_{k=1}^{K} R_{ki} \tag{4}$$

Then the GDoF region of the X channel as represented in (3) is given by

$$\begin{aligned} \mathcal{D}(\alpha, \theta) \triangleq \Big\{ & (d_{11}, d_{12}, ..., d_{KK}) : \ d_{ki} = \lim_{P \to \infty} \frac{R_{ki}}{\log P}, \\ & \forall i, k \in \{1, 2, ..., K\}, \\ & (R_{11}, R_{12}, ..., R_{KK}) \in \mathcal{C}(P, \alpha, \theta) \Big\}, \end{aligned} \tag{5}$$

and its sum-GDoF value is

$$d_{\Sigma,X} = \max_{\mathcal{D}(\alpha,\theta)} \sum_{i=1}^{K} \sum_{k=1}^{K} d_{ki} \tag{6}$$

B. On the Optimality of TIN for Interference Channel

Let us first review the optimality of TIN for the K-user interference channel from the perspective of GDoF.

Theorem 1: (Theorem 1 in [12]) In a K-user interference channel, where the channel strength level from transmitter i to receiver j is equal to α_{ji}, $\forall i, j \in \{1, ..., K\}$, if the following condition is satisfied

$$\alpha_{ii} \geq \max_{j: j \neq i} \{\alpha_{ji}\} + \max_{k: k \neq i} \{\alpha_{ik}\}, \quad \forall i, j, k \in \{1, 2, ..., K\},$$

then power control and treating interference as noise achieves the whole GDoF region. Moreover, the GDoF region is the set of all K-tuples $(d_1, d_2, ..., d_K)$ satisfying

$$\text{individual bounds: } 0 \leq d_i \leq \alpha_{ii}, \quad \forall i \in \{1, 2, ..., K\},$$

$$\text{cycle bounds: } \sum_{j=1}^{m} d_{i_j} \leq \sum_{j=1}^{m} (\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}),$$
$$\forall (i_1, ..., i_m) \in \Pi_K, \quad \forall m \in \{2, 3, ..., K\},$$

where Π_K is the set of all possible cyclic sequences of all subsets of $\{1, ..., K\}$, and the modulo-m arithmetic is implicitly used on the user indices, e.g., $i_m = i_0$.

Remark: The above theorem claims that in the K-user interference channel, if *for each user the desired signal strength is no less than the sum of the strengths of the strongest interference from this user and the strongest interference to this user (all values in dB scale)*, then TIN is GDoF-optimal. Furthermore, it is shown in [12] that under the same condition, TIN achieves the entire channel capacity region to within a gap no larger than $\log_2(3K)$ bits. Note that the gap is bounded by a constant for a fixed number of users, i.e., it does not depend on the channel strength parameters α_{ij} and P.

III. Results

The main result of this paper is the following theorem.

Theorem 2: In a K-user interference channel, where the channel strength level from transmitter i to receiver j is equal to α_{ji}, $\forall i, j \in \{1, 2, ..., K\}$, when the following condition is satisfied,

$$\alpha_{ii} \geq \max_{j: j \neq i} \{\alpha_{ji}\} + \max_{k: k \neq i} \{\alpha_{ik}\}, \quad \forall i, j, k \in \{1, 2, ..., K\}, \tag{7}$$

then even if the message set is increased to the X channel setting, operating the new channel as the original interference channel and treating interference as noise at each receiver still achieves the sum-GDoF. Furthermore, the same scheme is also optimal for the sum channel capacity up to a constant gap of no more than $K \log_2[K(K+1)]$ bits.

The proof of Theorem 2 is presented in Section IV.

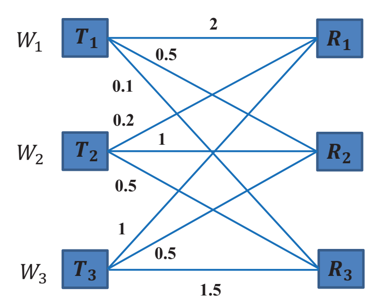

Fig. 1. A 3-user interference channel, where the value on each link denotes its channel strength level.

Example 1: First, consider the 3-user interference channel illustrated in Fig. 1, where transmitter i intends to send an independent message to its desired receiver i, $\forall i \in \{1, 2, 3\}$. Note there are 3 messages in this setting. It's easy to check that the TIN-optimal condition (7) is satisfied for each user. Then according to Theorem 1, it is not hard to verify that the sum-GDoF value of this interference channel is $d_{\Sigma,IC} = d_1 + d_2 + d_3 = 2.5$, which is achieved by power control and TIN.

