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CAPACITY AND DUALITY OF AF RELAY MAC AND BC

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ABSTRACT

We consider multi-hop multiple access (MAC) and broadcast channels (BC) where communication takes place with the assistance of relays that amplify and forward (AF) their received signals. For a two hop parallel AF relay MAC, assuming a sum power constraint across all relays we characterize optimal relay amplification factors and the resulting capacity regions. We find that the parallel AF relay MAC with total transmit power of the two users $P_1 + P_2 = P$ and total relay power P_R is the dual of the parallel AF relay BC where the MAC source nodes become the BC destination nodes, the MAC destination node becomes the BC source node, the dual BC source transmit power is P_R and the total transmit power of the AF relays is P.

I INTRODUCTION

AF relay optimization for dual hop communications has been the focus of much recent research. [1–3] consider the case of orthogonal relay transmissions. While orthogonal relay schemes are attractive for wideband communications, Maric and Yates [4] have shown that for amplify and forward relays, shared bandwidth transmission schemes can provide higher capacity. Maric and Yates also find closed form solutions for the relay amplification factor and the point to point AF relay channel capacity with shared band transmission. Optimum and near optimum power allocation schemes for single branch multi-hop relay networks have been considered in [5].

While much of the work on relay networks has focused on point to point communications, multiuser relay networks are increasingly gaining attention as seminal work in this area [6–10] has shown the remarkable advantages of multiuser relaying. In [11] gains for AF relays in a multiuser parallel network are determined to achieve a joint minimization of the MMSE of all the source signals at the destination. Tang et. al. [8] consider a MIMO relay broadcast channel, where a multiple antenna transmitter sends data to multiple users via a relay with multiple antennas over two hops. Capacity bounds are used to establish that the performance loss is not significant. Capacity with cooperative relays has been explored for the multicast problem by Maric and Yates [12, 13], for the broadcast problem by Liang and Veeravalli [10], and for the mixed multiple access and broadcast problem by Host-Madsen [14]. Maric and Yates explore an accumulative multicast strategy where nodes collect energy from previous transmissions, while Liang and Veeravalli [10] and Host-Madsen [14] address the general question of optimal relay functionality which may not be an amplify and forward scheme.

In this work, we pursue two related objectives. The first is to investigate the capacity optimal relay amplification factors for two hop multiple access and broadcast channels where communication takes place via parallel AF relay links and no direct link exists between source(s) and destination(s). With a sum power constraint on all the relays, we characterize the optimal relay amplification factor for the parallel AF relay MAC for all points on the boundary of the MAC capacity region. We obtain the sum capacity and the individual user capacities in closed form and present a couple of simultaneous equations whose numerical solution yields the optimal rate pair (R_1, R_2) that maximizes $\mu_1 R_1 + \mu_2 R_2$ for any $\mu_1, \mu_2 \geq 0$. The second objective of this work is to identify duality relationships in AF relay networks. We obtain a general duality result for multi-hop multiple access and broadcast channels where each hop may consist of parallel AF relays and the relays may be equipped with multiple antennas. The duality allows us to compute the capacity region of the two hop parallel AF relay BC as the union of the two hop relay MAC capacity regions.

II PARALLEL AF RELAY MAC AND BC

We consider two hop multiple access and broadcast channels where communication takes place with the assistance of multiple parallel AF relays. All channels are known and fixed, there is one sum power constraint across all relays, and all channel, input, output, noise variables and the relay scaling factors are real. For simplicity we focus on the two user case. The AF relay MAC and its dual BC models are depicted in Fig. 1.

A Two User Parallel AF Relay MAC and BC

We use the notation MAC$(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R)$ to denote the two user multiple access channel described above, i.e. with the channels between the transmitters and relays $\mathbf{F}^{[1]}, \mathbf{F}^{[2]}$, the corresponding source transmit powers P_1, P_2 (respectively), the channel between the relays and the destination $\mathbf{G}$, the relay amplification vector $\mathbf{D}$, and total transmit power at all relays P_R. To distinguish the MAC resulting from a specific choice of $\mathbf{D}$ from the MAC where all $\mathbf{D}$ from the feasible set are allowed, we denote the former as MAC$\left(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \mathbf{D}, P_R\right)$ and the latter as MAC$\left(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R\right)$

