Interference Alignment and Spatial Degrees of Freedom for the K User Interference Channel

Viveck R. Cadambe, Syed A. Jafar
Electrical Engineering and Computer Science
University of California Irvine,
Irvine, California, 92697, USA
Email: vcadambe@uci.edu, syed@uci.edu

***Abstract*— We provide tight inner and outerbounds to characterize the degrees of freedom region of the K user wireless interference channel where the channel coefficients take distinct values across frequency slots but are fixed in time[1]. We show that this interference channel almost surely has $K/2$ spatial degrees of freedom per orthogonal time and frequency dimension *without any cooperation* in the form of data sharing between transmitters or between receivers. Inspite of the interference, each user is simultaneously able to achieve $(1/2-\epsilon)$ degrees of freedom, for any $\epsilon > 0$. The conventional wisdom prior to this work has been that wireless interference networks have only one degree of freedom. The results of this work indicate that the capacity of wireless networks has been grossly underestimated. For example, we find that at high SNR the capacity of the K user interference channel is $50\%, 900\%$ and 4900% higher than prior belief for $K = 3, 20$ and 100 users, respectively. We also show that the linear schemes of interference alignment and zero forcing with single user decoding suffice to achieve all the degrees of freedom of wireless interference networks.**

I. Introduction

The capacity of ad-hoc wireless networks is the much sought afer “holy-grail” of network information theory [1]. While capacity characterizations have been found for centralized networks (Gaussian multiple access and broadcast networks with multiple antennas), similar capacity characterizations for most distributed communication scenarios (e.g. interference networks) remain long standing open problems. In the absence of precise capacity characterizations, researchers have pursued asymptotic and/or approximate capacity characterizations. Recent work has found the asymptotic scaling laws of network capacity as the number of nodes increases in a large network [2], [3]. However, very little is known about the capacity region of smaller (finite) decentralized networks. An important step in this direction is the recent approximate characterization of the capacity region of the 2 user interference channel that is accurate within one bit of the true capacity region [4]. Approximate characterizations of capacity regions would also be invaluable for most open problems in network information theory and may be the key to improving our understanding of wireless networks.

It can be argued that the most preliminary form of capacity characterization for a network is to characterize its degrees of freedom [5]–[11]. The degrees of freedom represent the rate of growth of network capacity with the $\log$ of the signal to noise ratio (SNR). In most cases, the spatial degrees of freedom turn out to be the number of non-intefering paths that can be created in a wireless network through signal processing at the transmitters and receivers. While time, frequency and space all offer degrees of freedom in the form of orthogonal dimensions over which communication can take place, spatial degrees of freedom are especially interesting in a distributed network. Potentially a wireless network may have as many spatial dimensions as the number of transmitting and receiving antennas. However, the ability to access and resolve spatial dimensions is limited by the distributed nature of the network. Therefore, characterizing the degrees of freedom for distributed wireless networks is by itself a non-trivial problem. For example, consider an interference network with K single-antenna transmitters and K single-antenna receivers where each transmitter has a message for its corresponding receiver. For $K = 2$ it is known that this interference network has only 1 degree of freedom [6], [8]. There are no known results to show that more than 1 degrees of freedom are achievable on the interference channel with any number of users. Some of the fundamental early results on the degrees of freedom of wireless interference networks were obtained by Host-Madsen and Nosratinia in [9]. It is conjectured in [9] that the K user interference channel with constant channel coefficients (i.e. when channel coefficients do not change with time or frequency slot) has only 1 degree of freedom[2]. Yet, the best known outerbound for the number of degrees of freedom with K interfering nodes is $K/2$, also presented in [9].

[1] All the results in this work are equally applicable to time-varying channels with only one frequency slot.

[2] We refer to this conjecture as the Host-Madsen-Nosratinia conjecture in this paper.

The unresolved gap between the inner and outerbounds highlights our lack of understanding of the capacity of wireless networks because even the number of degrees of freedom, which is the most basic characterization of the network capacity, remains an open problem. It is this open problem that we pursue in this paper.

