Does Beamforming achieve Outage Capacity with Direction Feedback?

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Abstract

We explore the outage capacity of an isotropic fading vector channel with multiple transmit antennas at the base station and a single antenna at the mobile receiver. Perfect channel knowledge is assumed to be available at the receiver while the transmitter has only partial knowledge of the direction of the user's channel vector based on quantized feedback. We provide a proof technique to determine whether transmit beamforming is optimal in terms of outage capacity for a quantized channel direction feedback system. The technique is used to establish the optimality of beamforming for several cases, e.g for a two antenna system with any number of feedback bits. Extensions to more than two transmit antennas are also provided.

I. Extended Summary

a) Introduction: The availability of channel knowledge at the transmitter and at the receiver significantly improves the performance of multiple antenna systems. While channel state information at the receiver can be obtained by training techniques, it is impractical to feedback complete and accurate channel state information to the transmitter in frequency division duplex (FDD) systems. Consequently, several partial feedback strategies dealing with different channel feedback parameters have been proposed over the recent years [1]–[10]. Of the various channel information feedback schemes, channel direction feedback has received considerable attention because direction information at the transmitter allows the transmitter to focus the available power along the channel and utilize the multiple transmit antennas more effectively to increase system throughput. Seminal work in [1]–[3] and the recent results of [4], [5] on quantized direction feedback systems form the background for this paper.

b) Optimality of beamforming (Ergodic and Outage Capacity): The quantized direction feedback system is examined using lower bounds for outage probability in [1] while [2], [3] analyze the system from an average receive SNR perspective. However, the transmit strategy assumed throughout [1]–[3] is beamforming. Beamforming is a desirable transmit strategy owing to the ease of its implementation and the ability to use scalar codecs. But beamforming does not always achieve capacity when only partial channel knowledge is available at the transmitter. Expressions for the ergodic capacity of the quantized direction feedback system have been derived and necessary and sufficient conditions have been established for beamforming to the optimal transmit strategy in terms of ergodic capacity [4], [5]. However, the delays involved with ergodic capacity can be quite excessive. For practical systems with more stringent delay constraints, outage capacity is a more relevant metric [11], [12]. Even so, it is generally harder to mathematically analyze outage capacity than it is to analyze ergodic capacity. The input covariance matrix that maximizes the ergodic capacity does not necessarily maximize the outage capacity and vice versa. Unlike the ergodic capacity maximization, the optimization problem for outage capacity is not convex [13]. Consequently, this precludes using well known convex optimization techniques. This work is motivated by the need to determine whether beamforming is optimal, in the context of outage capacity, for quantized direction feedback systems.

c) Channel Model: We explore the outage capacity of a partial feedback wireless communication system with M antennas at the transmitter and a single antenna at the receiver. We assume that the receiver has perfect channel state information while only a quantized estimate of the user's channel direction is available at the transmitter. The input-output relationship for the system under consideration is given by $Y = \mathbf{H}^{\dagger} \cdot \mathbf{X} + Z$, where $\mathbf{X}$, $\mathbf{H}$, Z and Y are the $(M \times 1)$ input, $(M \times 1)$ channel vector, AWGN noise and output respectively. The power constraint at the transmitter is given by P. The finite rate feedback path can support a rate of B bits per frame and is assumed to be error free. A predetermined set $Q = \{\mathbf{q}_1,\ \mathbf{q}_2,\ \cdots,\ \mathbf{q}_N\}$ of $N = 2^B$ unit norm channel quantization vectors is available to both the transmitter and receiver. The quantization vectors are chosen according to the Grassmannian criterion detailed in [1]–[3]. The receiver then sends back the index k of the quantization vector closest to the channel, i.e, $|\langle \mathbf{H}, \mathbf{q}_k \rangle| \geq |\langle \mathbf{H}, \mathbf{q}_j \rangle|$ for all $j \in \{1,\ \cdots,\ N\}$, $\ j \neq k$. We define the 'decision region' D_k corresponding to the quantization vector $\mathbf{q}_k$ as the set of all channel vectors closest to $\mathbf{q}_k$, i.e, $D_k = \{\mathbf{H} : |\langle \mathbf{H}, \mathbf{q}_k \rangle| \geq |\langle \mathbf{H}, \mathbf{q}_j \rangle|\}$ for all $j \in \{1, \cdots, N\}, \ \ j \neq k$.

