Degrees of Freedom of Wireless X Networks

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***Abstract*— We study the degrees of freedom characterization of wireless X networks, i.e. networks of M distributed single antenna transmitters and N distributed single antenna receivers where every transmitter has an independent message to every receiver. We provide an outerbound on the capacity region of X networks within $o(\log(\mathbf{SNR}))$. If the channel co-efficients are time-varying/frequency selective, we show that the *total* number of degrees of freedom is equal to $\frac{MN}{M+N-1}$ using a coding scheme based on the idea of interference alignment.**

I. Introduction

Of late, there is increased interest in approximate and/or asymptotic capacity characterizations of wireless networks as a means to understanding their performance limits. The capacity regions of relay networks and the 2 user interference networks, which have eluded information theorists for decades, have been recently approximated to within a constant number of bits in [1], [2]. A coarser approximation to the capacity $C(\text{SNR})$ of a network is the degrees of freedom[1] [3] approximation which maybe expressed as

$$C(\text{SNR}) = d\log(\text{SNR}) + o(\log(\text{SNR}))$$

where d is the number of degrees of freedom of the network and SNR represents the signal-to-noise ratio (SNR). The accuracy of the approximation approaches 100% as the SNR grows arbitrarily large, since the $o(\log(\text{SNR}))$ term becomes negligible in comparison to $\log(\text{SNR})$. By de-emphasizing the noise level relative to signal (and interference) level, the degrees of freedom perspective addresses the issue of interference - the primary bottleneck of rates of communication in wireless networks. Note that the degrees of freedom approximation of a network is, in general, a weaker approximation than those presented in [1], [2], since the $o(\log(\text{SNR}))$ term may not necessarily be bounded by a constant.

Recently, the degrees of freedom of the 2 user MIMO X channel [4] and the K user interference networks [5] have been characterized. An interesting insight emerged from the study of the K user interference network with time-varying/frequency-selective channel gains which

[1]Also known as multiplexing-gain or capacity pre-log.

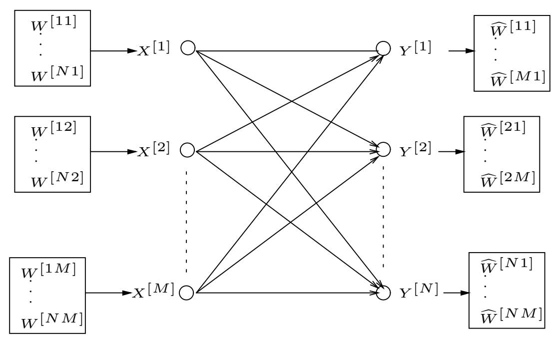


Fig. 1. The $M \times N$ X network

was shown to have $K/2$ degrees of freedom in [5]. The result implied that each user in the K user interference network can achieve a rate of $1/2\log(\text{SNR}) + o(\log(\text{SNR}))$ - half the rate achievable in the absence of interference at high SNR. The result was achieved using a coding scheme based on interference alignment - the idea that signals are constructed so that they cast overlapping shadows at the receivers where they constitute interference while they remain distinguishable at the receivers where they are desired [4], [6]–[8].

In this paper, we generalize the interference network to the X network where each source node has a message to every destination node in the network so that a K user network has K^2 messages. Since there are several more messages in the X network as compared to the interference network, the constraints of interference alignment problem are more strict. For example, at any receiver, there are $K(K-1)$ undesired signals which should overlap and K desired signals which should remain distinguishable from each other, and from the interference. The main contribution of this paper is the degrees of freedom characterization of the time-varying X channel with an achievable scheme solving the optimal interference alignment problem (over random channels). We now formally introduce the X network.

A. *The X network*

The $M \times N$ X network (Figure 1) is a single-hop communication network with M transmitters and N

receivers where each transmitter has an independent message to each of the N receivers. Thus, there are a total of MN independent messages in the system. Transmitters are not allowed to receive and receivers are not allowed to transmit so that relaying, feedback and transmit/receive cooperation are not allowed. The K *user* X *network* is an $M \times N$ X network with $M = N = K$. The $M \times N$ X network is described by input-output relations

$$Y^{[j]}(\kappa) = \sum_{i \in \{1,2\ldots M\}} H^{[ji]}(\kappa) X^{[i]}(\kappa) + Z^{[j]}(\kappa)$$

for $j \in \{1, 2, \ldots N\}$, where $\kappa \in \mathbb{N}$ represents the time index[2]. At time slot κ, $X^{[i]}(\kappa)$ is the signal transmitted by transmitter i, $Y^{[j]}(\kappa)$ is the signal received by receiver j and $Z^{[j]}(\kappa)$ represents the additive white Gaussian noise at receiver j. The noise variance at all receivers is assumed to be equal to unity. $H^{[ji]}(\kappa)$ represents the channel gain at time slot κ between transmitter i and receiver j. All nodes are assumed to have causal knowledge of all the channel gains. We assume that all channel fade coefficients are drawn from a continuous distribution whose support lies between a non-zero minimum value and a finite maximum value. Physically, this translates to the assumption that channel gains are time-varying (or frequency-selective if κ represents the frequency index).

