GDoF of the MISO BC: Bridging the Gap between Finite Precision CSIT and Perfect CSIT

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***Abstract*—This work bridges the gap between sharply contrasting results on the degrees of freedom of the K user broadcast channel where the transmitter is equipped with K transmit antennas and each of the K receivers is equipped with a single antenna. This channel has K DoF when channel state information at the transmitter (CSIT) is perfect, but as shown recently, it has only 1 DoF when the CSIT is limited to finite precision. By considering the full range of partial CSIT assumptions parameterized by $\beta \in [0,1]$, such that the strength of the channel estimation error terms scales as $\sim SNR^{-\beta}$ relative to the channel strengths which scale as $\sim SNR$, it is shown that this channel has $1-\beta+K\beta$ DoF. For $K=2$ users with arbitrary β_{ij} parameters, the DoF are shown to be $1+\min_{i,j}\beta_{ij}$. To explore diversity of channel strengths, the results are further extended to the symmetric Generalized Degrees of Freedom setting where the direct channel strengths scale as $\sim SNR$ and the cross channel strengths scale as $\sim SNR^{\alpha}$, $\alpha \in [0,1], \beta \in [0,\alpha]$. Here, the roles of α and β are shown to counter each other on equal terms, so that the sum GDoF value in the K user setting is $(\alpha-\beta)+K(1-(\alpha-\beta))$ and for the 2 user setting with arbitrary β_{ij}, is $2-\alpha+\min_{i,j}\beta_{ij}$.**

I. Introduction

As the first step in the path towards progressively refined capacity approximations, degrees of freedom (DoF) studies of wireless networks have turned out to be surprisingly useful. By exposing large gaps where they exist in our understanding of the capacity limits, DoF studies have been the catalysts for numerous discoveries over the past decade [1]. One of the most striking contrasts brought to light by recent DoF studies is between settings where the channel state information at the transmitters (CSIT) is assumed to be perfect, and where it is assumed to be available only with finite precision. Consider the MISO BC, i.e., the broadcast channel where the transmitter is equipped with K antennas and each of the K receivers has a single antenna. If the channel state information at the transmitter (CSIT) is perfect then the MISO BC has K DoF almost surely. This is achievable by zero-forcing which eliminates all interference. However, if the CSIT is available only within finite precision, then the MISO BC has only 1 DoF, i.e., the DoF collapse as conjectured by Lapidoth et al. nearly a decade ago in [2]. The conjecture was proved recently in [3]. Since the MISO BC contains within it the K user interference and X channels, the collapse of DoF under finite precision CSIT implies that neither zero-forcing nor interference alignment is robust enough to provide a DoF advantage under finite precision CSIT. Bridging this large gap between perfect and finite precision CSIT is the motivation for this work. It involves studying partial channel knowledge settings where the channel estimation error scales as $\sim SNR^{-\beta}$ for arbitrary exponents β.

As exemplified by the conjecture of Lapidoth et al. which remained unresolved for nearly a decade, a key hurdle in studying partial CSIT settings tends to be the outer bounds. DoF outer bounds under channel uncertainty have until recently been limited mostly to compound channel arguments [4]. Remarkably, compound channel arguments produce tight outer bounds in several settings of interest that have been successfully explored in prior work. For instance, it is known that in order to maintain the full DoF (i.e., the same as with perfect CSIT), the channel estimation error should scale as $O(SNR^{-1})$ [5], [6], [7]. Compound channel arguments also produce tight outer bounds for various settings involving retrospective [8] and blind interference alignment [9]. However, outer bounds based on compound channel arguments are evidently not strong enough to bridge the gap between perfect CSIT and finite precision CSIT. For instance, although the collapse of DoF of the MISO BC was originally conjectured under the compound setting by Weingarten et al. in [4], this conjecture was settled in the negative by [10] and [11]. Therefore, in order to bridge this gap, we appeal to the combinatorial accounting of the size of aligned image sets, in short the AIS approach, that was introduced in [3] to settle the conjectured collapse of DoF under finite precision CSIT. We are also inspired by recent works that successfully apply the AIS approach beyond DoF settings, to generalized degrees of freedom (GDoF) characterizations under finite precision CSIT [12]. As such, in this work our goal is not only to apply the AIS approach to bridge the gap between finite precision and perfect CSIT, but also to go beyond DoF toward GDoF characterizations. The main results of this work are presented and discussed in Section III.

