1 Introduction

Multiple input multiple output (MIMO) systems have assumed great importance in recent times because of their remarkably higher capacity compared to single input single output (SISO) systems. One of the most celebrated results in this context is that the capacity of a point-to-point MIMO system with $M$ inputs and $N$ outputs increases linearly as $\min(M, N)$ at high SNR. For power and bandwidth limited wireless systems this opens up another dimension - “space” that can be exploited in a similar way as time and frequency. Similar to time division and frequency division multiplexing, MIMO systems present the possibility of multiplexing signals in space. For example, using the singular value decomposition of a MIMO channel, one can generate parallel channels in space similar to the multiple channels created by dividing time or frequency into orthogonal slots.

Inspite of the theoretical duality of time, frequency, and space, the contextual constraints of a communication system create important differences that determine whether the degrees of freedom that exist in each of these dimensions are available or not. In particular, the availability of the spatial degrees of freedom depends upon two factors: cooperation within inputs and outputs, and channel knowledge. Previous work has shown that in the absence of channel knowledge the spatial degrees of freedom are lost [1, 2]. In this paper we will focus on communication scenarios with constrained cooperation between inputs and outputs.

The traditional view of MIMO systems is that of a point-to-point multiple antenna system with multiple inputs representing the collocated antennas at the transmitter and the multiple outputs representing the collocated antennas at the receiver. In this paper we take a generalized view of MIMO systems where the inputs and outputs are distributed across users. The distributed MIMO model captures most multiuser communication scenarios. While multiuser communications have always been interesting topic, there has been a recent surge of interest in this area because of the growing popularity and applications of ad-hoc and sensor networks. What makes the generalized MIMO systems especially challenging is that unlike the point-to-point case, joint processing is not possible at the inputs or the outputs controlled by different users. The available spatial degrees of freedom are affected by the inability to jointly process the signals at the distributed inputs and outputs. In this paper our goal is to quantify the loss in the available degrees of freedom under the cooperation constraints imposed by various multiuser communication scenarios.

2 Degrees of Freedom Measure

In order to isolate the impact of distributed processing from the channel uncertainty we assume that the channel state is fixed and perfectly known at all transmitters and receivers. Also, we assume that the channel matrices are sampled from a rich scattering environment. Therefore we can ignore the measure zero event that some channel matrices are rank deficient. It is well known that the capacity of a scalar AWGN channel scales as $\log(SNR)$ at high SNR. On the other hand, for a single user MIMO channel with $M$ inputs and $N$ outputs, the capacity growth rate can be shown to be $\min(M, N) \log(SNR)$ at high SNR. This motivates the natural definition of the spatial degrees of freedom as:

$$\eta \triangleq \lim_{\rho \to \infty} \frac{C_{\infty}(\rho)}{\log(\rho)}$$

where $C_{\infty}(\rho)$ is the sum capacity (just the capacity in case of point to point (P2P) communications) at SNR $\rho$. In other words, the degrees of freedom $\eta$ represent the maximum multiplexing gain [3] of the generalized MIMO
system. For the point to point case, the \((M, N)\) degrees of freedom are easily seen to correspond to the parallel channels that can be isolated using the SVD operation, involving joint processing at the \(M\) inputs and joint processing at the \(N\) outputs.

As a simple illustration of the results, we will use the elemental model of a (2,2) MIMO system as a running example throughout this paper, i.e. a system with two inputs and two outputs. The generalized (2,2) MIMO system gives rise to the following distributed scenarios with different degrees of cooperation.

### 3 The Point to Point Channel

As stated above, the degrees of freedom in a \(2 \times 2\) point-to-point MIMO channel can be expressed as \(\eta(\text{P2P}) = 2\). In general,

\[
\eta(\text{P2P}) = \lim_{\rho \to \infty} \frac{C(\rho)}{\log(\rho)} = \lim_{\rho \to \infty} \frac{1}{\log(\rho)} \sum_{i=1}^{\min(M,N)} \log \left( 1 + \frac{\rho}{\min(M,N) \sigma_i^2} \right)
\]

Here, \(\sigma_i^2\) are the singular values of the MIMO channel and (2) follows because uniform power allocation is optimal at high SNR.