1) MAC$\left(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R\right)$

For the two user parallel AF relay MAC$\left(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R\right)$, the received signals at the relays and the common destination are as follows:

$$\mathbf{R} = \mathbf{F}^{[1]}x_1 + \mathbf{F}^{[2]}x_2 + \mathbf{N}_R \quad (1)$$

$$y = \text{Tr}[\mathbf{G}\mathbf{D}\left(\mathbf{F}^{[1]}x_1 + \mathbf{F}^{[2]}x_2 + \mathbf{N}_R\right)] + n \quad (2)$$

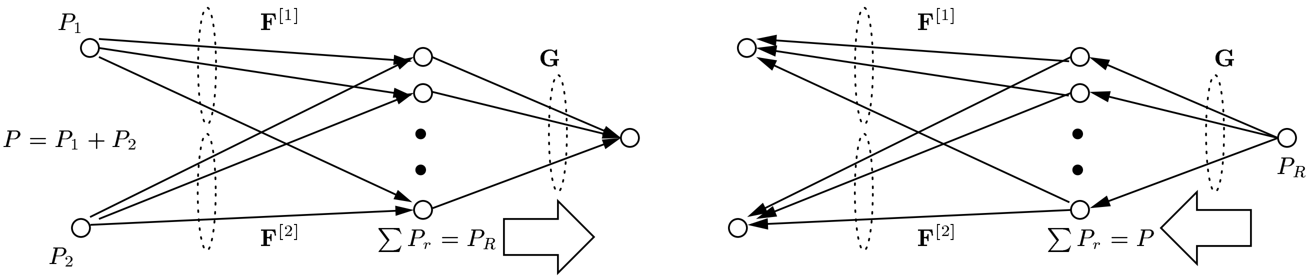

Figure 1: Dual Parallel AF Relay MAC and BC channels

Variable	AF MAC	Dual AF BC
R	Number of parallel AF relays	Number of parallel AF relays
P_1	Transmit power for source 1	
P_2	Transmit power for source 2	
P	$P_1 + P_2$	Total transmit power of R relays
P_R	Total transmit power of R relays	Source transmit power
$f_i^{[j]}$	Channel from j^{th} source to i^{th} relay	Channel from i^{th} relay to j^{th} destination.
g_i	Channel from i^{th} relay to destination	Channel from source to i^{th} relay.
d_i	Amplification factor at the i^{th} relay	Amplification factor at the i^{th} relay
x		Transmitted symbol for common source
x_i	Transmitted symbol for source i	
y	Received symbol at common destination	
y_i		Received symbol at i^{th} destination
r_i	Received symbol at the i^{th} relay	Received symbol at the i^{th} relay
$d_i r_i$	Symbol transmitted by the i^{th} relay	Symbol transmitted by the i^{th} relay
$n_{R,i}$	Unit power AWGN at the i^{th} relay	Unit power AWGN at the i^{th} relay
n_i		Unit power AWGN at i^{th} destination
n	Unit power AWGN at common destination	

Normalizing the noise to unit variance, the destination output can be expressed as

$$y' = \frac{\mathrm{Tr}(\mathbf{GDF}^{[1]})}{\sqrt{1+\mathrm{Tr}(\mathbf{D}^2\mathbf{G}^2)}} x_1 + \frac{\mathrm{Tr}(\mathbf{GDF}^{[2]})}{\sqrt{1+\mathrm{Tr}(\mathbf{D}^2\mathbf{G}^2)}} x_2 + n' \quad (3)$$

The power constraints are:

$$P_1 = \mathrm{E}[x_1^2], \quad P_2 = \mathrm{E}[x_2^2] \quad (4)$$

$$P_R = \mathrm{Tr}\left(\mathbf{D}^2\left(I + P_1\mathbf{F}^{[1]2} + P_2\mathbf{F}^{[2]2}\right)\right). \quad (5)$$

2) BC$\left(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \{\mathbf{D}\}, P\right)$