II. System Model

A. Problem Formulation

Under the GDoF framework, the channel model for the K user MISO BC is defined by the following input-output equations.

$$Y_k(t) = \sum_{l=1}^{K} \sqrt{P^{\alpha_{kl}}} G_{kl}(t) X_l(t) + Z_k(t), \quad \forall k \in [K]. \qquad (1)$$

The channel uses are indexed by $t \in \mathbb{N}$, $X_l(t)$ is the symbol sent from Transmitter l subject to a unit power constraint, $Y_k(t)$ is the symbol observed by Receiver k, $Z_k(t)$ is the zero mean unit variance additive white Gaussian noise (AWGN) at Receiver k, and $G_{kl}(t)$ are the channel fading coefficients between Transmitter l and Receiver k. P is the nominal SNR parameter that is allowed to approach infinity. The channel strengths are represented in α_{kl} parameters. We focus on the symmetric setting, where for all $k, l \in [K]$, we set

$$\alpha_{kl} = \begin{cases} \alpha, & k \neq l \\ 1, & k = l. \end{cases}$$

The definitions of achievable rates $R_i(P)$ and capacity region $\mathcal{C}(P)$ are standard. The GDoF region is defined as

$$\mathcal{D} = \{(d_1, \cdots, d_K) : \exists (R_1(P), \cdots, R_K(P)) \in \mathcal{C}(P) \text{ s. t. } d_k = \lim_{P \to \infty} \frac{R_k(P)}{C_o(P)}, \forall k \in [K]\} \quad (2)$$

where $C_o(P)$ is a reference capacity of an additive white Gaussian noise channel $Y = X + N$ with transmit power P and unit variance additive white Gaussian noise. For real settings, $C_o(P) = 1/2\log(P) + o(\log(P))$ and for complex settings $C_o(P) = \log(P) + o(\log(P))$.

An important definition for this work is the notion of a "bounded density" assumption.

Definition 1 (Bounded Density): A set of random variables, $\mathcal{A}$, is said to satisfy the bounded density assumption if there exists a finite positive constant $f_{\max}$,

$$0 < f_{\max} < \infty$$

such that for all finite cardinality disjoint subsets $\mathcal{A}_1, \mathcal{A}_2$ of $\mathcal{A}$,

$$\mathcal{A}_1 \subset \mathcal{A}, \mathcal{A}_2 \subset \mathcal{A}, \mathcal{A}_1 \cap \mathcal{A}_2 = \phi, |A_1| < \infty, |\mathcal{A}_2| < \infty$$

the conditional probability density functions exist and are bounded as follows,

$$\forall A_1, A_2, \quad f_{\mathcal{A}_1|\mathcal{A}_2}(A_1|A_2) \leq f_{\max}^{|\mathcal{A}_1|}.$$

B. Partial CSIT

Under partial CSIT, the channel coefficients may be represented as

$$G_{kl}(t) = \hat{G}_{kl}(t) + \sqrt{P^{-\beta_{kl}}}\tilde{G}_{kl}(t)$$

where $\hat{G}_{kl}(t)$ are the channel estimate terms and $\tilde{G}_{kl}(t)$ are the estimation error terms. To avoid degenerate conditions, the ranges of values are bounded away from zero and infinity as follows, i.e., there exist constants Δ_1, Δ_2 such that $0 < \Delta_1 \leq |G_{kl}(t)|$ and $|\tilde{G}_{kl}(t)|, |\tilde{G}_{kl}(t)| < \Delta_2 < \infty$. The channel variables $\hat{G}_{kl}(t), \tilde{G}_{kl}(t)$, $\forall k, l \in \{1, 2\}, t \in \mathbb{N}$, are subject to the bounded density assumption with the difference that the actual realizations of $\hat{G}_{kl}(t)$ are revealed to the transmitter, but the realizations of $\tilde{G}_{kl}(t)$ are not available to the transmitter. Note that under the partial CSIT model, the channel coefficients $G_{kl}(t)$ have variance that behaves as $\sim P^{-\beta_{kl}}$ and the peak of the probability density function that behaves as $\sim \sqrt{P^{\beta_{kl}}}$.