Next, let us expand the set of messages to the X channel setting, where each transmitter intends to send an independent message to each receiver as shown in Fig. 2. Therefore, there are totally 9 messages in this X channel. Theorem 2 claims

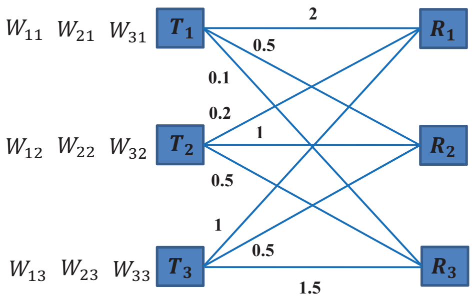

Fig. 2. A 3×3 X channel, which has the same channel strength levels as the 3-user interference channel in Fig. 1.

that for this 3×3 X channel, the sum-GDoF value is still $d_{\Sigma,X} = \sum_{i=1}^{3}\sum_{k=1}^{3} d_{ki} = 2.5$, which can be achieved by setting $W_{ki} = \phi$ for $i \neq k$ and $\forall i, k \in \{1, 2, 3\}$, sending only $\{W_{11}, W_{22}, W_{33}\}$ through the channel and treating interference as noise at each receiver. $\square$

IV. Proof of Theorem 2

Due to the limited space, we only provide the proof for the sum-GDoF of the TIN-optimal $K \times K$ X channel. The proof of the constant gap result for the sum channel capacity is relegated to the full paper [18].

The proof consists of two steps. In the first step, we show that for all individual and cycle bounds of a TIN-optimal K-user interference channel (see Theorem 1), if each d_i ($\forall i \in \{1, 2, ..., K\}$) is replaced by $\hat{d}_i = \sum_{j=1}^{K} d_{ij}$, these bounds still hold for its counterpart X channel.

In the following, we first give an example of the 3×3 X channel, then generalize the proof to the $K \times K$ X channel.

Example 2: Consider a 3-user TIN-optimal interference channel. According to Theorem 1, we can obtain the entire GDoF region, which is characterized by certain individual and cycle bounds. To extend the result to the X channel setting, each of these bounds will be extended. To illustrate the key ideas in this example, we consider the following two bounds,

$$d_3 \leq \alpha_{33}, \tag{8}$$

$$d_1 + d_2 \leq (\alpha_{11} + \alpha_{22}) - (\alpha_{12} + \alpha_{21}), \tag{9}$$

and intend to prove that in the counterpart 3×3 X channel, if we replace each d_i by $\hat{d}_i = \sum_{j=1}^{3} d_{ij}$, $\forall i \in \{1, 2, 3\}$, the above two bounds still hold, i.e.,

$$\hat{d}_3 = d_{31} + d_{32} + d_{33} \leq \alpha_{33} \tag{10}$$

$$\begin{aligned} \hat{d}_1 + \hat{d}_2 &= d_{11} + d_{12} + d_{13} + d_{21} + d_{22} + d_{23} \\ &\leq (\alpha_{11} + \alpha_{22}) - (\alpha_{12} + \alpha_{21}) \end{aligned} \tag{11}$$

All the remaining bounds can be extended to the X channel similarly.

To prove (10), we just need to consider the MAC consisting of all the transmitters and the receiver 3, then we have

$$R_{31} + R_{32} + R_{33} \leq \log_2(1 + P^{\alpha_{31}} + P^{\alpha_{32}} + P^{\alpha_{33}}) \tag{12}$$

Because (7) is satisfied, i.e., $\alpha_{33} \geq \alpha_{32}$ and $\alpha_{33} \geq \alpha_{31}$, in the GDoF sense we have

$$\hat{d}_3 = d_{31} + d_{32} + d_{33} \leq \alpha_{33} \tag{13}$$

To prove (11), consider the subnetwork consisting of all the transmitters and the receivers 1 and 2, where we have eliminated the third receiver and its desired messages W_{31}, W_{32}, W_{33}. This cannot hurt the rates of the remaining messages, so the outer bound arguments remain valid. Define

$$S_1(t) = h_{21}X_1(t) + Z_2(t) \tag{14}$$

$$S_2(t) = h_{12}X_2(t) + Z_1(t) \tag{15}$$

For receiver 1, we provide S_1^n, W_{21} and W_{23} through a genie. From Fano's inequality, we have the inequalities at the top of the next page, where (17) follows because all the messages are independent, (20) holds since adding conditioning does not increase entropy and (21) holds because dropping conditioning (in the first and third terms) does not reduce entropy.