A. Overview of Results

The main result of this work may be summarized in general terms as follows:

"*Regardless of how many speakers and listeners are located within earshot of each other, each speaker can speak half the time and be heard without any interference by its intended listener*"

This result may seem impossible at first. For example, how can a total duration of 1 hour be shared by 100 speakers such that each speaker speaks for 30 minutes and is heard interference free by its intended listener when all systems are located within earshot of each other? And yet, this seemingly impossible result is made possible by the concept of interference alignment as explained through a toy example in Section I-B where we assume the speakers and listeners can choose their locations as needed.

We show that the K user interference channel with single antennas at all nodes has (almost surely) a total of $K/2$ degrees of freedom per orthogonal time and frequency dimension when the channels are drawn randomly from a continuous distribution. The implications of this result for our understanding of the capacity of wireless networks are quite profound. While we do not settle the Host-Madsen-Nosratinia conjecture (because we assume frequency selective channels) we do show that this conjecture is not representative of the capacity of wireless networks where the channel fading is typically time and/or frequency selective. Moreover, since conventional wisdom for wireless networks has been consistent with the Host-Madsen-Nosratinia conjecture, it is clear that the capacity of wireless networks has been grossly underestimated. For example, our result in this paper shows that at high SNR the true capacity is higher by 50%, 900%, and 4900% than previously believed for $K = 3$, 20, and 100 interfering users, respectively.

Interference is one of the principal challenges faced by wireless networks. However, we show that with perfect channel knowledge the frequency selective interference channel is not interference limited. In fact, after the first two users, additional users do not compete for degrees of freedom and each additional user is able to achieve $1/2$ degree of freedom without reducing the degrees of freedom available to previously existing users.

It is worthwhile to place our result in perspective with the well known result of [3]. In [3], Ozgur, Leveque and Tse show that the capacity of wireless interference networks increases linearly in the number of users. While the linear dependence on the number of users may suggest a close relationship with our result, the two results address completely different regimes. First, [3] is asymptotic in the number of users $K \to \infty$ but deals with finite SNR. Our result on the other hand deals with a finite number of users (e.g. $K = 3$) but is asymptotic in SNR. Thus, [3] does not address the capacity of a relatively small wireless network and our result does not describe the low SNR capacity of an interference network. Second, the result of [3] is based on a hierarchical cooperative scheme where transmitters and receivers are able to achieve MIMO behavior by sharing message information among themselves. On the other hand, our model does not rely on any cooperation in the form of sharing of messages because of the assumption that the transmitters never receive signals and the receivers never transmit. The fact that we achieve $K/2$ degrees of freedom without message sharing between transmitters and receivers is quite remarkable. Note that it has been shown previously for the 2 user interference channel that unidirectional message sharing (e.g. from transmitter 1 to transmitter 2) does not allow higher degrees of freedom [10], [12] and even bi-directional message sharing (through full duplex noisy channels between the transmitters and full duplex noisy channels between the receivers) will not increase the degrees of freedom if the cost of message sharing is considered [9], [13]. Therefore it is quite surprising that the K user interference channel has $K/2$ degrees of freedom even without any message sharing.

B. Interference Alignment

Interference alignment is a powerful scheme that has emerged out of recent work in [12], [14]–[18]. Interference alignment refers to the simple idea that signal vectors can be aligned in such a manner that they cast overlapping shadows at the receivers where they constitute interference while they continue to be distinct at the receivers where they are desired. Interference alignment was first discovered in the context of the MIMO X channel. In their seminal work [14], [15] on the MIMO X channel Maddah-Ali, Motahari and Khandani propose an elegant coding scheme (the MMK scheme) based on dirty paper coding and successive decoding that takes advantage of the multiple access and broadcast channels contained in the X channel. Interference is implicitly aligned in the MMK scheme by iteratively optimizing the transmit precoding and receive combining vectors. The first explicit interference alignment scheme was presented in [16]. [16] also showed that interference alignment is optimal for degrees of freedom, i.e., dirty paper coding or successive decoding are not needed. Interference alignment was subsequently used in [12], [17] to show achievability of all points within the degrees of freedom region of the MIMO X channel. Interference alignment was also independently discovered in the context of the compound broadcast channel in [18].

We introduce interference alignment through the following toy example. In order to create the simplest example possible, we deviate a little from our system model to allow propagation delays in the toy example. The rest of the paper assumes no propagation delays which is the more conventional information theoretic model of the interference channel.

