On the Capacity of the Vector MAC and BC with Feedback

Syed Ali Jafar
Qualcomm Incorporated
San Diego, CA 92121 USA
E-mail : sjafar@qualcomm.com

Andrea Goldsmith
Department of Electrical Engineering
Stanford University, Stanford CA 94305, USA
E-mail : andrea@wsl.stanford.edu

Abstract

We determine the feedback capacity region of a two user Gaussian multiple access channel (MAC) with multiple antennas at the base station and a single antenna at each user. The vector MAC and broadcast channels (BC) with a single antenna at the base station and multiple antennas at each user are shown to be equivalent to scalar MACs and BCs, respectively. We also determine the capacity enhancement due to feedback at high SNR for the vector MAC and BC.

The capacity benefits of multiple antenna systems are highly dependent on the amount of channel knowledge at the transmitter and receiver [1] [2]. In practice, the channel state is learned at the receiver from a pilot signal and conveyed back to the transmitter through a feedback channel. The existence of a feedback channel opens up several interesting possibilities in terms of the information that can be conveyed from the receiver to the transmitter to enhance the forward channel capacity. Channel state feedback is just one of these possibilities. In this work we explore the potential capacity benefits of a feedback channel *beyond* providing channel state information. Therefore we assume perfect channel knowledge at all transmitters and receivers. We consider the feedback model that information theorists have traditionally been interested in: a feedback channel that makes the received signal available to the transmitter instantaneously. Feeding back the channel output to the transmitter establishes the absolute limit of how much the forward channel capacity can be enhanced by relaying information besides channel state information on a feedback channel. In particular, we explore the feedback capacity of a two user vector Gaussian MAC and BC with either multiple antennas at the base station and a single antenna at each mobile or with multiple antennas at each mobile and a single antenna at the base station.

For a single user discrete memoryless channel it is well known that feedback does not increase the channel capacity [3]. However, for multiuser channels feedback can increase the channel capacity. The feedback capacity of a two user scalar Gaussian MAC is found by Ozarow in [4]. Ozarow also shows that feedback increases the capacity of a scalar Gaussian BC [5]. However, the capacity region of even the scalar Gaussian BC with feedback remains unknown.

1 The System Model

We begin with the channel model for the vector Gaussian MAC. Figure 1 shows two kinds

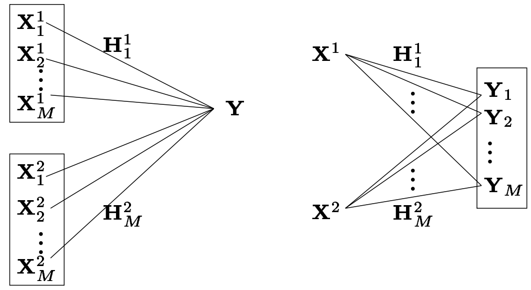

Figure 1: Vector MAC

of vector multiple access channels. In the first case each of the users has multiple transmit antennas while the base station has only one receive antenna. The second case is when each of the users has a single transmit antenna each and the base station has M receive antennas. Note that while the additive noise is not shown in the figure for simplicity, each output component has a Gaussian noise component added to it. The noise components are normalized so that they are zero mean, unit variance Gaussian stochastic processes uncorrelated in space and time. The input-output equations for the two cases are:

1. Multiple Antennas at each User

$$\begin{aligned} \mathbf{Y} &= \mathbf{H}^{1^T}\mathbf{X}^1 + \mathbf{H}^{2^T}\mathbf{X}^2 + \mathbf{N} \\ \mathbf{N} &\sim \mathcal{N}(0,1) \end{aligned} \tag{1}$$

2. Multiple Antennas at Base Station

$$\begin{aligned} \mathbf{Y} &= \mathbf{H}^1\mathbf{X}^1 + \mathbf{H}^2\mathbf{X}^2 + \mathbf{N} \\ \mathbf{N} &\sim \mathcal{N}(\mathbf{0},\boldsymbol{I}) \end{aligned} \tag{2}$$

Similarly, the vector Gaussian broadcast channels we consider can have multiple antennas at either the base station or at each user, as shown in figure 2.