Due to symmetry, for the receiver 2, we similarly obtain

$$\begin{aligned} &n\left(R_{21} + R_{22} + R_{23} - \epsilon\right) \\ &\leq h(S_2^n|W_{12}) - h(Z_1^n) + h(Y_2^n|S_2^n) - h(S_1^n|W_{21}) \end{aligned}$$

Thus the sum rate is bounded as follows.

$$\begin{aligned} &n(\sum_{i=1}^{2}\sum_{j=1}^{3} R_{ij} - 2\epsilon) \\ &\leq h(Y_1^n|S_1^n) + h(Y_2^n|S_2^n) - h(Z_1^n) - h(Z_2^n) \\ &\leq \sum_{t=1}^{n}[h(Y_1(t)|S_1(t)) + h(Y_2(t)|S_2(t)) - h(Z_1(t)) - h(Z_2(t))] \end{aligned}$$

where the second inequality follows from the chain rule and the fact that dropping conditioning does not reduce entropy. Finally, because the circularly symmetric complex Gaussian distribution maximizes conditional differential entropy for a given covariance constraint, we obtain

$$\begin{aligned} \sum_{i=1}^{2}\sum_{j=1}^{3} R_{ij} - 2\epsilon \leq \log_2\left(1 + P^{\alpha_{13}} + P^{\alpha_{12}} + \frac{P^{\alpha_{11}}}{1 + P^{\alpha_{21}}}\right) \\ + \log_2\left(1 + P^{\alpha_{23}} + P^{\alpha_{21}} + \frac{P^{\alpha_{22}}}{1 + P^{\alpha_{12}}}\right) \end{aligned} \tag{22}$$

Due to the condition (7), in the GDoF sense we obtain

$$\hat{d}_1 + \hat{d}_2 \leq (\alpha_{11} + \alpha_{22}) - (\alpha_{12} + \alpha_{21}) \tag{23}$$

which is the desired extension, (11), to the X channel setting of the original bound, (9), for the interference channel. $\square$

Now let us consider the proof for the general $K \times K$ X channel. For the individual bounds in the K-user interference channel

$$d_i \leq \alpha_{ii}, \quad \forall i \in \{1, 2, ..., K\}, \tag{24}$$

in its counterpart X channel, the corresponding bound comes from the MAC consisting of all the transmitters and the

$$
\begin{aligned}
n(R_{11}+R_{12}+R_{13}-\epsilon) &\leq I(W_{11},W_{12},W_{13};Y_1^n,S_1^n,W_{21},W_{23}) && (16)\\
&= I(W_{11},W_{12},W_{13};Y_1^n,S_1^n|W_{21},W_{23}) && (17)\\
&= I(W_{11},W_{12},W_{13};S_1^n|W_{21},W_{23}) + I(W_{11},W_{12},W_{13};Y_1^n|S_1^n,W_{21},W_{23}) && (18)\\
&= h(S_1^n|W_{21},W_{23}) - h(S_1^n|W_{21},W_{23},W_{11},W_{12},W_{13})\\
&\quad + h(Y_1^n|S_1^n,W_{21},W_{23}) - h(Y_1^n|S_1^n,W_{21},W_{23},W_{11},W_{12},W_{13}) && (19)\\
&\leq h(S_1^n|W_{21},W_{23}) - h(S_1^n|W_{21},W_{23},W_{11},W_{12},W_{13},X_1^n)\\
&\quad + h(Y_1^n|S_1^n,W_{21},W_{23}) - h(Y_1^n|S_1^n,W_{21},W_{23},W_{11},W_{12},W_{13},X_1^n,X_3^n) && (20)\\
&\leq h(S_1^n|W_{21}) - h(Z_2^n) + h(Y_1^n|S_1^n) - h(S_2^n|W_{12}) && (21)
\end{aligned}
$$

receiver i,

$$\sum_{j=1}^{K} R_{ij} \leq \log_2(1+\sum_{j=1}^{K} P^{\alpha_{ij}}) \quad (25)$$

According to (7), in the GDoF sense we have

$$\hat{d}_i = \sum_{j=1}^{K} d_{ij} \leq \alpha_{ii} \quad (26)$$

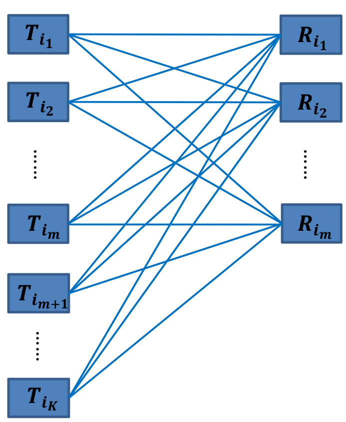

Fig. 3. A $K \times m$ X channel ($K \geq m$)