1) Motivating example - Can everyone speak half the time with no interference?: Consider the K user interference channel where there is a propagation delay from each transmitter to each receiver. Let T_{ij} represent the signal propagation delay from transmitter i to receiver j. Suppose the locations of the transmitters and receivers can be configured such that the delay T_{ii} from each transmitter to its intended receiver is an even multiple of a basic symbol duration T_s, while the signal propagation delays $T_{ij}, (i \neq j)$ from each transmitter to all unintended receivers are odd multiples of the symbol duration. The communication strategy is the following. All transmissions occur simultaneously at even symbol durations. Note that with this policy, each receiver sees its own transmitter's signal interference-free over even time periods, while it sees all interfering signals simultaneously over odd time periods. Thus *each speaker is able to talk half the time and be heard interference-free by its desired audience*.

While interference alignment on the 2 user X channel is quite natural, it becomes increasingly non-intuitive as the number of users increases in a wireless interference network. While the number of signal vectors to be designed is proportional to the number of users K, at each receiver we need the signal vectors from the $K-1$ undesired transmitters to align. Since there are K receivers, the number of constraints is of the order of K^2. The problem appears to be overly constrained, which would typically preclude the existence of a solution. Indeed we can show that exact interference alignment is not possible for even $K = 3$ single antenna users (see full paper [19] for the context and details of this argument). Given the infeasibility of perfect interference alignment, it is clear that interference alignment on the interference network is not a simple extension of the 2 user X channel result. In this paper we circumvent this problem by adopting a novel approach of *partial* interference alignment where we do not seek to exactly align all interference terms but allow partial overlaps with the goal of restricting the dimensionality of the space spanned by interference terms to be as small as possible. It turns out that while this imperfect alignment cannot exactly achieve $K/2$ degrees of freedom, it is capable of achieving arbitrarily close to $K/2$ degrees of freedom. With the degrees of freedom defined as a limit superior we say that the network has $K/2$ degrees of freedom.

II. System Model

Consider the K user interference channel, comprised of K transmitters and K receivers. We assume coding may occur over multiple orthogonal frequency and time dimensions and the rates as well as the degrees of freedom are normalized by the number of orthogonal time and frequency dimensions. Each node is equipped with only one antenna. The channel output at the k^{th} receiver over the f^{th} frequency slot and the t^{th} time slot is described as follows:

$$Y^{[k]}(f,t)=H^{[k1]}(f)X^{[1]}(f,t)+\cdots+H^{[kK]}(f)X^{[K]}(f,t)+Z^{[k]}(f,t)$$

where, $k \in \{1, 2, \cdots, K\}$ is the user index, $f \in \mathbb{N}$ is the frequency slot index, $t \in \mathbb{N}$ is the time slot index, $Y^{[k]}(f,t)$ is the output signal of the k^{th} receiver, $X^{[k]}(f,t)$ is the input signal of the k^{th} transmitter, $H^{[kj]}(f)$ is the channel fade coefficient from transmitter j to receiver k over the f^{th} frequency slot and $Z^{[k]}(f,t)$ is the additive white Gaussian noise (AWGN) term at the k^{th} receiver. The channel coefficients vary across frequency slots but are assumed constant in time. We assume all noise terms are i.i.d. (independent identically distributed) zero mean complex Gaussian with unit variance. We assume all channel coefficients $H^{[kj]}(f)$ are known a-priori to all transmitters and receivers. To avoid degenerate channel conditions (e.g. all channel coefficients are equal or channel coefficients are equal to either zero or infinity) we assume that the channel coefficient values are drawn i.i.d. from a continuous distribution and the absolute value of all the channel coefficients is bounded between a non-zero minimum value and a finite maximum value. Since the channel values are assumed constant in time, the time index t is sometimes suppressed for compact notation.

We assume that transmitters $1, 2, \cdots, K$ have independent messages $W_1, W_2, \cdots, W_K$ intended for receivers $1, 2, \cdots, K$, respectively. The total power across all transmitters is assumed to be equal to ρ per orthogonal time and frequency dimension. We indicate the size of the message set by $|W_i(\rho)|$. For codewords spanning $f_0 \times t_0$ channel uses (i.e. using f_0 frequency slots and t_0 time slots), the rates $R_i(\rho) = \frac{\log|W_i(\rho)|}{f_0 t_0}$ are achievable if the probability of error for all messages can be simultaneously made arbitrarily small by choosing an appropriately large $f_0 t_0$.