We use the shorthand notation BC$\left(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \mathbf{D}, P\right)$ to indicate the dual BC, i.e. broadcast channel with transmit power P_R, channel vector $\mathbf{G}$ from transmitter to relays, channel vectors $\mathbf{F}^{[1]}$ and $\mathbf{F}^{[2]}$ from the relays to receiver 1 and 2 respectively, relay amplification factor given by $\mathbf{D}$ and total transmit power used by the relays P. As for the MAC, we use $\{\mathbf{D}\}$ to indicate all feasible relay amplification factors are allowed and $\mathbf{D}$ to indicate a specific choice. For the dual broadcast channel, received signals at the relays and the two destinations are as follows:

$$\mathbf{R} = \mathbf{G}x + \mathbf{N}_R \quad (6)$$

$$y_1 = \mathrm{Tr}[\mathbf{F}^{[1]}\mathbf{D}(\mathbf{G}x + \mathbf{N}_R)] + n_1 \quad (7)$$

$$y_2 = \mathrm{Tr}[\mathbf{F}^{[2]}\mathbf{D}(\mathbf{G}x + \mathbf{N}_R)] + n_2 \quad (8)$$

$$P_R = \mathrm{E}[x^2], \;\; P = \mathrm{Tr}\left(\mathbf{D}^2\left(I + P\mathbf{G}^2\right)\right). \quad (9)$$

Notice that the relays are associated with power P and the source with power P_R.

III CAPACITY AND RELAY OPTIMIZATION FOR PARALLEL AF RELAY MAC

Given a relay amplification vector $\mathbf{D}$ the capacity region of the resulting scalar Gaussian MAC is the well known pentagon. Taking the union over all $\mathbf{D}$ that satisfy the relay sum power constraint gives us a characterization of the capacity region of MAC$\left(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R\right)$. Note that in the absence of any further characterization of $\mathbf{D}$ we are left with optimization over the entire space of feasible $\mathbf{D}$, i.e. a R dimensional space. A brute force solution to such an optimization may be difficult as the number of relays increases. Theorem 1 solves this problem.

A Relay Optimization

The following Theorem reveals the structure of the optimal relay amplification factor $\mathbf{D}$ for rate pairs on the boundary of the capacity region.

Theorem 1 *The optimal relay amplification matrix $\mathbf{D}$ to maximize any weighted sum of users' rates $\mu_1 R_2 + \mu_2 R_2$ with $\mu_1, \mu_2 \geq 0$, has the following form:* $\mathbf{D}(\theta) =$

$$\gamma\mathbf{G}\left(P_1\mathbf{F}^{[1]}\sin\theta + P_2\mathbf{F}^{[2]}\cos\theta\right)\left(I + P_1\mathbf{F}^{[1]2} + P_2\mathbf{F}^{[2]2} + P_R\mathbf{G}^2\right)^{-1}$$

and γ is a constant whose value is easily calculated from the relay power constraint.

The proof of Theorem 1 is presented in [15]. Note that the optimization space over $\mathbf{D}$ is now only one dimensional, as opposed to the original R dimensional space. To identify a point on the boundary of the capacity region one only needs the corresponding θ. Therefore, the angle θ in Theorem 1 plays a very important role. As we will establish in the following Theorems, $|\theta|$ going from 0 to $\pi/2$ describes the boundary of the capacity region, with $\theta = 0$ corresponding to the point where user 2 achieves his maximum rate, θ^{11} (defined in [15]) corresponds to the points where the sum rate is maximized, and $|\theta| = \pi/2$ corresponds to the point where user 1 achieves his maximum rate. This is also depicted in Fig 2.

Next we characterize the capacity region explicitly through a system of equations whose solution is the rate pair (R_1, R_2) that maximizes $\mu_1 R_1 + \mu_2 R_2$ for any positive $\mu_1 + \mu_2$.