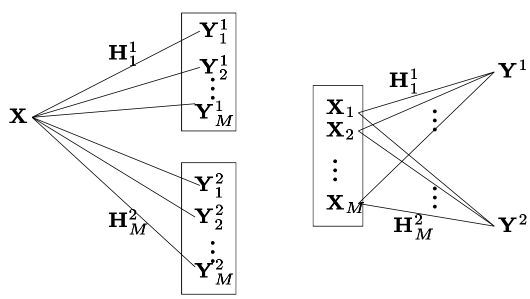

Figure 2: Vector BC

The input-output equations for the i^{th} user $(i = 1, 2)$ in the two cases are:

1. Multiple Antennas at each User

$$\begin{aligned} \mathbf{Y}^i &= \mathbf{H}^i\mathbf{X} + \mathbf{N}^i \\ \mathbf{N}^i &\sim \mathcal{N}(\mathbf{0},\mathbf{I}) \end{aligned} \tag{3}$$

2. Multiple Antennas at Base Station

$$\begin{aligned} \mathbf{Y}^i &= \mathbf{H}^{i^T}\mathbf{X} + \mathbf{N}^i \\ \mathbf{N}^i &\sim \mathcal{N}(0,1) \end{aligned} \tag{4}$$

2 Vector MAC and BC with Single Antenna at Base Station

In this section we show that with a single antenna at the base station, the vector MAC and BC are equivalent to the scalar MAC and BC obtained by a maximum ratio combining (MRC) of the antennas at each user. As illustrated in Figure 3, each user can premultiply his input signal

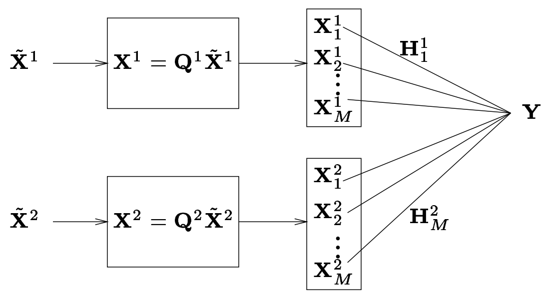

Figure 3: Transformation of Degenerate Vector MAC to Scalar MAC

with a unitary matrix $\mathbf{Q}^i$. Note that $\text{Trace}(\tilde{\mathbf{X}}^i\tilde{\mathbf{X}}^{i\dagger}) = \text{Trace}(\mathbf{X}^i\mathbf{X}^{i\dagger})$, so the power constraint is unchanged. Since a unitary transformation is an invertible transformation the capacity region is also unchanged. Now, choosing the first column of $\mathbf{Q}^i$ as $\frac{\mathbf{H}^i}{||\mathbf{H}^i||}$ makes the remaining columns of $\mathbf{Q}^i$ orthogonal to $\mathbf{H}^i$ and the vector channel becomes a scalar channel with input-output equation:

$$\mathbf{Y} = ||\mathbf{H}^1||\tilde{\mathbf{X}}_1^{\ 1} + ||\mathbf{H}^2||\tilde{\mathbf{X}}_1^{\ 2} + \mathbf{N} \quad (5)$$

Thus the vector MAC with a single antenna at the base station is equivalent to a scalar MAC.

Similarly, it is easily seen that the vector BC with a single antenna at the base station can also be transformed into a scalar BC. The users multiply the received vector with a unitary matrix $\mathbf{Q}^i$ that transforms the channel into a scalar channel while the noise statistics are unaffected by multiplication with a unitary matrix.

Next we proceed to determine the capacity region of the non-degenerate vector Gaussian MAC.

3 Vector MAC Capacity Region

We start with the non-degenerate vector Gaussian MAC with multiple antennas at the base station. To simplify the problem we first apply a unitary transformation at the base station that reduces user 1's channel to a scalar channel. Thus, user 1's input only affects the first receive antenna at the base station. Then we apply another unitary transformation on the remaining $M-1$ receive antennas so that user 2's signal affects only the first two receive antennas at the base station. For simplicity of notation we scale the transmit powers of the users so that the channel gains from the first user to the first receive antenna and the second user to the second receive antenna are unity. Figure 4 shows the transformations. Henceforth, based on this transformation, we use the following model for the non-degenerate vector Gaussian MAC:

$$\begin{aligned}\mathbf{Y}_1 &= \mathbf{X}^1 + \gamma\mathbf{X}^2 + \mathbf{N}_1 \\ \mathbf{Y}_2 &= \mathbf{X}^2 + \mathbf{N}_2\end{aligned}$$