The capacity region $\mathcal{C}(\rho)$ of the three user interference channel is the set of all *achievable* rate tuples $\mathbf{R}(\rho) = (R_1(\rho), R_2(\rho), \cdots, R_K(\rho))$.

A. Degrees of Freedom

Similar to the degrees of freedom region definition for the MIMO X channel in [12] we define the degrees of freedom region $\mathcal{D}$ for the K user interference channel as follows:

$$\mathcal{D} = \Bigg\{(d_1, \cdots, d_K) \in \mathbb{R}_+^K : \forall (w_1, \cdots, w_K) \in \mathbb{R}_+^K$$
$$w_1 d_1 + w_2 d_2 + \cdots + w_K d_K \leq \tag{1}$$
$$\limsup_{\rho\to\infty}\left[\sup_{\mathbf{R}(\rho)\in\mathcal{C}(\rho)} \frac{[w_1 R_1(\rho) + \cdots + w_K R_K(\rho)]}{\log(\rho)}\right]\Bigg\}$$

III. Degrees of Freedom for the K User Interference Channel

The following theorem presents our main result.

Theorem 1: The number of degrees of freedom for the K user interference channel with single antennas at all nodes is $K/2$.

$$\max_{\mathbf{d}\in\mathcal{D}} d_1 + d_2 + \cdots + d_K \quad = \quad K/2 \tag{2}$$

The converse argument for the theorem follows directly from the outerbound for the K user interference channel presented in [9]. The achievability proof is presented next. For simplicity of exposition, we present here the constructive proof for $K = 3$. The proof for general $K \geq 3$ is provided in the full paper [19].

A. Achievability Proof for Theorem 1 with $K = 3$

We show that $(d_1, d_2, d_3) = (\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ lies in the degrees of freedom region $\forall n \in \mathbb{N}$. Since the degrees of freedom region is closed, this automatically implies that

$$\max_{(d_1,d_2,d_3)\in\mathcal{D}} d_1 + d_2 + d_3 \geq \sup_n \frac{3n+1}{2n+1} = \frac{3}{2}$$

This result, in conjunction with the converse argument proves the theorem[3].

To show that $(\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ lies in $\mathcal{D}$, we construct an interference alignment scheme using only $2n+1$ frequency slots. We collectively denote the $2n+1$ symbols transmitted over the first $2n+1$ frequency slots at each time instant as a supersymbol. We call this the $(2n+1)$ symbol extension of the channel. With the extended channel, the signal vector at the k^{th} user's receiver can be expressed as

$$\bar{\mathbf{Y}}^{[k]} = \bar{\mathbf{H}}^{[k1]}\bar{\mathbf{X}}^{[1]} + \bar{\mathbf{H}}^{[k2]}\bar{\mathbf{X}}^{[2]} + \bar{\mathbf{H}}^{[k3]}\bar{\mathbf{X}}^{[3]} + \bar{\mathbf{Z}}^{[k]}, \;\; k = 1, 2, 3.$$

where $\bar{\mathbf{X}}^{[k]}$ is a $(2n+1)\times 1$ column vector representing the $2n+1$ symbol extension of the transmitted symbol $X^{[k]}$, i.e

$$\bar{\mathbf{X}}^{[k]}(t) \triangleq \begin{bmatrix} X^{[k]}(1,t) \\ X^{[k]}(2,t) \\ \vdots \\ X^{[k]}(2n+1,t) \end{bmatrix}$$

Similarly $\bar{\mathbf{Y}}^{[k]}$ and $\bar{\mathbf{Z}}^{[k]}$ represent $2n+1$ symbol extensions of the $Y^{[k]}$ and $Z^{[k]}$ respectively. $\bar{\mathbf{H}}^{[kj]}$ is a diagonal $(2n+1)\times(2n+1)$ matrix representing the $2n+1$ symbol extension of the channel i.e

$$\bar{\mathbf{H}}^{[kj]} \triangleq \begin{bmatrix} H^{[kj]}(1) & 0 & \cdots & 0 \\ 0 & H^{[kj]}(2) & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & H^{[kj]}(2n+1) \end{bmatrix}$$

Recall that we assume that the channel coefficient values for each frequency slot are chosen independently from a continuous distribution. Thus, all the diagonal channel matrices $\bar{\mathbf{H}}^{[kj]}$ are comprised of all distinct diagonal elements with probability 1.