d) Contributions and Overview of Results: [1]–[3] show that if the transmit strategy is constrained to beamforming (unit rank input covariance matrix), then the optimal beamforming direction is along the closest quantization vector. This leaves open the possibility that higher throughputs may be obtained by using multiple beams (higher rank input covariance matrix). Thus, the optimality of beamforming with respect to outage capacity is not known. It is this problem that we address to in this paper.

The main contribution of this paper is a technique to determine whether beamforming along the closest quantization vector is optimal in terms of outage capacity for quantized direction feedback systems. We begin with outage capacity expressions for the case when the index fed back to the transmitter is k. The outage capacity to support an outage probability of P_{out} for the channel in consideration when the quantization vector index that is fed back is k can be mathematically expressed as

$$C_{out,k}\left(P_{out}\right) = \max_{\mathbf{K}:\mathrm{Tr}[\mathbf{K}]\leq P} \left(\sup\left\{R: \ \text{Prob}\left[\ \mathbf{H}^{\dagger}\cdot\mathbf{K}\cdot\mathbf{H} \leq \tfrac{2^R-1}{P} \ \middle| \ \mathbf{H}\in D_k \ \right] \leq P_{out}\right\}\right) \tag{1}$$

where $\mathbf{K} = \mathrm{E}\left[\mathbf{X}\cdot\mathbf{X}^{\dagger}|\mathbf{H}\in D_k\right]$ is the covariance matrix of the input. Let the input covariance matrix that maximizes the outage capacity be $\mathbf{K}_o = \mathbf{U}_o\cdot\mathbf{\Lambda}_o\cdot\mathbf{U}_o^{\dagger}$, where $\mathbf{U}_o$ is the matrix of the eigenvectors and $\mathbf{\Lambda}_o$ is a diagonal matrix containing the eigenvalues $\mathbf{\Lambda}_o\left(i\right)$. Note that transmission with a covariance matrix $\mathbf{K}_o$ can be thought of as power allocation (based on the eigenvalues of $\mathbf{K}_o$) along the eigenvectors of $\mathbf{K}_o$. The outage capacity is then

$$C_{out,k}\left(P_{out}\right) = \sup\left\{R: \ \text{Prob}\left[\ \left(\mathbf{H}^{\dagger}\cdot\mathbf{U}_o\right)\cdot\mathbf{\Lambda}_o\cdot\left(\mathbf{U}_o^{\dagger}\cdot\mathbf{H}\right) \leq \tfrac{2^R-1}{P} \ \middle| \ \mathbf{H}\in D_k \ \right] \leq P_{out}\right\} \tag{2}$$

From equation (2), it can be seen that $\mathbf{H}^{'} = \mathbf{U}_o^{\dagger}\cdot\mathbf{H}$ is just a rotation of the channel vector $\mathbf{H}$ by the unitary matrix $\mathbf{U}_o^{\dagger}$. If $\mathbf{H}\in D_k$, then $\mathbf{H}^{'}\in\mathbf{U}_o^{\dagger}\cdot D_k = D_k^{'}$. In this light, the outage capacity expression can be rewritten as

$$C_{out,k}\left(P_{out}\right) = \sup\left\{R: \ \text{Prob}\left[\ \mathbf{H}^{'\dagger}\cdot\mathbf{\Lambda}_o\cdot\mathbf{H}^{'} \leq \tfrac{2^R-1}{P} \ \middle| \ \mathbf{H}^{'}\in D_k^{'} \ \right] \leq P_{out}\right\} \tag{3}$$

Now that we have the necessary expressions for outage capacity, we present the procedure to test the optimality of beamforming with respect to outage capacity. Without loss of generality, we assume henceforth that $k=1$, i.e, $\mathbf{q}_1 = [1 \ 0 \ \cdots \ 0]$ is the quantization vector fed back to the transmitter.