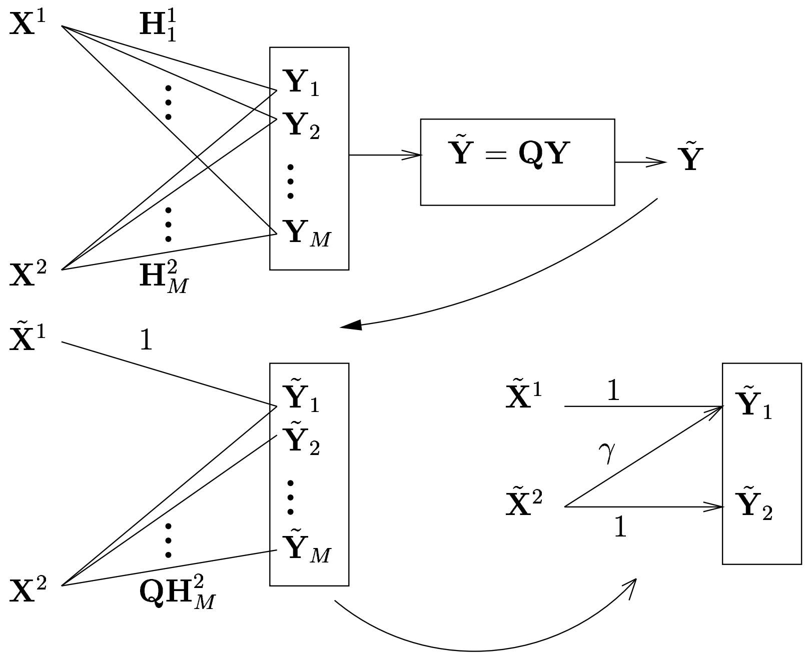

Figure 4: Simplification of the non-degenerate Vector MAC

We wish to show that the capacity region of this non-degenerate vector Gaussian MAC with feedback is given by:

$$
\begin{aligned}
C_{fb} = \quad & \bigcup_{0 \le \rho \le 1} \{(R_1, R_2) : \\
& 0 \le R_1 \le \frac{1}{2} \log\left(1 + P_1(1-\rho^2)\right) \\
& 0 \le R_2 \le \frac{1}{2} \log\left(1 + (1+\gamma^2)P_2(1-\rho^2)\right) \\
& 0 \le R_1 + R_2 \le \frac{1}{2} \log\left(1 + P_1 + (1+\gamma^2)P_2 + 2\gamma\rho\sqrt{P_1 P_2} + P_1 P_2 (1-\rho^2)\right)\}
\end{aligned} \tag{6}
$$

We start with the converse.

3.1 Converse

As found by Ozarow for the scalar MAC, the capacity region of the vector MAC with feedback is contained within C_o, where C_o is given by

$$
\begin{aligned}
C_o = \quad \bigcup_{p(\mathbf{X}^1, \mathbf{X}^2)} \{(R_1, R_2) : \quad & 0 \le R_1 \le I(\mathbf{X}^1; \mathbf{Y}_1, \mathbf{Y}_2 | \mathbf{X}^2) \\
& 0 \le R_2 \le I(\mathbf{X}^2; \mathbf{Y}_1, \mathbf{Y}_2 | \mathbf{X}^1) \\
& 0 \le R_1 + R_2 \le I(\mathbf{X}^1, \mathbf{X}^2; \mathbf{Y}_1, \mathbf{Y}_2)\}
\end{aligned}
$$

Since Ozarow's proof for C_o applies directly, we do not repeat the proof.

Next we obtain the Gaussian version of C_o. Assuming that $\mathbf{X}^1$ and $\mathbf{X}^2$ have variances σ_1^2

and σ_2^2, and correlation coefficient ρ, we bound the differential entropies as follows:

$$\begin{aligned} h(\mathbf{Y}_1,\mathbf{Y}_2) &\leq \frac{1}{2}\log(2\pi e)^2 \det\Bigg(I+ \\ &\left[\begin{array}{cc} 1 & \gamma \\ 0 & 1 \end{array}\right]\left[\begin{array}{cc} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ \gamma & 1 \end{array}\right]\Bigg) \\ h(\mathbf{Y}_1,\mathbf{Y}_2|\mathbf{X}^1,\mathbf{X}^2) &= \frac{1}{2}\log(2\pi e)^2 \\ h(\mathbf{Y}_1,\mathbf{Y}_2|\mathbf{X}^1) &= h(\mathbf{Y}_1|\mathbf{X}^1) + h(\mathbf{Y}_2|\mathbf{Y}_1,\mathbf{X}^1) \end{aligned} \tag{7}$$