We assume that transmitter i has message $W^{[ji]}$ for receiver j, for each $i \in \{1, 2 \ldots M\}, j \in \{1, 2 \ldots N\}$, resulting in a total of MN messages in the system. The total power across all transmitters in this system is assumed to be ρ per time/frequency slot. We denote the size of the message set by $|W^{[ji]}(\rho)|$. Let $R_{ji}(\rho) = \frac{|\log\left(W^{[ji]}(\rho)\right)|}{\kappa_0}$ denote the rate of the codeword encoding the message $W^{[ji]}$, where the codeword spans κ_0 slots. A rate-matrix $[(R_{ji}(\rho))]$ is said to be *achievable* if messages $W^{[ji]}$ can be encoded at rates $R_{ji}(\rho)$ so that the probability of error can be made arbitrarily small simultaneously for all messages by choosing appropriately long κ_0. Let $C(\rho)$ represent capacity region of the X network i.e it represents the set of all achievable rate-matrices $[(R_{ji}(\rho))]$. Then the degrees of freedom region of this channel is defined by

$$\mathcal{D} = \left\{ [d_{ij}] \in \mathbb{R}_+^{MN} : d_{ij} = \lim_{\rho \to \infty} \frac{R_{ij}(\rho)}{\log(\rho)} \right.$$
$$\left. [R_{ij}(\rho)] \in C(\rho), \forall (i,j) \in \{1, 2 \ldots N\} \times \{1, 2 \ldots M\} \right\}$$

X networks are interesting because they encompass most one-way single hop communication scenarios. For example, the multiple access, broadcast, and interference channels can be derived from the X network by setting appropriate messages to null. Due to the generic nature of the X channel, its degrees of freedom characterization reveal several interesting insights into wireless networks which are summarized in the next section

[2]For the purposes of this paper, κ maybe equivalently interpreted as the frequency index as well, if the coding occurs over multiple frequency slots.

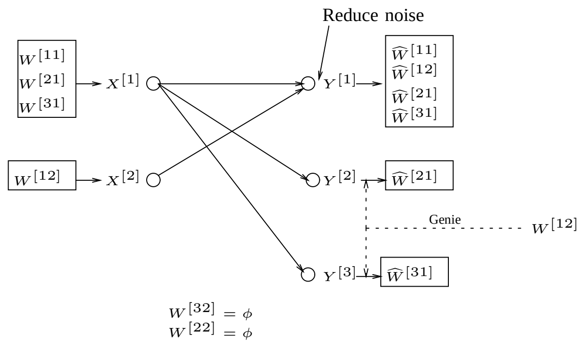
II. Summary and Discussion of Results

Outerbound : We provide an outerbound to the capacity *region* of the $M \times N$ X network within $o(\log(\text{SNR}))$ in Theorem 1, i.e, we provide an outerbound for the the degrees of freedom *region* of the network. This outerbound is fairly general and applies to fully connected single-hop networks with time-varying or constant non-zero channel gains. The degrees of freedom *region* outerbound is important since it can be used to bound for the degrees of freedom of most fully connected distributed one-way single hop network communications scenarios. A corollary to the theorem shows that also the *total* number of degrees of freedom of the $M \times N$ X network cannot exceed $\frac{MN}{M+N-1}$.

Achievable Scheme - Interference Alignment : The outerbound of $\frac{MN}{M+N-1}$ is shown to be tight for time-varying/frequency-selective channels using an achievable scheme based on interference alignment over multiple-symbol extensions of the channel extensions. In the process of showing achievability, we also show a useful *reciprocity* property of achievable schemes based on interference alignment and zero-forcing.

Propagation Delay Example : Reference [5] demonstrates the idea of interference alignment by considering a toy example of a K user interference channel with non-negligible delays. We construct a similar scheme achieving the outerbound of $4/3$ degrees of freedom in the 2 user X channel based on TDMA by carefully choosing the propagation delays between nodes. The construction conveys the idea of interference alignment in a simple manner. It must be noted that section IV is the only section of this paper that considers propagation delays in its model. All other sections use the classical X network model which assumes zero propagation delays between transmitting and receiving nodes. The propagation delay example is shown to have application over the X channel with constant (i.e. not time-varying) channel gain in [9].