For any cycle bound in the interference channel

$$\sum_{j=1}^{m} d_{i_j} \leq \sum_{j=1}^{m} (\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}),$$
$$\forall (i_1, ..., i_m) \in \Pi_K, \quad \forall m \in \{2,3,...,K\},$$

consider the subnetwork consisting of all the transmitters and the receivers $\{i_1, i_2, ..., i_m\}$ as shown in Fig. 3. Eliminate all other receivers and their desired messages, which cannot hurt the rates of the remaining messages. For such a $K \times m$ X channel, define $\mathcal{W} \triangleq \{W_{i_j i_k}\}$, $\mathcal{W}_{i_j}^* \triangleq \{W_{i_j i_1}, W_{i_j i_2}, ..., W_{i_j i_K}\}$, $\mathcal{W}_{i_k}^\dagger \triangleq \{W_{i_1 i_k}, W_{i_2 i_k}, ..., W_{i_m i_k}\}$, and $\mathcal{W}_\mathcal{S}^c \triangleq \mathcal{W}/\mathcal{W}_\mathcal{S}$, where $\forall j \in \{1,2,...,m\}$, $\forall k \in \{1,2,...,K\}$, and $\mathcal{S}$ is any subset of message indices. In words, the sets $\mathcal{W}$, $\mathcal{W}_{i_j}^*$, and $\mathcal{W}_{i_k}^\dagger$ represent all the remaining messages delivered in the channel, all the messages intended to receiver i_j, and all the messages coming from transmitter i_k, respectively, and $\mathcal{W}_\mathcal{S}^c$ is the complement of $\mathcal{W}_\mathcal{S}$ in $\mathcal{W}$. For instance, when $j,k \in \{1,2\}$ and $\mathcal{S} = \{i_1 i_1, i_1 i_2\}$, then $\mathcal{W}_\mathcal{S} = \{W_{i_1 i_1}, W_{i_1 i_2}\}$ and $\mathcal{W}_\mathcal{S}^c = \{W_{i_2 i_1}, W_{i_2 i_2}\}$. Modulo-$m$ arithmetic is used on the receiver indices, e.g., $i_0 = i_m$. Lastly, to complete the setup, define

$$S_{i_j}(t) = h_{i_{j-1} i_j} X_{i_j}(t) + Z_{i_{j-1}}(t), \quad \forall j \in \{1,2,...,m\} \quad (27)$$

Then for receiver i_1, we provide $S_{i_1}^n$, $\mathcal{W}_{i_2 i_2}^c / \mathcal{W}_{i_1}^*$ through a genie. From Fano's inequality, we have the inequalities at the top of the next page, where (29) follows because all the messages are independent, and in (32) we use the fact that dropping conditioning does not reduce entropy.

Similarly, for other receivers i_j, $\forall j \in \{2,3,...,m-1\}$, by providing $S_{i_j}^n$, $\mathcal{W}_{i_{j+1} i_{j+1}}^c / \mathcal{W}_{i_j}^*$ through a genie we have

$$
\begin{aligned}
n(\sum_{k=1}^{K} R_{i_j i_k} - \epsilon) \leq & h(S_{i_j}^n | \mathcal{W}_{i_j}^\dagger / W_{i_j i_j}) - h(Z_{i_{j-1}}^n)\\
&+ h(Y_{i_j}^n | S_{i_j}^n) - h(S_{i_{j+1}}^n | \mathcal{W}_{i_{j+1}}^\dagger / W_{i_{j+1} i_{j+1}})
\end{aligned}
$$

Finally for receiver i_m, we can provide $S_{i_m}^n$, $\mathcal{W}_{i_1 i_1}^c / \mathcal{W}_{i_m}^*$ through a genie and obtain

$$
\begin{aligned}
n(\sum_{k=1}^{K} R_{i_m i_k} - \epsilon) \leq & h(S_{i_m}^n | \mathcal{W}_{i_m}^\dagger / W_{i_m i_m}) - h(Z_{i_{m-1}}^n)\\
&+ h(Y_{i_m}^n | S_{i_m}^n) - h(S_{i_1}^n | \mathcal{W}_{i_1}^\dagger / W_{i_1 i_1})
\end{aligned}
$$