In order to bound the conditional entropies we upper bound the conditional variances by the variances around the linear estimates. Thus,

$$\begin{aligned} h(\mathbf{Y}_1|\mathbf{X}^1) &= h(\mathbf{X}^1+\gamma\mathbf{X}^2+\mathbf{N}_1|\mathbf{X}^1) \\ &\leq \frac{1}{2}\log(2\pi e)\left(1+\gamma^2\sigma_2^2-\frac{(\gamma\rho\sigma_1\sigma_2)^2}{\sigma_1^2}\right) \\ &= \frac{1}{2}\log(2\pi e)\left(1+\gamma^2\sigma_2^2(1-\rho^2)\right) \\ h(\mathbf{Y}_2|\mathbf{Y}_1,\mathbf{X}^1) &= h(\mathbf{X}^2+\mathbf{N}_2|\mathbf{X}^1+\gamma\mathbf{X}^2+\mathbf{N}_1,\mathbf{X}^1) \end{aligned}$$

$$\leq \tfrac{1}{2}\log(2\pi e)\Bigg(1+\sigma_2^2-\left[\begin{array}{cc} \rho\sigma_1\sigma_2 & \sigma_2^2+\rho\sigma_1\sigma_2 \end{array}\right]$$

$$\left[\begin{array}{cc} \sigma_1^2 & \sigma_1^2+\rho\sigma_1\sigma_2 \\ \sigma_1^2+\rho\sigma_1\sigma_2 & \sigma_1^2+\gamma^2\sigma_2^2+1+2\gamma\rho\sigma_1\sigma_2 \end{array}\right]^{-1}\left[\begin{array}{c} \rho\sigma_1\sigma_2 \\ \sigma_2^2+\rho\sigma_1\sigma_2 \end{array}\right]\Bigg)$$

Upon simplification, these bounds give us

$$\begin{aligned} R_1 &\leq \frac{1}{2}\log\left(1+\sigma_1^2(1-\rho^2)\right) \\ R_2 &\leq \frac{1}{2}\log\left(1+(1+\gamma^2)\sigma_2^2(1-\rho^2)\right) \\ R_1+R_2 &\leq \frac{1}{2}log\left(1+\sigma_1^2+(1+\gamma^2)\sigma_2^2+2\gamma\rho\sigma_1\sigma_2+\sigma_1^2\sigma_2^2(1-\rho^2)\right) \end{aligned}$$

Note that the region for all values of $\rho, \sigma_1^2, \sigma_2^2$ such that $-1 \leq \rho \leq 1, 0 \leq \sigma_1^2 \leq P_1$ and $0 \leq \sigma_2^2 \leq P_2$ is contained within that for which $\rho = |\rho|$ and $\sigma_1^2 = P_1, \sigma_2^2 = P_2$.

This completes the converse for the capacity region specified in (6).

3.2 Achievability

The achievability is based on the Kailath Schalkwijk coding scheme. The transmitters proceed exactly as in the scalar MAC with feedback [4]. To avoid repetition we only describe computations that are different from the scalar MAC case. Note that unlike the scalar MAC where a scalar value is received for each channel use, we now have a two dimensional received vector.

Following the notation of [4] θ^i is the mapping of user i's message into the unit interval, $\hat{\theta}^i(k)$ is the receiver's estimate of θ^i at the kth transmission, $\epsilon^i(k) = \hat{\theta}^i(k) - \theta^i$ is the estimation error, $\alpha^i(k) = \text{var}(\epsilon^i(k))$, and $\rho(k)$ is the correlation coefficient of $\epsilon^1(k)$ and $\epsilon^2(k)$.

