Multiple Access Outerbounds and the Inseparability of Parallel Interference Channels

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***Abstract*— It is known that the capacity of parallel (multi-carrier) Gaussian point-to-point, multiple access and broadcast channels can be achieved by separate encoding for each subchannel (carrier) subject to a power allocation across carriers. In this paper we show that such a separation does not apply to parallel Gaussian interference channels in general. A counter-example is provided in the form of a 3 user interference channel where separate encoding can only achieve a sum capacity of $\log(\mathbf{SNR})+o(\log(\mathbf{SNR}))$ per carrier while the actual capacity, achieved only by joint-encoding across carriers, is $3/2\log(\mathbf{SNR}))+o(\log(\mathbf{SNR}))$ per carrier. As a byproduct of our analysis, we propose a class of multiple-access-outerbounds on the capacity of the 3 user interference channel.**

I. Introduction

It is well known that over the parallel Gaussian point-to-point channel, coding separately over the individual subchannels (carriers) achieves capacity subject to optimal power allocation. Thus the capacity of the parallel Gaussian point-to-point channel is equal to the sum of the capacities of the point-to-point Gaussian subchannels with corresponding powers chosen through the water-filling algorithm. Similarly, it has also been shown that separate coding over each carrier is optimal for parallel Gaussian multiple access (MAC) and broadcast (BC) channels [1], [2]. The separability of parallel Gaussian point-to-point, MAC and BC is useful because the coding schemes designed for classical (single carrier) models can be applied directly to multi-carrier systems subject to a power-allocation across carriers. A key question that remains open is whether such a separation holds for other Gaussian networks, and in particular, if separate encoding is optimal for multi-carrier interference networks.

Much work on parallel Gaussian interference network has focused on optimal power allocation across carriers under the assumption of *separate coding* over each carrier (For example, [3] and references therein). While separate coding has been shown to be optimal in the *strong* 2 user interference channel ([4]), it is not known whether separate coding is optimal when the interfering links are weak or if there are more than 2 users in an interference network. Recently, joint encoding of multiple-carriers has been used in [5] to characterize the sum capacity per carrier, of the K user multi-carrier Gaussian interference channel. The sum capacity (per carrier) is found to be

$$C(\text{SNR})=\frac{K}{2}\log(\text{SNR})+o(\log(\text{SNR}))$$

where SNR represents the signal to noise power ratio. In other words, the K user interference channel has $K/2$ degrees of freedom[1] per orthogonal time and frequency dimension. The interference alignment constructions proposed in [5] are based on *joint encoding* over multiple frequencies. Interestingly, another recent work in [6] has constructed interference alignment schemes over a single-carrier interference channel, i.e., with separate encoding. Thus, it remains unclear whether the capacity of multi-carrier interference channels can be achieved by separate encoding over each carrier and a power allocation across carriers. It is this open problem that we address in this paper.

The main result of this paper is that unlike the point-to-point, multiple-access and broadcast channels, in general, separate coding *does not suffice* to achieve the capacity of the interference channel. We establish this result by constructing a counterexample. We first construct a classical (single-carrier) interference channel with a capacity of $\log(\text{SNR})+o(\log(\text{SNR}))$, in Theorem 1. We then use this result to construct to construct a 3 user frequency-selective interference channel where separate coding can only achieve a sum rate of $\log(\text{SNR})+o(\log(\text{SNR}))$ per carrier while a joint coding scheme is shown to achieve $3/2\log(\text{SNR})+o(\log(\text{SNR}))$ per carrier. Thus, parallel interference channels are, in general, inseparable. As a byproduct of the construction of Theorem 1, we also propose a class of outerbounds on the capacity of the single-carrier 3 user interference channel.

We start with the classical (single-carrier) Gaussian 3 user interference channel.

II. The Gaussian 3 User Interference channel

We study the 3 user (single-carrier) Gaussian interference channel whose input-output relations are described

[1]Also known as multiplexing-gain or capacity pre-log.