We show that $(d_1, d_2, d_3) = (n+1, n, n)$ is achievable on this extended channel implying that $(\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ lies in the degrees of freedom region of the original channel.

In the extended channel, message W_1 is encoded at transmitter 1 into $n+1$ independent streams $x_m^{[1]}(t), m = 1, 2, \ldots, (n+1)$ sent along vectors $\mathbf{v}_m^{[1]}$ so that $\bar{\mathbf{X}}^{[1]}(t)$ is

$$\bar{\mathbf{X}}^{[1]}(t) = \sum_{m=1}^{n+1} x_m^{[1]}(t)\mathbf{v}_m^{[1]} = \bar{\mathbf{V}}^{[1]}\mathbf{X}^{[1]}(t)$$

where $\mathbf{X}^{[1]}(t)$ is a $(n+1)\times 1$ column vector and $\bar{\mathbf{V}}^{[1]}$ is a $(2n+1)\times(n+1)$ dimensional matrix. Similarly W_2 and W_3 are each encoded into n independent streams by transmitters 2 and 3 as $\mathbf{X}^{[2]}(t)$ and $\mathbf{X}^{[3]}(t)$ respectively.

$$\bar{\mathbf{X}}^{[2]}(t) = \sum_{m=1}^{n} x_m^{[2]}(t)\mathbf{v}_m^{[2]} = \bar{\mathbf{V}}^{[2]}\mathbf{X}^{[2]}(t)$$

$$\bar{\mathbf{X}}^{[3]}(t) = \sum_{m=1}^{n} x_m^{[3]}(t)\mathbf{v}_m^{[3]} = \bar{\mathbf{V}}^{[3]}\mathbf{X}^{[3]}(t)$$

The received signal at the i^{th} receiver is

$$\bar{\mathbf{Y}}^{[i]}(t) = \bar{\mathbf{H}}^{[i1]}\bar{\mathbf{V}}^{[1]}\mathbf{X}^{[1]}(t) + \bar{\mathbf{H}}^{[i2]}\bar{\mathbf{V}}^{[2]}\mathbf{X}^{[2]}(t) + \bar{\mathbf{H}}^{[i3]}\bar{\mathbf{V}}^{[3]}\mathbf{X}^{[3]}(t) + \bar{\mathbf{Z}}^{[i]}(t)$$

[3]Note that for any $\epsilon > 0$ we can choose an appropriately large n so that each user is able to achieve more than $1/2-\epsilon$ degrees of freedom.

In this achievable scheme, receiver i eliminates interference by zero-forcing all $\bar{\mathbf{V}}^{[j]}, j \neq i$ to decode W_i. At receiver 1, $n+1$ desired streams are decoded after zero-forcing the interference to achieve $n+1$ degrees of freedom. To obtain $n+1$ interference free dimensions from a $2n+1$ dimensional received signal vector $\bar{\mathbf{Y}}^{[1]}(t)$, the dimension of the interference should be not more than n. This can be ensured by perfectly aligning the interference from transmitters 2 and 3 as follows.

$$\bar{\mathbf{H}}^{[12]}\bar{\mathbf{V}}^{[2]} = \bar{\mathbf{H}}^{[13]}\bar{\mathbf{V}}^{[3]} \tag{3}$$

At the same time, receiver 2 zero-forces the interference from $\bar{\mathbf{X}}^{[1]}$ and $\bar{\mathbf{X}}^{[3]}$. To extract n interference-free dimensions from a $2n+1$ dimensional vector, the dimension of the interference has to be not more than $n+1$. i.e.

$$\text{rank}\left(\left[\bar{\mathbf{H}}^{[21]}\bar{\mathbf{V}}^{[1]} \quad \bar{\mathbf{H}}^{[23]}\bar{\mathbf{V}}^{[3]}\right]\right) \leq n+1$$