At time $k+1$, users 1 and 2 send,

$$\begin{aligned}\mathbf{X}^1(k+1) &= \sqrt{\frac{P_1}{\alpha^1(k)}}\epsilon^1(k)\text{sgn}(\rho(k))\\ \mathbf{X}^2(k+1) &= \sqrt{\frac{P_2}{\alpha^2(k)}}\epsilon^2(k)\end{aligned}$$

After the $k+1$st transmission, the receiver forms the following estimate

$$\hat{\theta}^i(k+1) = \hat{\theta}^i(k) - K^i_{Y\epsilon}(k+1)K^{-1}_{YY}(k+1)\begin{bmatrix}\mathbf{Y}_1(k+1)\\ \mathbf{Y}_2(k+1)\end{bmatrix}$$

where

$$\begin{aligned}K^i_{Y\epsilon}(k+1) &= \mathrm{E}\left\{[\ \mathbf{Y}_1(k+1) \quad \mathbf{Y}_2(k+1)\ ]\epsilon^i(k)\right\}\\ K_{YY}(k+1) &= \mathrm{E}\left\{\begin{bmatrix}\mathbf{Y}_1(k+1)\\ \mathbf{Y}_2(k+1)\end{bmatrix}[\ \mathbf{Y}_1(k+1) \quad \mathbf{Y}_2(k+1)\ ]\right\}\end{aligned}$$

Thus

$$\begin{aligned}\epsilon^i(k+1) &= \epsilon^i(k) - K^i_{y\epsilon}(k+1)K^{-1}_{YY}(k+1)\begin{bmatrix}\mathbf{Y}_1(k+1)\\ \mathbf{Y}_2(k+1)\end{bmatrix}\\ \alpha^i(k+1) &= \alpha^i(k) - K^i_{Y\epsilon}(k+1)K^{-1}_{YY}(k+1)K^{iT}_{Y\epsilon}(k+1)\end{aligned}$$

$K^i_{Y\epsilon}(k+1)$ and $K_{YY}(k+1)$ can be explicitly computed as

$$\begin{aligned}K^1_{Y\epsilon}(k+1) &= \text{sgn}(\rho(k))\sqrt{\alpha^1(k)}\left[\ \sqrt{P_1}+\gamma\sqrt{P_2}|\rho(k)| \quad \sqrt{P_2}|\rho(k)|\ \right]\\ K^2_{Y\epsilon}(k+1) &= \sqrt{\alpha^2(k)}\left[\ \sqrt{P_1}|\rho(k)|+\gamma\sqrt{P_2} \quad \sqrt{P_2}\ \right]\\ K_{YY}(k+1) &= \begin{bmatrix}P_1+\gamma^2P_2+2\gamma\sqrt{P_1P_2}|\rho(k)|+1 & \gamma P_2+\sqrt{P_1P_2}|\rho(k)|\\ \gamma P_2+\sqrt{P_1P_2}|\rho(k)| & 1+P_2\end{bmatrix}\end{aligned}$$

Substituting back into the previous equations we obtain after some simplification,

$$\begin{aligned}\alpha^1(k+1) &= \alpha^1(k)\frac{1+(1+\gamma^2)P_2(1-\rho^2(k))}{1+P_1+(1+\gamma^2)P_2+2\gamma|\rho(k)|\sqrt{P_1P_2}+P_1P_2(1-\rho^2(k))}\\ \alpha^2(k+1) &= \alpha^2(k)\frac{1+P_1(1-\rho^2(k))}{1+P_1+(1+\gamma^2)P_2+2\gamma|\rho(k)|\sqrt{P_1P_2}+P_1P_2(1-\rho^2(k))}\end{aligned}$$

Next we wish to obtain $\rho(k+1)$ in terms of $\rho(k)$. Note that

$$\mathrm{E}[\epsilon^1(k+1)\epsilon^2(k+1)] = \mathrm{E}[\epsilon^1(k)\epsilon^2(k)] - K^1_{Y\epsilon}(k+1)K^{-1}_{YY}(k+1)K^2_{Y\epsilon}(k+1)$$

which gives us the following relationship between $\rho(k+1)$ and $\rho(k)$:

$$\rho(k+1) = \frac{\rho(k)-\text{sgn}(\rho(k))\gamma\sqrt{P_1P_2}(1-\rho^2(k))}{\sqrt{(1+P_1(1-\rho^2(k)))(1+(1+\gamma^2)P_2(1-\rho^2(k)))}}$$

Now, for all values of P_1, P_2 the series $\rho(k)$ converges to $\rho(k) = (-1)^k\rho^\star$ where $\rho^\star$ is the solution in (0,1) of $1+P_1+(1+\gamma^2)P_2+2\gamma|\rho(k)|\sqrt{P_1P_2}+P_1P_2(1-\rho^2(k)) =$
$(1+(1+\gamma^2)P_2(1-\rho^2(k)))(1+P_1(1-\rho^2(k)))$

As for the scalar MAC the existence of a solution in $(0,1)$ is easy to see, since for $\rho = 0$ the LHS is smaller than the RHS while for $\rho = 1$ the RHS is smaller than the LHS.