III. Main Result

Let us denote the sum-GDoF value as $\mathcal{D}_\Sigma(\alpha, [\beta_{kl}])$. Further, let us assume that

$$\alpha \in [0, 1] \quad (3)$$

$$\beta_{kl} \in [0, \alpha] \quad (4)$$

For the $K = 2$ users, there is no loss of generality in assuming $\alpha \in [0, 1]$ because following the reasoning in [14][1], we have

$$\mathcal{D}_\Sigma(\alpha, [\beta_{kl}]) = \alpha \mathcal{D}_\Sigma(1/\alpha, [\beta_{kl}/\alpha]) \quad (5)$$

There is no loss of generality in assuming $\beta_{kl} \in [0, \alpha]$ because for $\alpha \in [0, 1]$, we will see that this range of values spans the entire space between finite precision CSIT ($\beta = 0$) and perfect CSIT ($\beta = \alpha$).

Our main results are the sum GDoF characterizations for the K user MISO BC defined above. We start with $K = 2$ users, where we allow arbitrary β_{ij} parameters, so that each channel is associated with its own level of channel uncertainty. The sum GDoF for this setting are characterized in the following theorem.

Theorem 1: Under partial CSIT, the sum GDoF value of the 2-user MISO BC is

$$\mathcal{D}_\Sigma(\alpha, [\beta_{kl}]) = 2 - \alpha + \min_{kl} \beta_{kl} \quad (6)$$

The general K user setting is considered next, where additional assumptions of symmetry are made on the β_{kl} values in order to control the explosion of parameters. Specifically we set all $\beta_{kl} = \beta$, so that every channel is subject to the same level of channel uncertainty. The GDoF characterization in this setting is presented in the following theorem.

Theorem 2: Under partial CSIT, the sum GDoF value of the K-user MISO BC is

$$\mathcal{D}_\Sigma(\alpha, [\beta]) = (\alpha - \beta) + K(1 - (\alpha - \beta)) \quad (7)$$

Recall that the DoF are obtained as a special case of GDoF, by setting $\alpha = 1$. With this specialization, we note that Theorem 2 shows that the K user MISO BC has $1 - \beta + K\beta$ DoF. This covers the extremes of perfect CSIT ($\beta = 1$) where the DoF become equal to K and finite precision CSIT ($\beta = 0$) where the DoF collapse to 1. It also shows that $\beta \geq 1$ is necessary to achieve the full K DoF, thus matching the results of [6]. However, most significantly, it goes well beyond these specializations, for the first time *bridging* these divergent extremes by characterizing the DoF for all intermediate values of β as well.

The DoF value of $1 - \beta + K\beta$ has a simple intuitive interpretation. Using terminology analogous to [13], the signal power levels split into the bottom β levels where CSIT is perfect and the remaining top $1 - \beta$ levels where CSIT is only available to finite precision. This is because transmission in a direction orthogonal to estimated channel vector of undesired user (zero-forcing) with power up to $\sim P^\beta$ leaks no

[1]This is easily shown by a change of variables argument, re-casting the GDoF framework so that instead of P we use $P' = P^\alpha$.

power above the noise floor at the undesired receiver. Due to essentially perfect zero-forcing, the bottom β levels contribute $K\beta$ DoF. The top $1-\beta$ levels, which cannot be zero-forced, contribute the remaining $1-\beta$ DoF.[2]

Another interesting aspect of our results is that they reveal how the CSIT requirement changes with channel strengths. It is clear that the CSIT requirements must depend on the channel strengths. For instance, it is obvious that if cross channels are too weak (below the noise floor) then no CSIT is needed to simultaneously achieve full interference-free transmission to both users (in the GDoF sense), but if the cross channels are as strong as direct channels then essentially perfect CSIT ($\beta \geq 1$) is required. The gap between these extremes is also bridged by Theorem 2. Unlike DoF which implicitly assume all channels are equally strong, by allowing $\alpha < 1$, the GDoF setting allows us in this work to characterize the impact of different channel strengths (albeit restricted within assumptions of symmetry). Remarkably here we find that cross-channel strength parameters α and channel uncertainty parameters β counter each other on equal terms, so that only their difference matters.

Finally, for the 2 user setting where the number of parameters is more manageable, we are able to study the impact of arbitrary channel uncertainty β_{kl} for each channel coefficient, as in Theorem 1. What is especially remarkable, perhaps even surprising, is that the GDoF are limited by the channel with the worst uncertainty, i.e., the smallest β_{kl}.

IV. Proof

We present the proof for the $K=2$ user setting (Theorem 1) here. The generalization to arbitrary K (Theorem 2) is relatively straightforward, and is relegated to the full paper. The most challenging aspect of the proof is to obtain a tight outer bound, for which we will generalize the Aligned Image Sets (AIS) argument of [3]. For this generalization we will skip the repetitive details and focus on the distinct aspects. We will focus on the real setting here. The extension to complex settings follows along the lines of similar extensions in [3].