This can be achieved by choosing $\bar{\mathbf{V}}^{[3]}$ and $\bar{\mathbf{V}}^{[1]}$ so that

$$\bar{\mathbf{H}}^{[23]}\bar{\mathbf{V}}^{[3]} \prec \bar{\mathbf{H}}^{[21]}\bar{\mathbf{V}}^{[1]} \tag{4}$$

where $\mathbf{P} \prec \mathbf{Q}$, means that the set of column vectors of matrix $\mathbf{P}$ is a subset of the set of column vectors of matrix $\mathbf{Q}$. Similarly, to decode W_3 at receiver 3, we wish to choose $\bar{\mathbf{V}}^{[2]}$ and $\bar{\mathbf{V}}^{[1]}$ so that

$$\bar{\mathbf{H}}^{[32]}\bar{\mathbf{V}}^{[2]} \prec \bar{\mathbf{H}}^{[31]}\bar{\mathbf{V}}^{[1]} \tag{5}$$

Thus, we wish to pick vectors $\bar{\mathbf{V}}^{[1]}$, $\bar{\mathbf{V}}^{[2]}$ and $\bar{\mathbf{V}}^{[3]}$ so that equations (3), (4), (5) are satisfied. Note that the channel matrices $\bar{\mathbf{H}}^{[ij]}$ have a full rank of $2n+1$ almost surely. Since multiplying by a full rank matrix (or its inverse) does not affect the conditions represented by equations (3), (4) and (5), they can be equivalently expressed as

$$\begin{aligned} \mathbf{B} &= \mathbf{TC} && (6)\\ \mathbf{B} &\prec \mathbf{A} && (7)\\ \mathbf{C} &\prec \mathbf{A} && (8) \end{aligned}$$

where

$$\begin{aligned} \mathbf{A} &= \bar{\mathbf{V}}^{[1]} && (9)\\ \mathbf{B} &= (\bar{\mathbf{H}}^{[21]})^{-1}\bar{\mathbf{H}}^{[23]}\bar{\mathbf{V}}^{[3]} && (10)\\ \mathbf{C} &= (\bar{\mathbf{H}}^{[31]})^{-1}\bar{\mathbf{H}}^{[32]}\bar{\mathbf{V}}^{[2]} && (11)\\ \mathbf{T} &= \bar{\mathbf{H}}^{[12]}(\bar{\mathbf{H}}^{[21]})^{-1}\bar{\mathbf{H}}^{[23]}(\bar{\mathbf{H}}^{[32]})^{-1}\bar{\mathbf{H}}^{[31]}(\bar{\mathbf{H}}^{[13]})^{-1} && (12) \end{aligned}$$

Note that $\mathbf{A}$ is a $(2n+1)\times(n+1)$ matrix. $\mathbf{B}$ and $\mathbf{C}$ are $(2n+1)\times n$ matrices. Since all channel matrices are invertible, we can choose $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ so that they satisfy equations (6)-(8) and then use equations (9)-(12) to find $\bar{\mathbf{V}}^{[1]}$,$\bar{\mathbf{V}}^{[2]}$ and $\bar{\mathbf{V}}^{[3]}$. $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are picked as follows. Let $\mathbf{w}$ be the $(2n+1)\times 1$ column vector

$$\mathbf{w} = \begin{bmatrix} 1\\ 1\\ \vdots\\ 1 \end{bmatrix}$$

We now choose $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as:

$$\begin{aligned} \mathbf{A} &= [\mathbf{w} \ \ \mathbf{Tw} \ \ \mathbf{T}^2\mathbf{w} \ \ \ldots \ \ \mathbf{T}^n\mathbf{w}]\\ \mathbf{B} &= [\mathbf{Tw} \ \ \mathbf{T}^2\mathbf{w} \ \ \ldots \ \ \mathbf{T}^n\mathbf{w}]\\ \mathbf{C} &= [\mathbf{w} \ \ \mathbf{Tw} \ \ \ldots \ \ \mathbf{T}^{n-1}\mathbf{w}] \end{aligned}$$

It can be easily verified that $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ satisfy the three equations (6)-(8). Therefore, $\bar{\mathbf{V}}^{[1]}$, $\bar{\mathbf{V}}^{[2]}$ and $\bar{\mathbf{V}}^{[3]}$ satisfy the interference alignment equations in (3), (4) and (5).