The remaining arguments in the proof of achievability are the same as in the scalar MAC [4].

4 MAC Feedback Capacity at High SNR

Henceforth we devote our attention to feedback capacity at high SNR. In particular we are interested in the sum capacity with feedback, C_Σ^{fb}. For the MAC, high SNR means that the users transmit powers can be represented as βP_1 and βP_2 with $\beta \to \infty$. First we consider the scalar MAC.

4.1 Scalar MAC

The capacity region of the scalar MAC with feedback was determined by Ozarow in [4] as:

$$\begin{aligned} C_{fb} = & \bigcup_{0 \le \rho \le 1} \{(R_1, R_2) : \\ & 0 \le R_1 \le \frac{1}{2} \log \left(1 + \beta P_1 (1-\rho^2)\right) \\ & 0 \le R_2 \le \frac{1}{2} \log \left(1 + \beta P_2 (1-\rho^2)\right) \\ & 0 \le R_1 + R_2 \le \frac{1}{2} \log \left(1 + \beta(P_1 + P_2 + 2\rho\sqrt{P_1 P_2})\right) \} \end{aligned} \tag{8}$$

The sum capacity is achieved for ρ that is the solution of

$$(1 + \beta P_1(1-\rho^2))(1 + \beta P_2(1-\rho^2)) = 1 + \beta(P_1 + P_2 + 2\rho\sqrt{P_1 P_2})$$

The solution of this equation in the limit as $\beta \to \infty$ is given by $\rho^\star = 1$. This gives us

$$\begin{aligned} \lim_{\beta\to\infty} \left[C_\Sigma^{\text{fb}} - C_\Sigma\right] &= \lim_{\beta\to\infty} \frac{1}{2} \log \left(1 + \beta(P_1 + P_2 + 2\sqrt{P_1 P_2})\right) - \log\left(1 + \beta(P_1 + P_2)\right) \\ &= \frac{1}{2} \log \left(1 + \frac{2\sqrt{P_1 P_2}}{P_1 + P_2}\right). \end{aligned} \tag{9}$$

Note that $\frac{2\sqrt{P_1 P_2}}{P_1+P_2}$ is the ratio of the geometric mean to the arithmetic mean. So it always lies between 0 and 1. This implies that

$$\lim_{\beta\to\infty} C_\Sigma^{\text{fb}} - C_\Sigma = \frac{1}{2} \log \left(1 + \frac{2\sqrt{P_1 P_2}}{P_1 + P_2}\right) \le \frac{1}{2} \text{ b/channel use}.$$

Thus, for the scalar MAC, at high SNR, the sum capacity with feedback is higher than that without feedback by a positive constant bounded above by 0.5 bits/channel use.

4.2 Non-Degenerate Vector MAC

Next we consider the capacity of the non-degenerate vector Gaussian MAC with feedback. From the capacity expression (6) notice that at high SNR all three bounds are maximized for $\rho = 0$. However, $\rho = 0$ corresponds to the capacity region of the vector Gaussian MAC *w*ithout feedback. Thus, we conclude that at high SNR, the entire capacity region of the non-degenerate vector Gaussian MAC with feedback is the same as that without feedback.

5 Non-degenerate Vector BC Sum Capacity

We wish to compare the sum capacity of the non-degenerate vector broadcast channel with and without feedback. Recall that the sum capacity with feedback is not known even for a scalar broadcast channel. For this reason we bound the sum capacity as follows.

$$C_{\Sigma}^{\text{fb}} \geq C_{\Sigma}, \tag{10}$$

i.e. the sum capacity with feedback is no less than the capacity without feedback.

$$C_{\Sigma}^{\text{fb}} \leq C_{\Sigma}^{\text{coop,fb}} = C_{\Sigma}^{\text{coop}} \tag{11}$$

i.e. sum capacity with feedback is no larger than the capacity of a single user who receives the signals received by both users with feedback. But for a single user feedback does not increase capacity. Thus, the sum capacity with feedback is no larger than the capacity of a single user who receives the signals received by both users *without* feedback.