1) The first step is to investigate whether the direction of the quantization vector fed back is one of the eigenbasis vectors of the optimal input covariance matrix. If $\mathbf{q}_1$ is the quantization vector closest to the channel, we need to examine whether $\mathbf{q}_1$ is one of the eigenvectors of $\mathbf{K}_o$. One of the approaches we will use is to compare the probabilities $\text{Prob}(R,P,D_1) = \text{Prob}\left[\ \mathbf{H}^{\dagger}\cdot\mathbf{\Lambda}_o\cdot\mathbf{H} \leq \tfrac{2^R-1}{P} \ \middle| \ \mathbf{H}\in D_1 \ \right]$ and $\text{Prob}(R,P,D_1^{'}) = \text{Prob}\left[\ \mathbf{H}^{'\dagger}\cdot\mathbf{\Lambda}_o\cdot\mathbf{H}^{'} \leq \tfrac{2^R-1}{P} \ \middle| \ \mathbf{H}^{'}\in D_1^{'} \ \right]$. If $\text{Prob}(R,P,D_1)$ turns out to be greater than $\text{Prob}(R,P,D_1^{'})$, then it can be easily seen from equation (3) that the optimal $\mathbf{U}_o$ is the identity matrix $\mathbf{I}$.
2) Once it is known that $\mathbf{q}_1$ is one of the power allocation directions, the next step is to check if the projection of the channel along the $\mathbf{q}_1$ is greater than the projection along the other orthogonal directions. This step just requires knowledge of the structure of D_1, the decision region involved. if the projection of the channel along $\mathbf{q}_1$ is greater than the projections along other directions, it is optimal to direct all the available power P along $\mathbf{q}_1$, and beamforming is optimal.

This technique can be used to prove the optimality of beamforming for the two transmit antenna case with an arbitrary number of beamforming vectors and also for other configurations. For a more intuitive understanding of the technique, we consider a simple two transmit antenna system example.

e) An illustrative example: Consider a channel with two transmit antennas ($M=2$), one receive antenna and one bit ($B=1$ of channel direction feedback. Let the power constraint at the transmitter be P. For simplicity of exposition, let the channel, the input and noise be real valued, i.e, the channel and the input are two dimensional vectors in the real space $\mathbb{R}^2$. Without any ambiguity, the two quantization vectors ($N = 2^B = 2$) can be chosen to be along the two axes as shown in Figure 1. The decision region D_1 for the quantization vector $\mathbf{q}_1$ is delimited by the lines A and B as shown in Figure 1. For any channel vector $\mathbf{H} = [H_1 \ \ H_2] \in D_1$, we have $|H_1| \geq |H_2|$.

Let the input covariance matrix that maximizes the outage capacity be $\mathbf{K}_o = \mathbf{U}_o\cdot\mathbf{\Lambda}_o\cdot\mathbf{U}_o^{\dagger}$. Without any loss of generality, we will assume that $\mathbf{\Lambda}_o\left(1\right) \geq \mathbf{\Lambda}_o\left(2\right)$. $D_1^{'} = \mathbf{U}_o^{\dagger}D_1$ represents the rotation of the decision region D_1 in the two dimensional plane, let $D_1^{'}$ be the region between the lines $A^{'}$ and $B^{'}$. The shaded region represents the channel vectors that are common to D_1 and $D_1^{'}$. We will compare the probabilities $\text{Prob}(R, P, D_1)$ and $\text{Prob}(R, P, D_1^{'})$. Note that every channel vector in the shaded region contributes equally to both $\text{Prob}(R, P, D_1)$ and $\text{Prob}(R, P, D_1^{'})$. We will therefore consider closely only those channel vectors that are not common to D_1 and $D_1^{'}$.