A. Outer Bound

For notational convenience, let us define

$$\bar{P} = \sqrt{P} \tag{8}$$

The first step in the AIS approach is the transformation into a deterministic setting such that a GDoF outer bound on the deterministic setting is also a GDoF outer bound on the original setting. Since the derivation of the deterministic setting is identical to [3], we directly present the deterministic model as follows.

[2]The achievability argument extends naturally to other settings. For example, we can show that in the corresponding K user interference channel the DoF value of $1-\beta+\frac{K}{2}\beta$ is similarly achievable.

1) Deterministic Channel Model: The deterministic channel model has inputs $\bar{X}_i(t) \in \mathbb{Z}$ and outputs $\bar{Y}_i(t) \in \mathbb{Z}$, $\forall t \in \mathbb{N}, i \in \{1,2\}$, such that

$$\bar{Y}_1(t) = \lceil G_{11}(t)\bar{X}_1(t)\rceil + \lceil \bar{P}^{\alpha-1}G_{12}(t)\bar{X}_2(t)\rceil \tag{9}$$
$$\bar{Y}_2(t) = \lceil \bar{P}^{\alpha-1}G_{21}(t)\bar{X}_1(t)\rceil + \lceil G_{22}(t)\bar{X}_2(t)\rceil \tag{10}$$

and $\bar{X}_i(t) \in \{0,1,\cdots,\lceil \bar{P}^{\max(1,\alpha)}\rceil\}$.

2) Functional Dependence and Aligned Image Sets: Following directly along the AIS approach [3], and omitting $o(\log(P))$ and $o(n)$ terms that are inconsequential for GDoF, we have $n(R_1+R_2)$

$$\leq H(\bar{Y}_1^{[n]}|W_2, G^{[n]}) + H(\bar{Y}_2^{[n]}|G^{[n]}) - H(\bar{Y}_2^{[n]}|G^{[n]}, W_2) \tag{11}$$
$$\leq n\log(\bar{P}) + H(\bar{Y}_1^{[n]}|W_2, G^{[n]}) - H(\bar{Y}_2^{[n]}|G^{[n]}, W_2) \tag{12}$$
$$\leq n\log(\bar{P}) + \log \mathrm{E}\left|S_{\nu^{[n]}}(G^{[n]}\right| \tag{13}$$

where $S_{\nu^{[n]}}(G^{[n]})$ is the aligned image set, i.e., the set of codewords that cast distinct images at Receiver 1 but the same image $\bar{Y}_2^{[n]} = \nu^{[n]}$ at Receiver 2. The set $S_{\nu^{[n]}}(G^{[n]})$ is comprised of all $(\bar{X}_1^{[n]}, \bar{X}_2^{[n]})$ such that $\forall t \in [n]$

$$\nu(t) = \lceil \bar{P}^{\alpha-1}G_{21}(t)\bar{X}_1(t)\rceil + \lceil G_{22}(t)\bar{X}_2(t)\rceil \tag{14}$$

Also as in [3], there is no loss of generality in assuming the following functional dependence

$$(\bar{X}_1^{[n]}, \bar{X}_2^{[n]}) = f(\bar{Y}_1^{[n]}, G_{11}^{[n]}, G_{12}^{[n]}) \tag{15}$$
$$\Rightarrow \bar{Y}_2^{[n]} = f(\bar{Y}_1^{[n]}, G^{[n]}) \tag{16}$$

where the notation $a = f(b)$ is used to indicate that a is *some* (not necessarily the same) function of b.

3) Bounding the Probability that Images Align: Given $G_{11}^{[n]}, G_{12}^{[n]}$, consider two distinct realizations of User 1's output sequence $\bar{Y}_1^{[n]}$, denoted as $\lambda^{[n]}$ and $\nu^{[n]}$, which are produced by the corresponding realizations of the codeword $(X_1^{[n]}, X_2^{[n]})$ denoted by $(\lambda_1^{[n]}, \lambda_2^{[n]})$ and $(\nu_1^{[n]}, \nu_2^{[n]})$, respectively.

$$\lambda(t) = \lfloor G_{11}(t)\lambda_1(t)\rfloor + \lfloor \bar{P}^{\alpha-1}G_{12}(t)\lambda_2(t)\rfloor \tag{17}$$
$$\nu(t) = \lfloor G_{11}(t)\nu_1(t)\rfloor + \lfloor \bar{P}^{\alpha-1}G_{12}(t)\nu_2(t)\rfloor \tag{18}$$