B Optimal Rate Pair to maximize $\mu_1 R_1 + \mu_2 R_2$

Theorem 2 *For any μ_1, μ_2 with $\mu_1 \geq \mu_2$, the weighted sum rate $\mu_1 R_1 + \mu_2 R_2$ for the parallel AF relay MAC$\left(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R\right)$ is maximized by the rate pair $(R_1^{\mu_1,\mu_2}, R_2^{\mu_1,\mu_2})$ given by*

$$R_1^{\mu_1,\mu_2} = \log(1 + SNR_1^\star), \; R_2^{\mu_1,\mu_2} = \log\left(1 + \frac{SNR_2^\star}{1 + SNR_1^\star}\right) \quad (10)$$

where $SNR_1^\star$ and $SNR_2^\star$ are the solutions to the following simultaneous equations:

$$\mu_1 = \frac{\mu_2}{SNR_1^\star + SNR_2^\star}(1 + P_R P_1 A_{11} + P_R P_2 A_{12}/\alpha) + \frac{\mu_1 - \mu_2}{1 + SNR_1^\star}(1 + P_R P_1 A_{11})$$

$$\mu_1 = \frac{\mu_2}{SNR_1^\star + SNR_2^\star}(1 + P_R P_1 A_{12}\alpha + P_R P_2 A_{22}) + \frac{\mu_1 - \mu_2}{1 + SNR_1^\star}(1 + P_R P_1 A_{12}\alpha)$$

$$\alpha = \sqrt{\frac{P_1 SNR_1^\star}{P_2 SNR_2^\star}}$$

$$A_{11} = \sum_{k=1}^{R} \frac{g_k^2 f_k^{[1]2}}{1 + P_1 f_k^{[1]2} + P_2 f_k^{[2]2} + P_R g_k^2}$$

$$A_{22} = \sum_{k=1}^{R} \frac{g_k^2 f_k^{[2]2}}{1 + P_1 f_k^{[1]2} + P_2 f_k^{[2]2} + P_R g_k^2}$$

$$A_{12} = \sum_{k=1}^{R} \frac{g_k^2 f_k^{[1]} f_k^{[2]}}{1 + P_1 f_k^{[1]2} + P_2 f_k^{[2]2} + P_R g_k^2}$$

Proof of Theorem 2 is provided in [15] where we also compute closed form solutions for the special cases of $\mu_1 = 1, \mu_2 = 0$ (individual capacity) and $\mu_1 = \mu_2 = 1$ (sum capacity) as discussed above. Recall that in the conventional Gaussian MAC (i.e. without AF relays) the maximum rate that user 1 can achieve is the same as his channel capacity *as if* user 2 is not transmitting.

Fig. 2 shows the typical shape of the capacity region. Optimal relay amplification factor is indicated on the figure in terms of the parameter $|\theta|$. θ is equal to zero between points A and B, it changes from 0 to θ^{11} in the curved portion from points B to C. θ is constant at θ^{11} between points C to D. θ changes from θ^{11} to sgn$(\theta^{11})\pi/2$ as we traverse the boundary from E, and it is again constant at $\theta = \pi/2$ from the point E to point F. The coordinates of the points A, B, C, D, E, F are all known in closed form provided in the full paper [15].

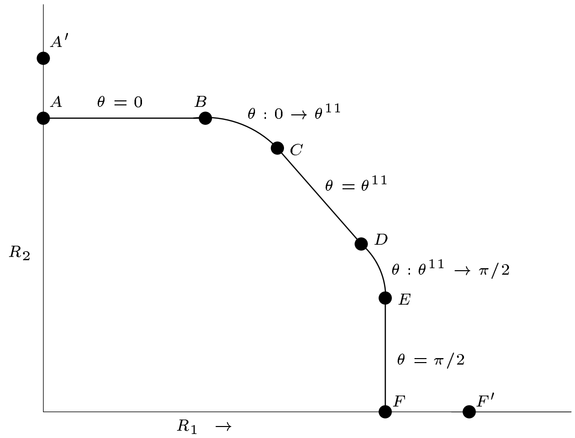

Figure 2: Capacity region of two user AF relay $\text{MAC}(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \{\mathbf{D}\}, P_R)$