X networks versus interference networks : The 2 user X network shown to have a total of $4/3$ degrees of freedom outperforms the 2 user interference channel which has only 1 degree of freedom. The corollary to Theorem 1 implies that for large values of K, the degrees of freedom outerbound of the K user X network is $\frac{K^2}{2K-1} \approx K/2$ for large K. Since the K user interference channel itself has $K/2$ degrees of freedom, the outerbound implies that the X network loses its degrees of freedom advantage over the interference network as K



(b)

Fig. 2. Converse argument in 2×3 X channel with for $m = n = 1$ in Theorem 1

becomes large.

Cost of Distributed Processing : The cost of distributed processing can easily be established from theorem 2. Compared to the $M \times N$ MIMO channel which represents joint processing at all transmitters and receivers, the $M \times N$ X network pays a degrees of freedom penalty of $\left(\min(M,N) - \frac{MN}{M+N-1}\right)$. It is interesting to note that this penalty vanishes if $M \gg N$ or if $N \gg M$, meaning that the number of transmitters (or resp. receivers) is much larger than the number of receivers (or resp. transmitters). Therefore, at high SNR, a small set of distributed nodes in a wireless communication network with no shared messages can emulate MIMO behavior, from a degrees of freedom perspective.

III. Outerbound for the Degrees of Freedom of the X network

Theorem 1: Let

$$\mathcal{D}^{out} \triangleq \Bigg\{[(d_{ji})] : \forall (m,n) \in \{1,2\ldots M\} \times \{1,2\ldots N\} \\ \sum_{q=1}^{N} d_{qm} + \sum_{p=1}^{M} d_{np} - d_{nm} \leq 1\Bigg\}$$

Then $\mathcal{D} \subseteq \mathcal{D}^{out}$ where $\mathcal{D}$ represents the degrees of freedom region of the $M \times N$ X network

Proof: Consider any $m \in \{1,2,\ldots M\}, n \in \{1,2\ldots N\}$. We first set to null, all messages which are not associated with either receiver m or transmitter n (see figure 2 (a)), i.e., we set

$$W^{[pq]} = \phi, \forall (p,q) \text{ s.t. } (p-n)(q-m) \neq 0$$

Consider *any* reliable coding scheme over the channel. Note that setting certain messages to null cannot deteriorate the performance of the non-null messages. We now bound the rates achieved by the coding scheme corresponding to the non-null messages as follows. Let a genie provides messages $W^{[nl]}, l \neq m$ to receivers $1,2,3,\ldots,n-1,n+1,\ldots N$. Then, receivers $1,2,3,\ldots,n-1,n+1,\ldots N$ can cancel the interference caused by transmitters $1,2\ldots(m-1),(m+1)\ldots M$ to obtain a noisy version of $X^{[m]}$. Notice that receiver $r \neq n$ is able to decode the message from transmitter m using the noisy version of $X^{[m]}$. Now, since we started with a reliable coding scheme, receiver n is able to decode the signal from $1,2\ldots m-1,m+1,\ldots M$ and thus cancel the interference from these transmitters to obtain a noisy version of $X^{[m]}$ as well. Therefore, by reducing noise sufficiently at this receiver, we can ensure that all other receivers are a degenerate version of receiver n (whose noise is sufficiently reduced). This implies that all messages in the system are decodable at receiver n meaning that the rates of all messages are achievable in the multiple access channel formed with receiver n. Note that the performance of the original coding scheme cannot deteriorate because of aid by the genie or reducing the noise and therefore the converse argument is not affected (similar argument in [4]). Since the multiple access channel with a single antenna has only 1 degree of freedom, the desired bound (below) automatically follows.

$$\sum_{q=1}^{N} d_{qm} + \sum_{p=1}^{M} d_{np} - d_{nm} \leq 1$$

■

Corollary 1: The number of spatial degrees of freedom of the X channel with M transmitters and N receivers is upper bounded by $\frac{MN}{M+N-1}$ i.e.