### 4 The Multiple Access Channel

The multiple access channel is an example of a MIMO system where cooperation is allowed only between the channel outputs. Let the MAC consist of \(N = 2\) outputs controlled by the same receiver and \(K = 2\) users, each controlling \(M_1 = M_2 = 1\) input for a total of \(M = M_1 + M_2 = 2\) inputs. By zero forcing at the MAC receiver, 2 parallel channels can be created, so that the total degrees of freedom are the same as with perfect cooperation between all the users.

\[
\eta(\text{MAC}) = \eta(\text{P2P}) = 2.
\]

In general, \(\eta(MAC) = \min(M_1 + M_2 + \cdots + M_K, N)\) and we show that zero forcing which requires joint processing at the MAC receiver is sufficient to exploit all the degrees of freedom.

### 5 The Broadcast Channel

The broadcast channel is an example of a MIMO system where cooperation is allowed only between the channel inputs. Let the BC consist of \(M = 2\) inputs controlled by the same transmitter and \(K = 2\) users, each controlling \(N_1 = N_2 = 1\) output for a total of \(N = N_1 + N_2 = 2\) outputs. It is possible to show that by zero forcing at the BC transmitter, 2 parallel channels can be created, so that the total degrees of freedom are the same as with perfect cooperation between all the users.

\[
\eta(\text{BC}) = \eta(\text{MAC}) = \eta(\text{P2P}) = 2.
\]

Again, in general, \(\eta(BC) = \min(M_1 + \cdots + M_K, N)\) and we show that zero forcing which requires joint processing only at the BC transmitter is sufficient to exploit all the degrees of freedom.
6  The Interference Channel

The interference channel is an example of a MIMO system where both the inputs and the outputs are distributed among multiple users. No cooperation is allowed between subsets of inputs and outputs controlled by different users. For our $2 \times 2$ MIMO system, there are two transmitters controlling one input each and two receivers controlling one output each. There are only two messages, one from transmitter 1 to receiver 1 and one from transmitter 2 to receiver 2. While the sum capacity of an interference channel is not known in general, we are able to show from the known upperbounds that only one degree of freedom is left in this case:

$$\eta(\text{INT}) = 1.$$  

Interestingly enough, for the interference channel with $M_1 = 2, N_1 = 1$ and $M_2 = 2, N_2 = 1$, we have $\eta = 2$. However, for the interference channel with $M_1 = 2, N_1 = 1$ and $M_2 = 1, N_2 = 2$, we have $\eta = 1$. With two users, for general $M_1, N_1, M_2, N_2$ we have

$$\eta(\text{INT}) = \min(M_1, N_1) + \min(M_2 - N_1, N_2)^+ 1(M_1 > N_1) + \min(M_2, N_2 - M_1)^+ 1(M_1 < N_1)$$

$$+ \min(M_2 - M_1, N_2 - N_1)^+ 1(M_1 = N_1)$$  

(5)

where $(x)^+$ is the notation for $\max(0, x)$ and $1(x)$ is the indicator function that takes values unity and zero when the boolean variable $x$ is true and false, respectively. Extensions to more than two users are included in the full paper.

7  The Z Channel, X Channel and the Relay Channel

The Z channel is similar to the interference channel in that both the inputs and the outputs are distributed among multiple users. The difference is that in a Z channel some users are protected from other users’ interference as the channel coefficients are assumed to be zero between certain users. Moreover, all users are assumed to carry messages for all receivers whenever a non-zero channel exists between them. No cooperation is allowed between subsets of inputs and outputs controlled by different users. For our $2 \times 2$ MIMO system, there are two transmitters controlling one input each and two receivers controlling one output each. No path is assumed to exist between input 1 and output 2. There are three messages, one from transmitter 1 to receiver 1, one from transmitter 1 to receiver 2, and one from transmitter 2 to receiver 2. While the sum capacity of the Z channel is also unknown in general, we can show that only one degree of freedom is available in this case:

$$\eta(Z) = 1.$$  

(6)

We also determine the degrees of freedom for the X channel (where all transmitters have messages for all receivers) and the relay channel. These results along with their extensions to more than two users and multiple antennas are included in the full paper.

References