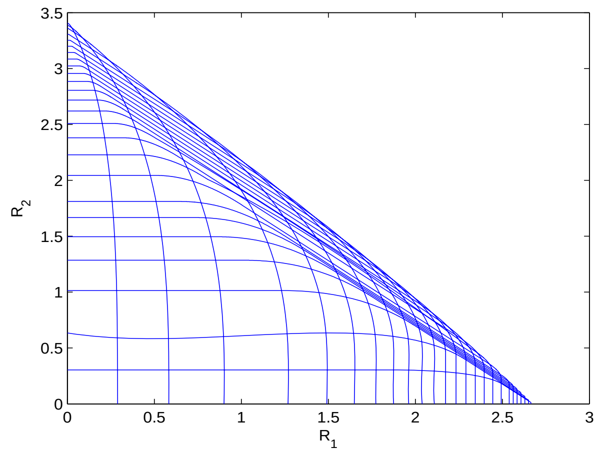

Figure 3: BC capacity region as the union of MAC capacity regions

IV DUALITY RELATIONSHIPS IN PARALLEL AF RELAY NETWORKS

A Duality of Parallel AF Relay MAC and BC

The following two theorems establish the duality relationship between the parallel AF relay MAC and BC.

Theorem 3 *Given any relay amplification matrix* $\mathbf{D}$ *that satisfies the power constraint on the parallel AF relay multiple access channel* $MAC(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \mathbf{D}, P_R)$, *there exists a dual parallel AF relay broadcast channel* $BC(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \kappa\mathbf{D}, P_1 + P_2)$ *such that any rate pair* (R_1, R_2) *that can be achieved on* $MAC(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \mathbf{D}, P_R)$ *can also be achieved on* $BC(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \kappa\{\mathbf{D}\}, P)$. κ *is chosen to satisfy the relay sum power constraint on* $BC(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \kappa\mathbf{D}, P_1 + P_2)$.

Theorem 4 *Given any relay amplification matrix* $\mathbf{D}$ *that satisfies the power constraint on the parallel AF relay broadcast channel* $BC(\mathbf{G}, P_R, \mathbf{F}^{[1]}, \mathbf{F}^{[2]}, \mathbf{D}, P)$ *and given a rate pair* (R_1, R_2) *that is achievable on this parallel AF relay broadcast channel, there exist* $P_1, P_2 \geq 0$ *such that* $P_1 + P_2 = P$ *and a dual multiple access channel* $MAC(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \kappa\mathbf{D}, P_R)$ *such that the rate pair* R_1, R_2 *is achievable on* $MAC(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \kappa\mathbf{D}, P_R)$. κ *is chosen to satisfy the relay sum power constraint on* $MAC(\mathbf{F}^{[1]}, P_1, \mathbf{F}^{[2]}, P_2, \mathbf{G}, \mathbf{D}, P_R)$.

The proof of Theorems 3 and 4 is presented in [15].

Note that the duality relationship is a strong duality in the sense that the parallel AF relay MAC and BC are duals not only for the optimal relay amplification matrix $\mathbf{D}$ but also for *any* feasible $\mathbf{D}$.

B Capacity Region of the Parallel AF Relay BC

The duality relationship described in the previous section provides a method to compute the capacity region of the parallel AF broadcast channel. As shown in Fig. 3 the BC capacity region may be found simply as the union of the MAC capacity regions.

The AF relay MAC-BC duality shown in section A is not limited to single antenna relays or two hop networks with parallel relays. Generalized versions of this duality are derived in the full paper [15]. The same duality relationship holds when some of the relays have multiple antennas. The duality also holds when more than 2 hops are considered. Thus, the MAC-BC duality holds for AF relay networks whether they are purely parallel (two hop), purely serial (multiple hops with a single relay at each hop), or several parallel AF relay clusters connected in series to form a multihop relay network. It holds whether the relays are distributed with a single antenna at each relay or they are able to cooperate as as a multiple antenna node.

V CONCLUSION

We explored the capacity and duality aspects of AF relay networks. The MAC-BC duality known for conventional one hop Gaussian channel was found to be applicable to multiple hop communication over AF relay networks where some of the relays may have multiple antennas. A unique aspect of the AF relay MAC-BC duality is that the powers of the transmitter and the relays are switched in the dual network. With distributed single antenna relay nodes we determined the optimal relay scaling factors for the entire capacity region of the relay multiple access channel. Closed form expressions were found for the sum rate and individual maximum rates while simultaneous equations were found that can be solved to determine any rate pair on the boundary of the relay MAC. The capacity region of the relay BC was evaluated using duality as the union of the relay MAC capacity regions over different power splits between user 1 and user 2 while keeping the total power constant and equal to the total relay transmit power on the BC.

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