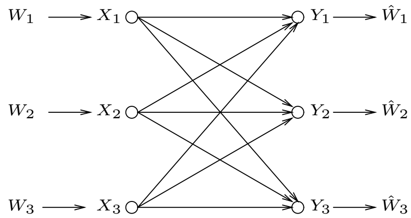

Fig. 1. The 3 user interference channel

as follows

$$Y_i(n) = \sum_{j=1}^{3} h_{i,j} X_j(n) + Z_i(n), i = 1, 2, 3$$

where at the nth symbol, $Y_i(n)$ and $Z_i(n)$ respectively represent the received signal and the noise at the ith receiver, and $X_j(n)$ represents the signal transmitted by the jth transmitter. $h_{i,j}$ represents the channel gain between transmitter j and receiver i. All channel gains are assumed to be non-zero and known to all the nodes in the network. Transmitter i has message W_i for receiver i for $i = 1, 2, 3$. The noise $Z_i(n)$ is a zero-mean unit-variance additive white Gaussian noise (AWGN), assumed independent identically distributed (i.i.d.) across users and symbols. The total transmit power can be expressed as $E\left[\frac{1}{N}\sum_{i=1}^{3}\sum_{n=1}^{N}|X_i(n)|^2\right] \leq \text{SNR}$, where N is the length of the codeword. The rate of the ith user is defined as $R_i(\text{SNR}) = \frac{\log(|W_i|)}{N}$ where $|W_i|$ is the cardinality of the message set corresponding to message W_i. A rate vector $\mathbf{R}(\text{SNR}) = (R_1(\text{SNR}), R_2(\text{SNR}), R_3(\text{SNR}))$ is said to be *achievable* if, for $i \in \{1, 2, 3\}$, message W_i can be encoded at rate $R_i(\text{SNR})$, so that the probability of decoding error can be made arbitrarily small by choosing an appropriately large N. The capacity region $\mathcal{C}(\text{SNR})$ represents the set of all achievable rate vectors in the network. The sum capacity $C_\Sigma(\text{SNR})$ of the network is defined as

$$C_\Sigma(\text{SNR}) = \max_{\mathbf{R}(\text{SNR}) \in \mathcal{C}(\text{SNR})} \sum_{i=1}^{3} R_i(\text{SNR})$$

The number of degrees of freedom of the network is defined as

$$d_\Sigma = \lim_{\text{SNR}\to\infty} \frac{C_\Sigma(\text{SNR})}{\log(\text{SNR})}$$

Equivalently, d_Σ is the total number of degrees of freedom of the network if and only if we can write

$$C_\Sigma(\text{SNR}) = d_\Sigma \log(\text{SNR}) + o(\log(\text{SNR})).$$

Theorem 1: Consider the 3 user interference channel where

$$\frac{h_{i,j}}{h_{i,i}} = \frac{h_{k,j}}{h_{k,i}}$$

for some $i, j, k \in \{1, 2, 3\}, j \neq k, k \neq i, i \neq j$. Then, this interference channel has 1 degree of freedom, i.e.,

$$C_\Sigma(\text{SNR}) = \log(\text{SNR}) + o(\log(\text{SNR}))$$

Proof: Achievability is trivial since setting $W_2 = W_3 = \phi$, we get a point-to-point Gaussian channel whose capacity is known to be of the form $\log(\text{SNR}) + o(\log(\text{SNR}))$. We show the converse for the special case where $k = 1, i = 2, j = 3$. i.e., we consider the case where

$$\frac{h_{2,3}}{h_{2,2}} = \frac{h_{1,3}}{h_{1,2}} = \gamma, \gamma \neq 0.$$

By symmetry, the converse extends to all other cases. Consider any achievable coding scheme. Let a genie give X_1 to receiver 2 (Figure 2(a)). Now, receiver 2 can cancel the interference from transmitter 1 to obtain $\tilde{Y}_2$ which may be written as

$$\begin{aligned} \tilde{Y}_2 &= h_{2,2}X_2 + h_{2,3}X_3 + Z_2 \\ &= h_{2,2}(X_2 + \gamma X_3) + Z_2 \qquad (1) \end{aligned}$$