We wish to bound the probability that the images of these two codewords align at User 2, i.e., $\nu^{[n]} \in S_{\lambda^{[n]}}$. For simplicity, consider first the single channel use setting, $n=1$. For $\nu \in S_\lambda$ we must have,

$$\lfloor \bar{P}^{\alpha-1}G_{21}\nu_1\rfloor + \lfloor G_{22}\nu_2\rfloor = \lfloor \bar{P}^{\alpha-1}G_{21}\lambda_1\rfloor + \lfloor G_{22}\lambda_2\rfloor \tag{19}$$

So for fixed value of G_{22} the random variable $\bar{P}^{\alpha-1}G_{21}(\nu_1 - \lambda_1)$ must take values within an interval of length no more than 4. If $\nu_1 \neq \lambda_1$, then G_{21} must take values in an interval of length no more than $\frac{4}{\bar{P}^{\alpha-1}|\nu_1-\lambda_1|}$, the probability of which is no more than $\frac{4f_{\max}\bar{P}^{\beta_{21}}}{\bar{P}^{\alpha-1}|\nu_1-\lambda_1|}$. Similarly, for fixed value of G_{21} the random variable $G_{22}(\nu_2-\lambda_2)$ must take values within an interval of length no more than 4. If $\nu_1 = \lambda_1$ then, because $\nu \neq \lambda$, we must have $\nu_2 \neq \lambda_2$, and the probability of alignment is similarly bounded by $\frac{4f_{\max}\bar{P}^{\beta_{22}}}{|\nu_2-\lambda_2|}$. Thus, based on (19), either the probabilty of alignment is zero or we have,

$$\bar{P}^{\alpha-1}\Delta_1|\nu_1-\lambda_1| \le \Delta_2|\nu_2-\lambda_2|+2 \quad (20)$$

$$\Delta_1|\nu_2-\lambda_2| \le \bar{P}^{\alpha-1}\Delta_2|\nu_1-\lambda_1|+2 \quad (21)$$

Next we will bound the max of $\bar{P}^{\alpha-1}|\nu_1-\lambda_1|$ and $|\nu_2-\lambda_2|$. From (17) and (18) we have

$$|\lambda-\nu| \le 2+|G_{11}||\lambda_1-\nu_1|+\bar{P}^{\alpha-1}|G_{12}||\lambda_2-\nu_2| \quad (22)$$

$$\le 2+2\Delta_2\bar{P}^{|1-\alpha|}\max(\bar{P}^{\alpha-1}|\nu_1-\lambda_1|,|\nu_2-\lambda_2|) \quad (23)$$

so, if $|\lambda-\nu| > \frac{4\Delta_2\bar{P}^{|1-\alpha|}}{\Delta_1}+2$, the probability of $\nu \in S_\lambda$ is no more than

$$\frac{4\Delta_2 f_{\max}\bar{P}^{\beta_{22}}}{\Delta_2|\nu_2-\lambda_2|} \quad (24)$$

$$\le \frac{4\Delta_2 f_{\max}\bar{P}^{\beta_{22}}}{\max(\bar{P}^{\alpha-1}\Delta_1|\nu_1-\lambda_1|-2,\Delta_2|\nu_2-\lambda_2|)} \quad (25)$$

$$\le \frac{4\Delta_2 f_{\max}\bar{P}^{\beta_{22}}}{\Delta_1\max(\bar{P}^{\alpha-1}|\nu_1-\lambda_1|,|\nu_2-\lambda_2|)-2} \quad (26)$$

$$\le \frac{4\Delta_2 f_{\max}\bar{P}^{\beta_{22}}}{\Delta_1\frac{|\lambda-\nu|-2}{2\Delta_2\bar{P}^{(1-\alpha)}}-2} \quad (27)$$

$$\le \frac{8\frac{\Delta_2^2}{\Delta_1}f_{\max}\bar{P}^{(1-\alpha+\beta_{22}}}{|\lambda-\nu|-\frac{4\Delta_2\bar{P}^{(1-\alpha)}}{\Delta_1}-2} \quad (28)$$