Then taking the sum of $n(\sum_{k=1}^{K} R_{i_j i_k} - \epsilon)$ for all $j \in \{1,2,...,m\}$, we have

$$
\begin{aligned}
n(\sum_{j=1}^{m}\sum_{k=1}^{K} R_{i_j i_k} - m\epsilon) &\leq \sum_{j=1}^{m} [h(Y_{i_j}^n | S_{i_j}^n) - h(Z_{i_j}^n)]\\
&\leq \sum_{t=1}^{n}\sum_{j=1}^{m} [h(Y_{i_j}(t) | S_{i_j}(t)) - h(Z_{i_j}(t))]
\end{aligned}
$$

where the second inequality follows the chain rule and the fact that dropping conditioning does not reduce entropy. Once again, using the fact that the circularly symmetric complex Gaussian distribution maximizes conditional differential entropy for a given covariance constraint and the condition (7), we can obtain the following desired outer bound in the GDoF sense, through the same set of manipulations as in Example

$$n(\sum_{k=1}^{K} R_{i_1 i_k} - \epsilon) \le I(\mathcal{W}_{i_1}^*; Y_{i_1}^n, S_{i_1}^n, \mathcal{W}_{i_2 i_2}^c/\mathcal{W}_{i_1}^*) \tag{28}$$

$$= I(\mathcal{W}_{i_1}^*; Y_{i_1}^n, S_{i_1}^n | \mathcal{W}_{i_2 i_2}^c/\mathcal{W}_{i_1}^*) \tag{29}$$

$$= I(\mathcal{W}_{i_1}^*; S_{i_1}^n | \mathcal{W}_{i_2 i_2}^c/\mathcal{W}_{i_1}^*) + I(\mathcal{W}_{i_1}^*; Y_{i_1}^n | S_{i_1}^n, \mathcal{W}_{i_2 i_2}^c/\mathcal{W}_{i_1}^*) \tag{30}$$

$$= h(S_{i_1}^n | \mathcal{W}_{i_2 i_2}^c/\mathcal{W}_{i_1}^*) - h(S_{i_1}^n | \mathcal{W}_{i_2 i_2}^c) + h(Y_{i_1}^n | S_{i_1}^n, \mathcal{W}_{i_2 i_2}^c/\mathcal{W}_{i_1}^*) - h(Y_{i_1}^n | S_{i_1}^n, \mathcal{W}_{i_2 i_2}^c) \tag{31}$$

$$\le h(S_{i_1}^n | \mathcal{W}_{i_1}^\dagger / W_{i_1 i_1}) - h(Z_{i_0}^n) + h(Y_{i_1}^n | S_{i_1}^n) - h(S_{i_2}^n | \mathcal{W}_{i_2}^\dagger / W_{i_2 i_2}) \tag{32}$$

2,

$$\sum_{j=1}^{m} \hat{d}_{i_j} = \sum_{j=1}^{m} \sum_{k=1}^{K} d_{i_j i_k} \le \sum_{j=1}^{m} (\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}) \tag{33}$$

Now we can proceed to the last step to prove that under condition (7), the K-user interference channel and its counterpart $K \times K$ X channel have the same sum-GDoF. According to Theorem 1, for the K-user interference channel, under condition (7), to obtain its sum-GDoF $d_{\Sigma,IC}$, we need to solve the following linear programming (LP) problem

$$\max \sum_{i=1}^{K} d_i \tag{34}$$

$$\text{s.t.} \quad 0 \le d_i \le \alpha_{ii}, \quad \forall i \in \{1, 2, ..., K\} \tag{35}$$

$$\sum_{j=1}^{m} d_{i_j} \le \sum_{j=1}^{m} (\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}), \quad \forall (i_1, ..., i_m) \in \Pi_K, \quad \forall m \in \{2, 3, ..., K\} \tag{36}$$

To get the sum-GDoF of its counterpart X channel $d_{\Sigma,X}$, we consider a similar LP problem. Note for this LP problem, with the objective function $\sum_{i=1}^{K} \hat{d}_i$, it needs to follow similar constraints to (35) and (36), in which each d_i is just replaced by $\hat{d}_i$. Thus we have $d_{\Sigma,IC} \ge d_{\Sigma,X}$. Obviously, in any case, the sum-GDoF of the K-user interference channel must be less than or equal to that of its counterpart X channel, i.e. $d_{\Sigma,IC} \le d_{\Sigma,X}$. Therefore, under condition (7), we have established that the K-user interference channel and its counterpart X channel have the same sum-GDoF.

V. Conclusion

In this paper, we extend the optimality of TIN to more general classes of message sets. The main result is that for the TIN-optimal K-user interference channel, even if the message set expands to include the X setting where each transmitter has one independent message to each receiver, operating the new channel as the original interference channal and treating interference as noise at each receiver is still optimal to achieve the sum channel capacity to within a constant gap.

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