$$\max_{d_{ij} \in \mathcal{D}} \sum_{ij} d_{ij} \leq \frac{MN}{M+N-1}$$

Equivalently, the sum-capacity of the X channel $C_\Sigma(\rho)$ may be bounded as

$$C_\Sigma(\rho) = \frac{MN}{M+N-1}\log(\rho) + o(\log(\rho))$$

The bound can be obtained by summing all the MN inequalities describing the outerbound of the degrees of freedom region. The outerbound of Theorem 1 can be used to bound most one-way distributed single hop communication scenarios. For example, consider a hypothetical channel with 3 transmitters and 3 receivers and 6 messages $W^{[ij]}, i \neq j$. i.e the 3×3 X channel with $W^{[11]} = W^{[22]} = W^{[33]} = \phi$. The solution to the following linear programming problem provides an outerbound for the total number of degrees of freedom of this channel.

$$\max_{d_{ij}} \sum_{i \neq j} d_{ij}$$
$$\text{s.t} \sum_{q=1}^{3} d_{mq} + \sum_{p=1}^{3} d_{pl} - d_{ml} \quad \leq \quad 1, \forall m,l = 1,2,3$$

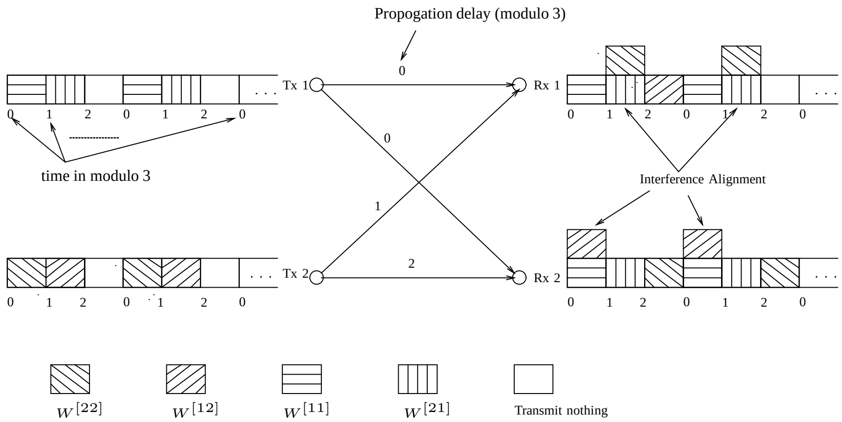


Fig. 3. TDMA scheme achieving 4/3 degrees of freedom over the 2 user X channel with propagation delays

Corollary 2: The number of degrees of freedom of the K user X channel is upper bounded by $\frac{K^2}{2K-1}$

IV. Achievable Schemes - Propagation Delay Example

The following example conveys the idea of interference alignment using a 2 user X channel with propagation delays. Consider a 2 user X channel with non-negligible propagation delays between the transmitters and the receivers. Let T_{ji} represent the propagation delay between transmitter i and receiver j, where $i, j \in \{1, 2\}$. As usual, there are 4 messages $W^{[ji]}$ in this X channel, with $W^{[ji]}$ representing the message from transmitter i to receiver j. Now, suppose the locations of the transmitters and receivers can be configured so that the propagation delays T_{ji} satisfy the following relations.

$$\begin{aligned}
T_{11} &= 3k, \quad \text{for some } k \in \mathbb{N} \\
T_{12} &= 3l+1, \quad \text{for some } l \in \mathbb{N} \\
T_{21} &= 3m, \quad \text{for some } m \in \mathbb{N} \\
T_{22} &= 3p+2, \quad \text{for some } p \in \mathbb{N}
\end{aligned}$$

Then, we construct a scheme (see Figure 3) achieving $4/3$ degrees of freedom over this X channel in the following manner.

Transmitter 1:

- Transmits a codeword corresponding to message $W^{[11]}$ starting at $t = 3n, \forall n$ for the duration of 1 unit of time
- Transmits a codeword corresponding to message $W^{[21]}$ starting at $t = 3n+1, \forall n$ for unit time.

Transmitter 2:

- Transmits a codeword corresponding to message $W^{[12]}$ starting at $t = 3n+1, \forall n$ for unit time
- Transmits a codeword corresponding to message $W^{[22]}$ starting at $t = 3n, \forall n$ for unit time

With this transmission strategy, it is easy to see (figure 3) that at both the receivers, the interference overlaps and the desired message remains free of interference. Since each of the 4 messages message is active for 1 in 3 time-slots, and each message can be decoded free of interference at the desired receivers, the transmission scheme achieves a total of $4/3$ degrees of freedom.

V. Achievable Schemes - Interference Alignment and Reciprocity

The following is the main result of this section

Theorem 2: The $M \times N$ X channel has $\frac{MN}{M+N-1}$ degrees of freedom ,i.e., the sum-capacity of the $M \times N$ X channel is

$$C_\Sigma(\rho) = \frac{MN}{M+N-1}\log(\rho) + o(\log(\rho))$$

The converse for the theorem is already stated in the corollary to Theorem 1.