Define $\Delta = \frac{4\Delta_2\bar{P}^{|1-\alpha|}}{\Delta_1}-2$, ($\Delta$ scales as $\bar{P}^{1-\alpha}$). Now let us return to the case of general n, where we similarly have,

$$\mathbb{P}(\lambda^{[n]}\in S_{\nu^{[n]}}) \le \prod_{t:|\lambda(t)-\nu(t)|\le\Delta} 1 \times \prod_{t:|\lambda(t)-\nu(t)|>\Delta} \frac{8\frac{\Delta_2^2}{\Delta_1}f_{\max}\bar{P}^{1-\alpha+\beta_{22}}}{|\lambda(t)-\nu(t)|-\Delta}$$

4) Bounding the Expected Size of Aligned Image Sets.:

$$\mathrm{E}(|S_{\nu^{[n]}}|) = \sum_{\lambda^n\in\{\bar{Y}_1^{[n]}\}} \mathbb{P}(\lambda^n\in S_{\nu^{[n]}})$$

$$\le \prod_{t=1}^{n}\left(\sum_{\lambda(t):|\lambda(t)-\nu(t)|\le\Delta} 1 + \sum_{\lambda(t):|\lambda(t)-\nu(t)|>\Delta}\frac{8\frac{\Delta_2^2}{\Delta_1}f_{\max}\bar{P}^{1-\alpha+\beta_{22}}}{|\lambda(t)-\nu(t)|-\Delta}\right)$$

$$\le \prod_{t=1}^{n}\left(2\Delta+1+8\frac{\Delta_2^2}{\Delta_1}f_{\max}\bar{P}^{1-\alpha+\beta_{22}}\times 2(1+\log(1+2\Delta_2\bar{P}))\right)$$

$$\le (8\frac{\Delta_2^2}{\Delta_1}f_{\max})^n\bar{P}^{n(1-\alpha+\beta_{22})}\times\left(\max(1,\alpha)\log(\bar{P})+o(\log(\bar{P}))\right)^n$$

5) The GDoF Bound: Substituting back into (13) we have

$$n(R_1+R_2) \le n\log(\bar{P})+\log\mathrm{E}|S_{\nu^{[n]}}|$$

$$\le n\left(1+1-\alpha+\beta_{22}\right)\log(\bar{P})$$

So that we obtain the GDoF bound

$$d_1+d_2 \le 2-\alpha+\beta_{22} \quad (29)$$

By symmetry we also have the GDoF bounds,

$$d_1+d_2 \le 2-\alpha+\beta_{11} \quad (30)$$

$$d_1+d_2 \le 2-\alpha+\beta_{21} \quad (31)$$

$$d_1+d_2 \le 2-\alpha+\beta_{12} \quad (32)$$

Together these bounds give us $d_1+d_2 \le 2-\alpha+\min_{ij}\beta_{ij}$, completing the proof of the outer bound for Theorem 1.

B. Achievability

Since the GDoF depend only on the worst channel uncertainty, i.e., the minimum β_{ij}, for the achievability proof, we can assume without loss of generality that all β_{ij} are equal to β. With this assumption we will prove that $2-\alpha+\beta$ is achievable.

Without loss of generality, we ignore measure zero events such as channel rank-deficiencies. This is because the channels are generated according to bounded densities, so that the probability mass that can be placed in a space whose measure approaches zero, must also approach zero.

First, consider $\alpha = 1$, i.e., the DoF setting. We wish to achieve the sum-DoF value of $d_1+d_2 = \min(1+\beta,2)$ through the tuple $d_1 = 1, d_2 = \min(\beta,1)$. Let us assume $\beta \le 1$, as the achievability for $\beta = 1$ suffices for all $\beta > 1$ as well. To achieve $1+\beta$ DoF, let us split User 1's message as $W_1 = (W_c, W_{1p})$ and User 2's message as $W_2 = W_{2p}$, where W_{1p} acts as a private sub-message to be decoded only by user 1, W_{2p} acts as a private sub-message to be decoded only by User 2, while W_c acts as a common submessage that can be decoded by both users. W_c, W_{1p} and W_{2p} carry $1-\beta,\beta,\beta$ DoF respectively. Messages W_c, W_{1p}, W_{2p} are encoded into independent Gaussian codebooks X_c, X_{1p}, X_{2p}, with unit powers, producing the transmitted symbols as follows.

