# Approaching the Capacity of Wireless Networks through Distributed Interference Alignment

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*Abstract*— **Recent results establish the optimality of interference alignment to approach the Shannon capacity of interference networks at high SNR. However, the extent to which interference can be aligned over a finite number of signalling dimensions remains unknown. Another important concern for interference alignment schemes is the requirement of global channel knowledge. In this work we provide examples of iterative algorithms that utilize the reciprocity of wireless networks to achieve interference alignment with only local channel knowledge at each node. These algorithms also provide numerical insights into the feasibility of interference alignment that are not yet available in theory.**

## I. INTRODUCTION

The recent emergence of the idea of interference alignment for wireless networks has shown that the capacity of wireless networks can be much higher than previously believed [1]. The canonical example of interference alignment is a communication scenario where, regardless of the number of interferers, every user is able to access one half of the spectrum free from interference from other users [1]. For the interference channel with $K$ transmitters and $K$ receivers and random, time varying channel coefficients drawn from a continuous distribution, reference [1] characterizes the network sum capacity as

$$C_\Sigma(SNR) = \frac{K}{2}\log(SNR) + o(\log(SNR)) \qquad (1)$$

so that the capacity per user is $\frac{1}{2}\log(SNR) + o(\log(SNR))$. Here SNR is defined as the total transmit power of all the transmitters in the network when the local noise power at each node is normalized to unity. The $o(\log(SNR))$ term, by definition, becomes negligible compared to $\log(SNR)$ at high SNR. Therefore the accuracy of the capacity approximation in (1) approaches $100\%$ at high SNR. Since the capacity of a single user in the absence of all interference is $\log(SNR) + o(\log(SNR))$, the main result of [1] may be summarized as:

*"At high SNR, every user in a wireless interference network is (simultaneously and almost surely) able to achieve (nearly) one half of the capacity that he could achieve in the absence of all interference."*

The capacity-optimal achievable scheme within $o(\log(\text{SNR}))$ is shown to be the interference alignment scheme. Interference alignment on the $K$ user interference channel refers to the idea of constructing signals in such a way that they cast overlapping shadows over one half of the signal space observed by each receiver where they constitute interference, leaving the other half of the signal space free of interference for the desired signal. This approach reveals the

sub-optimality of the cake-cutting view of spectrum allocation between co-existing wireless systems because, essentially, everyone gets "half the cake".

Interference alignment schemes are presented in [1] in the form of closed form expressions for the transmit precoding matrices. However, these closed form expressions require global channel knowledge which can be an overwhelming overhead in practice. Moreover, closed form solutions have only been found in certain cases. In general, analytical solutions to interference alignment problem are difficult to obtain and even the feasibility of interference alignment over a limited number of signaling dimensions is an open problem. In this paper we explore distributed interference alignment algorithms to accomplish the following objectives.

- Require only local channel knowledge at each node. Specifically, each receiver is assumed to know only the channel to its desired transmitter and the covariance matrix of its effective noise (consisting of the AWGN and the interference from all other users).
- Provide numerical insights into the feasibility of alignment.

We propose iterative algorithms that take a cognitive approach to interference management and utilize only the local side information available naturally due to the reciprocity of wireless networks. The two key properties can be summarized as follows.