The dependence on the symbol index n is dropped above for convenience. Since we started with an achievable coding scheme, receiver 1 can decode X_1 reliably and therefore, cancel the effect of X_1 from Y_1 to obtain

$$\tilde{Y}_1 = h_{1,2}(X_2 + \gamma X_3) + Z_1 \qquad (2)$$

Note that receiver 2 is able to decode W_2 from $\tilde{Y}_2$. Equations (2) and (1) imply that by reducing the variance of Z_1 sufficiently, we can ensure that $\tilde{Y}_2$ is a degraded version of $\tilde{Y}_1$ so that receiver 1 can decode W_2 as well. Note that reducing noise and the aid of a genie can only increase the capacity region of a channel and therefore the converse argument is not affected. Thus, in this genie-aided reduced-noise channel, any message which can be decoded at receiver 2 can be decoded at receiver 1 as well. Now, we can let transmitters 1 and 2 co-operate to form a MIMO two user interference channel as in Figure 2(b). Again, note that allowing transmitters to co-operate cannot reduce the sum capacity and thus, the MIMO interference channel of Figure 2(b) has a larger sum-capacity than the original channnel. Reference [7] has shown that the MIMO interference channel of Figure 2(b) has 1 degree of freedom meaning that its capacity is of the form $\log(\text{SNR}) + o(\log(\text{SNR}))$. Therefore, we have shown that

$$C_\Sigma(\text{SNR}) \leq \log(\text{SNR}) + o(\log(\text{SNR}))$$

and the converse argument is complete. ■

III. The Parallel Gaussian 3 User Interference Channel

The parallel Gaussian interference channel consisting of M parallel subchannels may be expressed as

$$\mathbf{Y}_i(n) = \sum_{j=1}^{3} \mathbf{H}_{i,j}\mathbf{X}_j(n) + \mathbf{Z}_i(n), i = 1, 2, 3$$

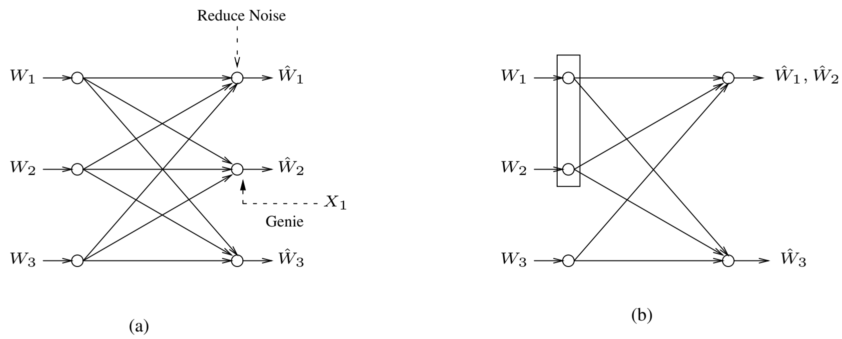

Fig. 2. The converse argument of Theorem 1

where, corresponding to the nth symbol $\mathbf{Y}_i(n), \mathbf{Z}_i(n), \mathbf{X}_j(n)$ are $M \times 1$ vectors whose M entries represent the signal received at receiver i over the M sub-channels, the i.i.d. AWGN experienced by receiver i over the M carriers, and the signal transmitted by the jth transmitter over the M carriers, respectively. $\mathbf{H}_{i,j}$ is a $M \times M$ diagonal matrix whose mth diagonal entry represents the channel gain between transmitter j and receiver i corresponding to the mth subchannel. All channel gains are assumed to be non-zero and known apriori to all nodes. Messages, achievable rates, power constraints, capacity and degrees of freedom are defined in the usual manner as described in the previous section.

Let $C_{\Sigma}^{[m]}(\text{SNR})$ denote the sum capacity of the interference channel over the m^{th} carrier and SNR_m denote the total transmit power constraint over the m^{th} carrier. The main question addressed in this correspondence is the following - Can the capacity (per carrier) of the parallel interference channel be expressed as the sum of the capacities achieved by the constituent interference channels over each carrier, i.e.,

$$C_{\Sigma}(\text{SNR}) = \frac{1}{M} \sum_{m=1}^{M} C_{\Sigma}^{[m]}(\text{SNR}_m) \quad (3)$$

for some power allocation vector $(\text{SNR}_1, \text{SNR}_2, \dots \text{SNR}_M)$ such that

$$\sum_{m=1}^{M} \text{SNR}_m \leq \text{SNR}. \quad (4)$$

The existence of a power vector satisfying the above equations would imply that a capacity-optimal scheme is to code separately over each carrier with power SNR_m allocated to the mth carrier. We will use the result of Theorem 1 to construct a parallel interference channel where independent coding over its subchannels is suboptimal.