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = c_o\sqrt{1-P^{\beta-1}}\mathbf{V}_c X_c + c_o\sqrt{P^{\beta-1}}\mathbf{V}_{1p}X_{1p} + c_o\sqrt{P^{\beta-1}}\mathbf{V}_{2p}X_{2p} \quad (33)$$

Here $\mathbf{V}_{1p}$, $\mathbf{V}_{2p}$ are unit vectors chosen so that

$$\begin{bmatrix} \hat{G}_{11} & \hat{G}_{12} \\ \hat{G}_{21} & \hat{G}_{22} \end{bmatrix}\begin{bmatrix} \mathbf{V}_{1p} & \mathbf{V}_{2p} \end{bmatrix} \quad (34)$$

is a diagonal matrix. In words, $\mathbf{V}_{1p}$ is a unit vector orthogonal to the estimated channel vector of User 2, and $\mathbf{V}_{2p}$ is a unit vector orthogonal to the estimated channel vector of User 1. $\mathbf{V}_c$ is a generic unit vector. Thus, the private messages are zero-forced to the estimated channels of the undesired users, whereas the common message is sent along a generic direction so it is heard by both users. c_o is a scaling factor, $O(1)$ in P, chosen to ensure that the transmit power constraint is satisfied. The signal seen at Receiver 1 is,

$$Y_1 = \bar{P}\begin{bmatrix} \hat{G}_{11} & \hat{G}_{12} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \bar{P}^{1-\beta}\begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + Z_1$$

$$= \sqrt{P}c_1X_c + \sqrt{P^\beta}c_2X_{1p} + c_3X_{2p} + Z_1$$

where the c_i are non-zero and bounded, i.e., $O(1)$ functions of P.

User 1 first decodes X_c while treating all other signals as white noise. This is possible because X_c is received with power $\sim P$, the effective noise has power $\sim P^\beta$, and X_c carries $1-\beta$ DoF. After decoding X_c, the receiver subtracts its contribution from its received signal and then proceeds to decode X_{1p} while treating remaining signals as noise. Since X_{1p} is received with power $\sim P^{\beta}$, the remaining signals and noise are received with only $O(1)$ power, and X_{1p} carries β DoF, this decoding is successful as well. Thus, User 1 achieves $1-\beta+\beta=1$ DoF. User 2 proceeds similarly to achieve β DoF, so that the total DoF achieved equal $1+\beta$.

Next, let us consider the general case where $\alpha \leq 1$. Here we prove the sum-GDoF value of $d_1+d_2=\min(2-\alpha+\beta,2)$ is achievable through the tuple $d_1=1, d_2=\min(1-\alpha+\beta,1)$. Let us assume $\beta \leq \alpha$, as the achievability for $\beta=\alpha$ suffices for all $\beta>\alpha$ as well. To do this, let us split User 1's message as $W_1=(W_c,W_{1p})$ where W_{1p} acts as a private sub-message to be decoded only by User 1, while W_c acts as a common message that can be decoded by both users. W_c, W_{1p} and W_2 carry $\alpha-\beta, 1-\alpha+\beta, 1-\alpha+\beta$ GDoF respectively. Messages W_c, W_{1p}, W_2 are encoded into independent Gaussian codebooks X_c, X_{1p}, X_{2p}, with unit powers. The transmitted symbols are constructed as follows.

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = c_o\sqrt{1-P^{\beta-\alpha}}\mathbf{V}_c X_c + c_o\sqrt{P^{\beta-\alpha}}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{P^{\alpha-1}} \end{bmatrix}\mathbf{V}_{1p}X_{1p} + c_o\sqrt{P^{\beta-\alpha}}\begin{bmatrix} \sqrt{P^{\alpha-1}} & 0 \\ 0 & 1 \end{bmatrix}\mathbf{V}_{2p}X_{2p}$$

where c_o, $\mathbf{V}_c$, $\mathbf{V}_{1p}$, $\mathbf{V}_{2p}$ are defined as before. The signal seen at Receiver 1 is,

$$\begin{aligned} Y_1 &= \sqrt{P}\begin{bmatrix} \hat{G}_{11} & \hat{G}_{12} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{P^{\alpha-1}} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \\ &+ \sqrt{P^{1-\beta}}\begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{P^{\alpha-1}} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + Z_1 \\ &= \sqrt{P}c_1 X_c + \sqrt{P^{1+\beta-\alpha}}c_2 X_{1p} + c_3 X_{2p} + Z_1 \end{aligned}$$

where the c_i are non-zero and bounded, i.e., $O(1)$ in P.