Now, consider the received signal vectors at Receiver 1. The desired signal arrives along the $n+1$ vectors $\bar{\mathbf{H}}^{[11]}\bar{\mathbf{V}}^{[1]}$ while the interference arrives along the n vectors $\bar{\mathbf{H}}^{[12]}\bar{\mathbf{V}}^{[2]}$ and the n vectors $\bar{\mathbf{H}}^{[13]}\bar{\mathbf{V}}^{[3]}$. As enforced by equation (3) the interference vectors are perfectly aligned. Therefore, in order to prove that there are $n+1$ interference free dimensions it suffices to show that the columns of the square, $(2n+1)\times(2n+1)$ dimensional matrix

$$\left[\bar{\mathbf{H}}^{[11]}\bar{\mathbf{V}}^{[1]} \quad \bar{\mathbf{H}}^{[12]}\bar{\mathbf{V}}^{[2]}\right] \tag{13}$$

are linearly independent almost surely. Multiplying by the full rank matrix $(\bar{\mathbf{H}}^{[11]})^{-1}$ and substituting the values of $\bar{\mathbf{V}}^{[1]}, \bar{\mathbf{V}}^{[2]}$, equivalently we need to show that almost surely

$$\mathbf{S} \stackrel{\triangle}{=} \left[\mathbf{w} \ \ \mathbf{Tw} \ \ \ldots \ \ \mathbf{T}^n\mathbf{w} \ \ \mathbf{Dw} \ \ \mathbf{DTw} \ \ \ldots \ \ \mathbf{DT}^{n-1}\mathbf{w}\right]$$

has linearly independent column vectors where $\mathbf{D} = (\bar{\mathbf{H}}^{[11]})^{-1}\bar{\mathbf{H}}^{[12]}$ is a diagonal matrix. In other words, we need to show $\det(\mathbf{S}) \neq 0$ with probability 1. The proof is obtained by contradiction. If possible, let $\mathbf{S}$ be singular with non-zero probability. i.e, $\Pr(|\mathbf{S}| = 0) > 0$. Further, let the diagonal entries of $\mathbf{T}$ be $\lambda_1, \lambda_2, \ldots \lambda_{2n+1}$ and the diagonal entries of $\mathbf{D}$ be $\kappa_1, \kappa_2 \ldots \kappa_{2n+1}$. Then the following equation is true with non-zero probability.

$$|\mathbf{S}| = 0.$$

But $|\mathbf{S}|$ is a polynomial in $\lambda_1, \lambda_2, \ldots \lambda_{2n+1}$, $\kappa_1, \kappa_2 \ldots \kappa_{2n+1}$ where each variable has a continuous distribution conditioned on all the rest. For this polynomial to be identically equal to zero with a probability greater than zero, all the coefficients must be equal to zero. It is easily verified that this is not the case. The details of the proof are provided in [19].

Thus, the $n+1$ vectors carrying the desired signal at receiver 1 are linearly independent of the n interference vectors which allows the receiver to zero force interference and obtain $n+1$ interference free dimensions, and therefore $n+1$ degrees of freedom for its message.

At receiver 2 the desired signal arrives along the n vectors $\bar{\mathbf{H}}^{[22]}\bar{\mathbf{V}}^{[2]}$ while the interference arrives along the $n+1$ vectors $\bar{\mathbf{H}}^{[21]}\bar{\mathbf{V}}^{[1]}$ and the n vectors $\bar{\mathbf{H}}^{[23]}\bar{\mathbf{V}}^{[3]}$. As enforced by equation (4) the interference vectors $\bar{\mathbf{H}}^{[23]}\bar{\mathbf{V}}^{[3]}$ are perfectly aligned within the interference vectors $\bar{\mathbf{H}}^{[21]}\bar{\mathbf{V}}^{[1]}$. Therefore, in order to prove that there are n interference free dimensions at receiver 2 it suffices to show that the columns of the square, $(2n+1)\times(2n+1)$ dimensional matrix

$$\left[\bar{\mathbf{H}}^{[22]}\bar{\mathbf{V}}^{[2]} \quad \bar{\mathbf{H}}^{[21]}\bar{\mathbf{V}}^{[1]}\right] \tag{14}$$

are linearly independent almost surely. This proof is identical to the proof for receiver 1. Using the same arguments we can show that both receivers 2 and 3 are able to zero force the $n+1$ interference vectors and obtain n interference free dimensions for their respective desired signals so that they each achieve n degrees of freedom.