Theorem 2: Parallel Gaussian interference channels are in general, not separable. Equivalently, in general there do not exist coding schemes such that equations (3) and (4) are satisfied

Proof: The proof is by counter-example i.e. we construct a multi-carrier interference channel where, assuming that the power constraint is ρ at each transmitter

1) Interference alignment achieves capacity of the channel is $\frac{3}{2}\log(\rho) + o(\log(\rho))$ per carrier. In particular, the capacity of the channel is $\frac{3}{2}\log(1+2\rho)$ per carrier.
2) Each subchannel has only 1 degree of freedom meaning that separate encoding over each carrier is suboptimal since it can only achieve a capacity of $\log(\rho)+o(\log(\rho))$ per carrier. In particular, separate coding can, at best, achieve a rate of $\log(1+3\rho)$ per carrier.

This is easily done as follows. Consider the case where we have 2 carriers, so $M = 2$. Let

$$\mathbf{H}_{i,j} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \forall i \neq j, i, j \in \{1, 2, 3\} \quad (5)$$

$$\mathbf{H}_{2,2} = \mathbf{H}_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (6)$$

$$\mathbf{H}_{3,3} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad (7)$$

It can be easily verified that each subchannel in this above channel satisfies the conditions of Theorem 1 so that each subchannel has 1 degree of freedom. In fact, assuming that the power ρ is split equally between the two carriers, and observing that each carrier only sees half the total noise variance, the capacity of each subchannel in this parallel channel can be shown to be $\log(1+3\rho)$ (see "Example 1" in Section IV). Note that the strategy of allocating power $\rho/2$ for each sub-channel is optimal from the perspective of sum-rate allocation. Therefore, separate coding achieves a rate of $\log(1+3\rho)$ per carrier over this channel.

However, the capacity of this channel is $\frac{3}{2}\log(1 + 2\rho)$ and can be achieved by an interference alignment based joint decoding scheme. In this achievable scheme, each transmitter beamforms its message along vector $[1/\sqrt(2) \quad 1\sqrt(2)]^T$. This ensures that at each receiver, all the interference aligns along $[1\sqrt(2) \quad 1/\sqrt(2)]^T$.

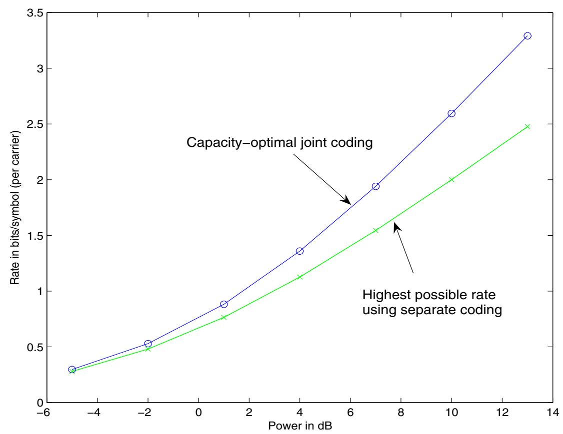

Fig. 3. A comparison of the performance of joint coding versus separate coding on the parallel 3 user interference channel

The desired messages can be decoded by projecting the recevied signal along $[1/\sqrt(2) \quad -1/\sqrt(2)]^T$ so that all the interference is cancelled. Thus, each user can achieve a rate of $\frac{1}{2}\log(1+2\rho)$ meaning that the sum-rate achieved is $\frac{3}{2}\log(1+2\rho)$. The outerbound of rates in the 3 user interference channel can be obtained by considering 2 users at a time ([5], [8]) . For example, setting $W_3 = \phi$, Carleil's interference channel outerbound for the 2 user interference channel formed by the first two users can simply be written as

$$R_1 + R_2 \leq \log(1+2\rho)$$

Writing similar outerbounds for $R_1 + R_3$ and $R_2 + R_3$ and summing all bounds together, we get

$$\begin{aligned} R_1 + R_2 + R_3 &\leq \frac{3}{2}\log(1+2\rho) \\ \Rightarrow C_\Sigma(\rho) &= \frac{3}{2}\log(1+2\rho) \end{aligned}$$