User 1 first decodes X_c while treating all other signals as white noise. This is possible because X_c is received with power $\sim P$, the effective noise has power $\sim P^{1+\beta-\alpha}$, and X_c carries $1-(1+\beta-\alpha)=\alpha-\beta$ GDoF. After decoding X_c, the receiver subtracts its contribution from its received signal and then proceeds to decode X_{1p} while treating remaining signals as noise. Since X_{1p} is received with power $\sim P^{1+\beta-\alpha}$, the remaining signals and noise are received with only $O(1)$ power, and X_{1p} carries $1+\beta-\alpha$ DoF, this decoding is successful as well. Thus, User 1 achieves $\alpha-\beta+1-\beta+\alpha=1$ GDoF. User 2 proceeds similarly to achieve $1-\beta+\alpha$ DoF, so that the total GDoF achieved equal $2+\beta-\alpha$.

V. Conclusion

Because of the coarse and asymptotic character of DoF and GDoF metrics, even small gaps in our understanding of these coarse approximations can hide the most consequential ideas. Numerous discoveries around interference alignment emerged from efforts to find new achievable schemes to bridge the gap between the best inner and outer bounds. Following in the same spirit, this work bridges the extremes of known DoF results between perfect and finite precision CSIT. In the process, it expands our understanding of a relatively new idea – the aligned image sets (AIS) approach. Interference alignment and AIS can be seen as two sides of the same coin. In the pursuit of DoF and GDoF characterizations, just as interference alignment enables powerful achievable schemes to close the gap from below, the AIS approach enables powerful outer bounds to close the gap from above. Whether these ideas are enough to close the GDoF gaps for all channels and regimes of interest, if so then what new insights emerge from the new GDoF characterizations, and if not, then what new ideas hide in the remaining gaps, are exciting questions for the future.

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References

[1] S. Jafar, "Interference alignment: A new look at signal dimensions in a communication network," in *Foundations and Trends in Communication and Information Theory*, vol. 7, 2011, pp. 1–136.

[2] A. Lapidoth, S. Shamai, and M. Wigger, "On the capacity of fading MIMO broadcast channels with imperfect transmitter side-information," in *Proceedings of 43rd Annual Allerton Conference on Communications, Control and Computing*, Sep. 28-30, 2005.

[3] A. G. Davoodi and S. Jafar, "Settling conjectures on the collapse of degrees of freedom under finite precision CSIT," *Globecom (full paper at arXiv:1403.1541)*, Dec. 2014.

[4] H. Weingarten, S. Shamai, and G. Kramer, "On the compound MIMO broadcast channel," in *Proceedings of Annual Information Theory and Applications Workshop UCSD*, Jan 2007.

[5] G. Caire, N. Jindal, and S. Shamai, "On the required accuracy of transmitter channel state information in multiple antenna broadcast channels," in *Proceedings of the Asilomar Conference on Signals, Systems and Computers*, 2007.

[6] N. Jindal, "MIMO broadcast channels with finite rate feedback," *IEEE Trans. on Information Theory*, vol. 52, no. 11, pp. 5045–5049, Nov. 2006.

[7] M. Kobayashi, G.Caire, and N. Jindal, "How much training and feedback are needed in MIMO broadcast channels?" *ISIT*, pp. 2663–2667, Aug 2008.

[8] T. Gou and S. A. Jafar, "Optimal use of current and outdated channel state information: Degrees of freedom of the MISO BC with mixed CSIT," *IEEE Communication Letters*, vol. 16, no. 7, pp. 1084 – 1087, July 2012.

[9] S. A. Jafar, "Blind Interference Alignment," *IEEE Journal of Selected Topics in Signal Processing*, vol. 6, no. 3, pp. 216–227, June 2012.

[10] T. Gou, S. Jafar, and C. Wang, "On the degrees of freedom of finite state compound wireless networks," *IEEE Transactions on Information Theory*, vol. 57, no. 6, pp. 3268–3308, June 2011.

[11] M. A. Maddah-Ali, "The degrees of freedom of the compound MIMO broadcast channels with finite states," *CoRR*, vol. abs/0909.5006, 2009. [Online]. Available: http://arxiv.org/abs/0909.5006

[12] A. G. Davoodi and S. Jafar, "Transmitter cooperation under finite precision CSIT: A GDoF perspective," *Globecom*, Dec. 2015.

[13] A. Avestimehr, S. Diggavi, C. Tian, and D. Tse, "An approximation approach to network information theory," in *Foundations and Trends in Communication and Information Theory*, vol. 12, 2015, pp. 1–183.

[14] C. Huang, V. Cadambe, and S. Jafar, "Interference alignment and the generalized degrees of freedom of the X channel," *IEEE Transactions on Information Theory*, vol. 58, no. 8, pp. 5130–5150, August 2012.