We assume that the noises seen by the two users are independent. In general if the noises n^1, n^2 are correlated with a coefficient of correlation ρ then

$$C_{\Sigma}^{\text{coop}}(\rho) = \max_{Q \geq 0, \text{Tr}(Q) \leq P} \frac{1}{2} \log \det \left(I + \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}^{-1} \mathbf{HQH}^{\dagger} \right) \tag{12}$$

Note that

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1^1 & \mathbf{H}_2^1 \\ \mathbf{H}_1^2 & \mathbf{H}_2^2 \end{bmatrix} \tag{13}$$

is non-singular for the non-degenerate vector Gaussian BC. Also note that for perfectly correlated noise $\rho = 1$ the capacity $C_{\Sigma}^{\text{coop}}(1)$ is infinite because noise can be perfectly cancelled out of one component of the received vector. From the proof of the sum rate capacity of the vector Gaussian BC [6] [7] [8] [9] we know that

$$C_{\Sigma} = \min_{\rho: -1 \leq \rho \leq 1} C_{\Sigma}^{\text{coop}}(\rho) \tag{14}$$

Let $\rho^{\star}$ denote the worst case noise correlation, i.e. the noise correlation that minimizes the RHS of (14). This implies that if the correlation coefficient of the users' noise terms $\mathbf{N}^1, \mathbf{N}^2$ is actually $\rho^{\star}$ then feedback does not improve the sum capacity. Note that this is true for any SNR. In other words, at any SNR, there is a possible correlation coefficient of the user's noise terms for which feedback does not increase the sum capacity.

6 BC Feedback Sum Capacity at High SNR

Now let us consider the sum capacity with feedback at high SNR. Recall that we assume that the users' noise terms $\mathbf{N}^1, \mathbf{N}^2$ are uncorrelated. From the bounds developed in the previous section we have

$$C_{\Sigma} \leq C_{\Sigma}^{\text{fb}} \leq C_{\Sigma}^{\text{coop}}(0) \tag{15}$$

At high SNR, the total power is βP, $\beta \to \infty$, and from (14) we have

$$\begin{aligned}
\lim_{\beta\to\infty} C_\Sigma &= \lim_{\beta\to\infty} \min_{\rho:-1\leq\rho\leq 1} \max_{\mathbf{Q}\geq 0, \mathrm{Tr}(\mathbf{Q})\leq\beta P} \\
&\quad \frac{1}{2}\log\det\left(I + \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}^{-1} \mathbf{HQH}^\dagger\right) \\
&= \lim_{\beta\to\infty} \max_{\mathbf{Q}\geq 0, \mathrm{Tr}(\mathbf{Q})\leq\beta P} \log\det\left(\mathbf{HQH}^\dagger\right) \\
&\quad - \max_{\rho:-1\leq\rho\leq 1} \log\det\left(\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right) \\
&= \lim_{\beta\to\infty} C_\Sigma^{\text{coop}}(0)
\end{aligned} \tag{16}$$

Combining (15) and (16) we conclude that

$$\lim_{\beta\to\infty} C_\Sigma^{\text{fb}} - C_\Sigma = 0. \tag{17}$$

That is, in the limit of high SNR, the sum capacity of the non-degenerate vector Gaussian BC with feedback is the same as the sum capacity without feedback.

7 Conclusions

We explore the feedback capacity region of two user vector Gaussian multiple access (MAC) and broadcast channels (BC) with either multiple antennas at the base station and a single antenna at each user or multiple antennas at each user and a single antenna at the base station. We show that the vector MAC and BC with a single antenna at the base station and multiple antennas at each user are degenerate vector channels as they are equivalent to a scalar MAC and BC. In the limit of high signal to noise ratio (SNR), we show that for the scalar Gaussian MAC (and for the degenerate vector Gaussian MAC), the difference between the sum capacity with and without feedback goes to a positive constant. We explicitly calculate this constant and show that it is no more than $\frac{1}{2}$ bit/channel use. For the non-degenerate vector Gaussian MAC we apply the Kailath Schalkwijk coding scheme to determine the previously unknown capacity region with feedback. Unlike the scalar Gaussian MAC, we show that for a non-degenerate vector Gaussian MAC the entire capacity region with feedback becomes the same as the capacity region without feedback in the limit of high SNR. For the non-degenerate two user vector Gaussian broadcast channel (BC) we show that there always exists a possible noise correlation for which feedback does not increase sum capacity. In the limit of high SNR, we show that the sum capacity of the non-degenerate vector BC with feedback is the same as that without feedback.

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