where the last equality follows from the fact that a rate of $\frac{3}{2}\log(1+2\rho)$ is achievable. Note that from Jensen's inequality, it can be inferred that

$$\forall \rho > 0, \log(1+3\rho) \leq \frac{3}{2}\log(1+2\rho)$$

Thus separate encoding is clearly suboptimal for all values of ρ. ■

The above theorem clearly implies the sub-optimality of separate coding over each carrier of the 3 user interference channel in general. Figure 3 illustrates the suboptimality of separate coding over each carrier in comparison with the optimal interference alignment based joint coding scheme for the channel.

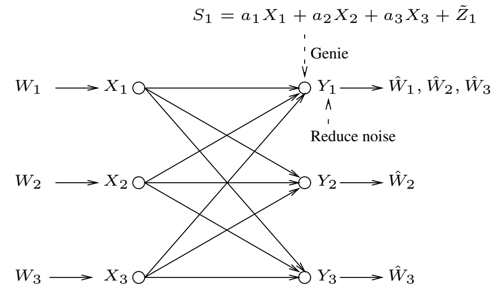

Fig. 4. Multiple access outerbound for the classical 3 user interference channel

IV. Multiple access outerbounds for the capacity of 3 user interference channel

In this section, we provide an interesting application of the result of Theorem 1 in the form of a class of outerbounds for the classical (single-carrier) 3 user interference channel. The outerbound argument goes as follows. Consider any achievable coding scheme. Using this coding scheme, receiver 1 can decode W_1. Our aim is to enhance receiver 1 with enough information so that it can decode W_2 and W_3 as well (See figure 4). Then the capacity region of the multiple access channel(MAC) formed by the three transmitters and the (enhanced) receiver 1 forms an outerbound for the capacity of the interference channel. The improvements to receiver 1 are described in the following steps

1) *Enhance receiver 1 to decode W_2 :* Let a genie provide receiver 1 with a $S_1 = a_1X_1 + a_2X_2 + a_3X_3 + \tilde{Z}_1$ where $\tilde{Z}_1$ is an AWGN term independent of $X_i, i = 1, 2, 3$. Note that this side information effectively acts as an additional antenna at receiver 1. The noise term $\tilde{Z}_1$ can possibly be correlated with other noise variables $Z_i, i = 1, 2, 3$. Now, receiver can form $U_1 = \alpha Y_1 + \beta S_1$ with α, β chosen such that the co-efficients of X_1 and X_2 in the U_1 satisfy the conditions of Theorem 1. The theorem then implies that by sufficiently reducing the noise at receiver 1, we can ensure that receiver 1 decodes W_2 as well. Note that aid of a genie is not necessary if the channel co-efficients already satisfy the conditions of Theorem 1
2) *Enhance receiver 1 to decode W_3 :* Receiver 1, enhanced as described in the previous step, can now decode W_1 and W_2. Receiver 1 can therefore choose $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ appropriately to form

$$\begin{aligned} V_1 &= \bar{\alpha}X_1 + \bar{\beta}X_2 + \bar{\gamma}Y_1' \\ &= h_{3,1}X_1 + h_{3,2}X_2 + h_{3,3}X_3 + \bar{\gamma}Z_1' \end{aligned}$$

Note that we use Y_1' and Z_1' above rather than Y_1 and Z_1 since the previous step involves reducing the noise at receiver 1. By further reducing the noise if required, receiver 1 can be enhanced to decode W_3.

Steps 1 and 2 above imply that the capacity region of the 3 user Gaussian interference channel is outer-bounded by the capacity region of the single-input-multiple-output (SIMO) Gaussian MAC where the receiver receives S_1 on one antenna and a reduced-noise version of Y_1 on the other. This class of bounds can be optimized over $a_i, i = 1, 2, 3$ and the statistics of $\tilde{Z}_1$. Further, similar outerbounds can be found by enhancing receiver 2 or receiver 3 rather than receiver 1. Note that, since a MAC with two antennas has 2 degrees of freedom, this class of outerbounds is loose from the perspective of degrees of freedom. A degrees of freedom outer-bound of $3/2$ has already been shown for the 3 user interference channel (See [5], [8]). We now provide examples of this class of outerbounds.