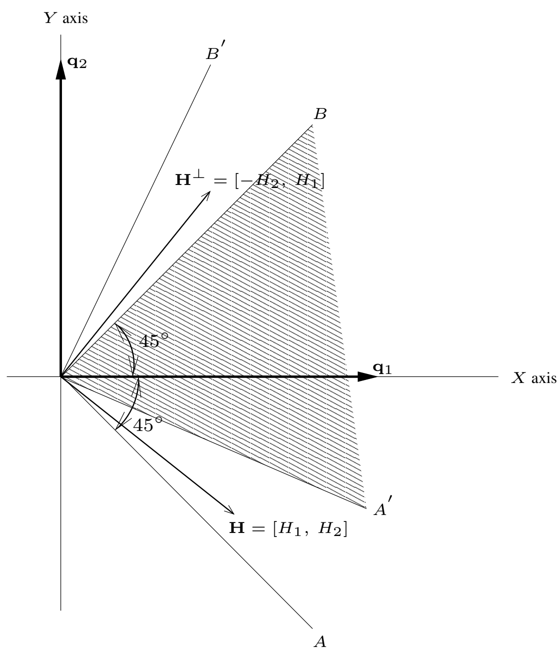

Fig. 1. $M = 2$ and $N = 2$ example.

For *every* channel vector $\mathbf{H} = [H_1 \ \ H_2] \in (D_1 \cap D_1^{'c})$, there exists a unique corresponding vector $\mathbf{H}^{\perp} = [-H_2 \ \ H_1] \in (D_1^{'} \cap D_1^{c})$ as shown in Figure 1. $\mathbf{H}$ and $\mathbf{H}^{\perp}$ occur with the same probability and are orthogonal. Given $|H_1| \geq |H_2|$ and $\mathbf{\Lambda}_o\,(1) \geq \mathbf{\Lambda}_o\,(2)$, it is easy to prove that $\mathbf{H}^{\dagger} \cdot \mathbf{\Lambda}_o \cdot \mathbf{H} \geq \mathbf{H}^{\perp\dagger} \cdot \mathbf{\Lambda}_o \cdot \mathbf{H}^{\perp}$. In other words, for every channel vector $\mathbf{H}$ in D_1, there is a corresponding equiprobable channel vector $\mathbf{H}^{\perp}$ in $D_1^{'}$ such that $\mathbf{H}^{\dagger} \cdot \mathbf{\Lambda}_o \cdot \mathbf{H} \geq \mathbf{H}^{\perp\dagger} \cdot \mathbf{\Lambda}_o \cdot \mathbf{H}^{\perp}$. Therefore we have $\text{Prob}(R, P, D_1) \leq \text{Prob}(R, P, D_1^{'})$. The optimal $\mathbf{U}_o$ is therefore the (2×2) identity matrix and the optimal input covariance matrix $\mathbf{Q}_o = \mathbf{\Lambda}_o$ is a diagonal matrix. Although the structure of the optimal input covariance matrix is now defined, the power allocation scheme along the eigenvectors has not been specified. It is straightforward to prove that for all $\mathbf{H} \in D_1$, $\mathbf{H}^{\dagger} \cdot \mathbf{\Lambda}_o \cdot \mathbf{H}$ (and therefore outage capacity) is maximized when the power allocation strategy is such that $\mathbf{\Lambda}_o\,(1) = P$ and $\mathbf{\Lambda}_o\,(2) = 0$. In other words, beamforming along the closest quantization vector is optimal (in terms of outage capacity) for the $M = 2$ quantized direction feedback system with $N = 2$. We point out that for this example, the optimal transmission direction for the maximization of outage capacity turns out to be the same as that for ergodic capacity [4].

f) Conclusion: We present a technique to explore the optimality of beamforming for outage capacity. Using the technique, we show optimality of beamforming for the two transmit antenna case with an arbitrary number $N \geq 2$ of quantization vectors. Detailed proofs for these cases and extensions to any number of transmit antennas are provided in the full paper.

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