The achievable scheme for the X channel is based on interference alignment and zero-forcing over multiple symbol extensions of the channel. For the general X channel, the achievable scheme involves a partial interference alignment scheme which approaches the outerbound as we arbitrarily increase the size of the channel extensions (super-symbols). However, for the special case where $M = 2$ or $N = 2$, a *perfect* interference alignment based scheme which achieves the degrees of freedom outerbound *exactly* using finite symbol extensions can be constructed. Due to paucity of space, we only place a sketch of the achievable schemes for the $M = 2$ and $N = 2$ cases here. The reader is referred to [10] for an achievable scheme over the general $M \times N$ X channel, and formal proofs for $M = 2$ and $N = 2$.

A. Achievability if $N = 2$

Consider the $M \times 2$ X channel. Now, we use a $M+1$ symbol extended super-symbol so that all inputs and outputs over this extended channel are $M+1$ dimensional vectors. The outerbound of $\frac{2M}{M+1}$ is achieved by creating

1 interference free stream from each transmitter to each of the 2 receivers over this extended channel. Now, let $\mathbf{v}^{[ij]}, j = 1, 2 \dots M, i = 1, 2$ represent the beamforming direction for the message stream from transmitter j to receiver i. Note that there are M interfering streams at each receiver. Interference alignment at receiver 1 is ensured by choosing directions $\mathbf{v}^{[2j]}, j = 1, 2 \dots M$ which, on being transformed by the channel from transmitter j to receiver 1, aligns with a particular pre-fixed interference direction $\mathbf{I}_1$. i.e. we choose

$$\mathbf{H}^{[1j]}\mathbf{v}^{[2j]} = \mathbf{I}_1$$

where $\mathbf{H}^{[1j]}$ represents the $(M+1) \times (M+1)$ channel matrix over the multiple symbol extension of the channel. This ensures that at receiver 1, the interference is of dimension 1 and the M desired streams can be decode in in an $M+1$ dimensional space. Similarly $\mathbf{v}^{[1j]}$ are designed to align with a pre-chosen direction at receiver 2. With random time-varying channels, the desired signal vectors can be shown to be distinguishable from each other, and from the interference vectors, and can hence be decoded free of interference by zero-forcing (see [10] for a proof).

B. Achievability for $M = 2$: Reciprocity of Zero-forcing and Interference Alignment

Consider an $M \times N$ X channel. We refer to this as the primal channel. Consider any achievable scheme on this channel based on beamforming and zero-forcing respectively. Specifically, consider any achievable scheme whose encoding and strategy are as follows.

Encoding : Transmitter i encodes a message to receiver j along linearly independent streams and beamforms the kth stream streams along directions $\mathbf{v}_k^{[ji]}$

Decoding : Receiver j decodes the kth stream from transmitter i by zero-forcing all undesired streams using a vector $\mathbf{u}_k^{[ji]}$.

The reciprocal (or dual) channel is the the channel formed when the transmitters and receivers of the primal channel are interchanged over the same physical channel. Therefore, the dual of an $M \times N$ X channel is a $N \times M$ X channel where the channel gain between transmitter i and receiver j in the primal channel is equal to the channel gain between transmitter j and receiver i in the dual channel. Now consider the following coding scheme over the dual X network.

Encoding : In the dual network, transmitter j encodes a message to receiver i along linearly independent streams and beamforms these streams along directions that were used for zero-forcing in the primal network $\mathbf{u}_k^{[ij]}$.

Decoding : Receiver i decodes all the desired streams through zero-forcing along directions that were the corresponding beamforming directions $\mathbf{v}_k^{[ji]}$in the primal network.

It can be easily verified that the above scheme maps every independent interference-free stream in the primal $M \times N$ X channel to an independent interference-free stream in the dual $N \times M$ X channel and thus achieves the same number of degrees of freedom in the dual network. The reciprocity property combined with the interference alignment scheme of the previous section describes achievability in the $M \times N$ X channel where $N = 2$.

VI. Conclusion

The main result of this paper is the degrees of freedom characterization of wireless X networks. Firstly, through the degrees of freedom *region* outerbound of the X network, we have established a bound on the total number of degrees of freedom of most distributed fully connected single-hop wireless ad-hoc networks. Secondly, using an achievable scheme based on interference alignment, we show that the outerbound on the *total* number of degrees of freedom is tight, if the channel-gains are time-varying. The study of the X channel helps characterize, at high SNR, the capacity benefits of joint processing as compared to distributed processing, and the capacity benefits of generalizing interference networks to X network.

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