Example 1: Here, we consider the interference channel formed on the first carrier of the parallel Gaussian interference channel described in Equations (5)-(7) in the previous section. In this channel, $h_{i,j} = 1, i \neq j, i, j \in \{1, 2, 3\}$. Also $h_{11} = h_{22} = 1, h_{33} = -1$. Let the average transmit power at all the transmitters be bounded by ρ. Consider any achievable coding scheme. Note that this channel already satisfies the conditions of Theorem 1. It can also be verified that both receiver 1 can decode both X_1 and X_2 without any noise reduction. Furthermore, receiver 1 can perform $X_1 + X_2 - Y_1 = h_{3,1}X_1 + h_{3,2}X_2 + h_{3,3}X_3 - Z_1$. Since $(-Z_1)$ is a AWGN term having the variance as Z_3, receiver 1 can decode W_3 without requiring any noise reduction. Thus, the capacity region of this channel is bounded by the capacity region of the multiple access channel formed at receiver 1 and we can write

$$C_\Sigma \leq \log(1 + 3\rho)$$

It can be verified that the above sum-rate is achievable with each user transmitting Gaussian codewords of the appropriate rate and all receivers decoding as they would in a MAC i.e. with multi-user detection and successive decoding . Further, it can also easily be verified that the sum-capacity of the interference channel corresponding to the second carrier of the parallel channel described by Equations (5)-(7) can also be determined as above.

Example 2: Consider the *perfectly symmetric* 3 user interference channel where $h_{i,i} = 1, \forall i = 1, 2, 3$ and $h_{i,j} = h > 1, \forall i \neq j, i, j \in \{1, 2, 3\}$. Also, let the power constraint at each node be equal to ρ. Since the channel does not satisfy the conditions of Theorem 1, a genie provides receiver 1 with information of $S_1 = a_1X_1 + (1-h)X_2 + \tilde{Z}_1$ where $\tilde{Z}_1$ is an i.i.d AWGN term correlated with Z_1 such that $E\left[(Z_1 + \tilde{Z}_1)^2)\right] = 1$. Note that the receiver can decode W_1 with any reliable coding scheme. It can also be verified that receiver 1 can now decode W_2 from $U_1 = (h-a_1-1)X_1 + Y_1 + S_1$. Now that receiver 1 is aware of X_1 and X_2, it can add appropriate terms to Y_1 to form $V_1 = h(hX_1 + hX_2 + X_3) + Z_1$. Since $h > 1$, Y_3 is a degraded version of V_1 which implies that receiver 1 can decode W_3 as well. Optimizing over parameters a_1 and the statistics of Z_1, we can bound the sum-capacity of the interference channel by the capacity of the SIMO MAC formed at receiver 1 as

$$C_\Sigma(\rho) \leq \min_{(a_1, \tilde{Z}_1)} C_{\text{MAC}}(\rho, a_1, \tilde{Z}_1)$$

where the minimization is carried over $\tilde{Z}_1$ which satisfies

$$E\left[(Z_1 + \tilde{Z}_1)^2\right] \leq 1, \tilde{Z}_1 \sim \mathcal{N}(0, \sigma^2)$$

and $C_{\text{MAC}}(\rho, a_1, \tilde{Z}_1)$ is the capacity of the 3 user SIMO MAC whose outputs are Y_1 and S_1.

V. Conclusion

We constructed a 3 user interference channel with constant (i.e., not frequency-selective or time-varying) channel coefficients such that it has 1 degree of freedom. Furthermore, we provided a mult-carrier extension of this channel such that separate coding over each carrier can only achieve sum rate $\log(\text{SNR}) + o(\log(\text{SNR}))$ per carrier, while the actual capacity is $3/2 \log(\text{SNR}) + o(\log(\text{SNR}))$ which can be achieved only through coding across carriers. The result implies that, in general, independent coding over the various channel states of the parallel Gaussian interference channel is not capacity optimal. Thus, unlike parallel Gaussian point to point, multiple access and broadcast channels, parallel Gaussian interference channels are, in general, not separable. We have also provided a class of multiple access outerbounds for the 3 user interference channel with constant channel co-efficients.

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