Thus we established the achievability of $d_1+d_2+d_3 = \frac{3n+1}{2n+1}$ for any n. This scheme, along with the converse automatically imply that

$$\sup_{(d_1,d_2,d_3)\in\mathcal{D}} d_1+d_2+d_3 = \frac{3}{2}$$

B. The Degrees of Freedom Region for the 3 User Interference Channel

Theorem 2: The degrees of freedom region of the 3 user interference channel is characterized as follows:

$$\begin{aligned}\mathcal{D} = \{(d_1,d_2,d_3) :& \\ d_1+d_2 &\le 1 \\ d_2+d_3 &\le 1 \\ d_1+d_3 &\le 1\}\end{aligned} \tag{15}$$

Proof: The converse argument is identical to the converse argument for Theorem 1 and is therefore omitted. We show achievability as follows. Let $\mathcal{D}^{'}$ be the degrees of freedom region of the 3 user interference channel. We need to prove that $\mathcal{D}^{'} = \mathcal{D}$. We show that $\mathcal{D} \subset \mathcal{D}^{'}$ which along with the converse proves the stated result.

The points $K=(0,0,1)$, $L=(0,1,0)$, $J=(1,0,0)$ can be verified to lie in $\mathcal{D}^{'}$ through trivial achievable schemes. Also, Theorem 1 implies that $N=(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ lies in $\mathcal{D}^{'}$ (Note that this is the only point which achieves a total of $\frac{3}{2}$ degrees of freedom and satisfies the inequalities in (15). Consider any point $(d_1,d_2,d_3)\in\mathcal{D}$ as defined by the statement of the theorem. The point (d_1,d_2,d_3) can then be shown to lie in a convex region

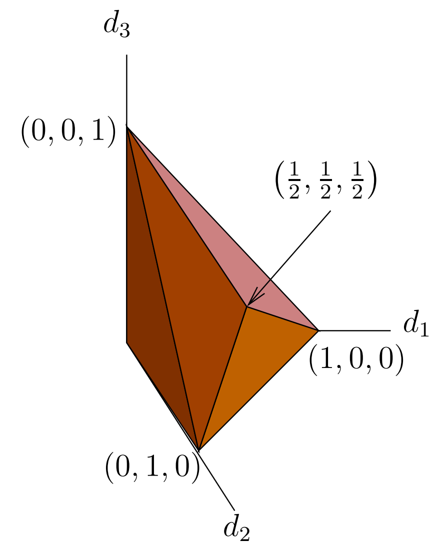

Fig. 1. Degrees of Freedom Region for the 3 user interference channel

whose corner points are $(0,0,0)$, J, K, L and N. i.e (d_1,d_2,d_3) can be expressed as a convex combination of the end points (see Fig. 1). Since convex combinations are achievable by time sharing between the end points, this implies that $\mathcal{D}\subset\mathcal{D}^{'}$ and the proof is complete. ∎

IV. Conclusion

We have shown that with perfect channel knowledge the K user interference channel has $K/2$ spatial degrees of freedom. Conventional wisdom has so far been consistent with the Host-Madsen-Nosratinia conjecture that distributed interfering systems cannot have more than 1 degree of freedom and therefore the best known outerbound $K/2$ has not been considered significant. This pessimistic outlook has for long invited researchers to try to prove that more than 1 degree of freedom is not possible while ignoring the $K/2$ outerbound. The present result shifts the focus onto the outerbound by proving that it is tight for fading channels if perfect and global channel knowledge is available. Thus, the present result could guide future research along an optimistic path in the same manner that MIMO technology has shaped our view of the capacity of a wireless channel.

Finally, there is increasing evidence that unlike the random coding based achievability schemes typically used in single user and many multiuser capacity theorems, structured codes (e.g. lattice codes) and random codes with a limited amount of structure may be necessary for network theorems in general [20]. Interference alignment over supersymbols is a clear example of the utility of structured random codewords for wireless interference networks. Further exploration of this concept will be especially useful to settle the Host-Madsen-Nosratinia conjecture. If structured codes can be designed to appropriately align interference in the codeword space (e.g. through lattice constructions) without the need for signal vector alignment over time/frequency varying channels then it may be possible to achieve more than 1 degree of freedom even over channel coefficients that do not vary in time or frequency.

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