- *Cognitive Principle:* Unlike selfish approaches studied in prior work where each transmitter tries to maximize his own rate by avoiding interference at his desired receiver, we follow an unselfish approach where each transmitter primarily tries to minimize the interference to unintended receivers. Since avoiding interference to unintended receivers is the defining feature of cognitive radio [2] the unselfish approach is a cognitive approach. The cognitive approach is found to lead to interference alignment, and is thus capable of approaching network capacity at high SNR.
- *Reciprocity:* For a given transmitter, learning how much interference is caused at unintended receivers can require too much side information, and is one of the key challenges for cognitive radio systems. However, this information is naturally available because of the reciprocity of the channel for networks where two-way communication is based on time-division duplex operation with synchronized time-slots. Due to reciprocity, the signalling dimensions along which a receiving node sees
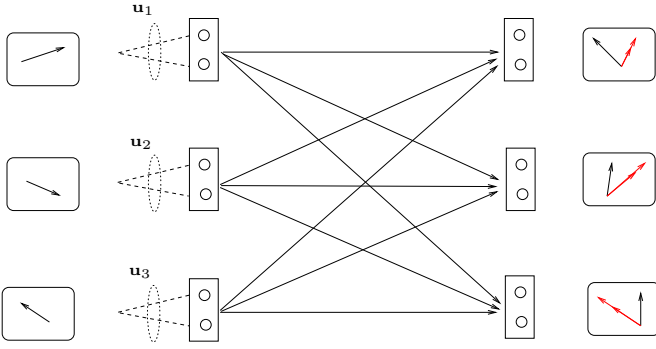
Fig. 1. Interference alignment solution for the three user two antenna case. The arrows in the red indicate the direction of the interference.

the least interference from other users are also the same signalling dimensions along which this node will cause the least interference to other nodes in the reciprocal network where all transmitters and receivers switch roles.

## II. INTERFERENCE OPTIMIZATION APPROACHES

The optimality of interference alignment schemes at high SNR is interesting because these schemes treat all interference as noise and require no multi-user detection. Achievable schemes based on treating interference as noise have been explored extensively over the last decade. Prominent among these are the interference avoidance and iterative waterfilling algorithms where each transmitter acts selfishly to align its transmissions along those directions where its desired receiver sees the least interference [3]–[5], and network duality approaches [6]–[8] that are based on the reciprocity of the wireless propagation channel. With iterative waterfilling/interference avoidance algorithms [3], [9], each transmitter tries to do what is best for his own receiver, i.e., each transmitter allocates its power in a manner best suited for his desired receiver. With interference alignment each transmitter tries to minimize the interference he causes to other receivers. The interference alignment schemes in [1], [10], [11] show that for interference networks, the "do no harm" approach is much more powerful, and is in fact capacity-optimal within $o(\log(\text{SNR}))$, than the "help yourself" approach of interference avoidance and iterative waterfilling schemes.

From a game-theoretic perspective, interference avoidance and iterative waterfilling algorithms lead to a stable operating point commonly known as the Nash equilibrium. At Nash equilibrium, there is no incentive for any user to unilaterally change his transmit strategy. Often these points are not optimum from a network perspective and indicate inefficiency in wireless network operation [12], [13]. Interestingly, interference alignment is not a Nash equilibrium point if the goal of each user is to maximize his own rate. Fig. 1 shows the interference alignment solution for the three user two antenna case. Notice that interfering signals are co-linear at each receiver while the desired signal may not be exactly orthogonal to the interference - a price paid for interference alignment. It can be easily observed that the interference alignment solution is not a Nash equilibrium point. Fixing the transmit strategy for users 2 and 3, the best strategy for user 1 is to choose $\mathbf{u}_1$ such that his signal is orthogonal to the interference at

receiver 1. Although this strategy is good for user 1, it will destroy interference alignment at receivers 2 and 3. Thus Fig. 1 clearly highlights the difference between the optimal strategy of interference alignment and the selfish strategy of iterative waterfilling or interference avoidance.

Iterative schemes have also been used to implicitly achieve interference alignment on the 2 user $X$ channel in [14], [15]. However, for the 2 user $X$ channel interference alignment can be explicitly achieved with roughly the same amount of channel knowledge as required by the iterative schemes, without the need for an iterative process [10]. The iterative schemes of [14] are specialized for the 2 user $X$ channel and generalizations to $X$ networks and interference networks with more than 2 users are not straightforward. Another approach taken in prior work is to exploit the duality relationships enabled by the reciprocity of the propagation channel. For example, network duality ensures that the same set of signal to interference and noise ratios (SINRs) can be achieved in the original and the reciprocal network with the same total transmit power [6], [7]. Network duality is used in [7] to minimize the total transmit power required to support a feasible rate vector. Reciprocity of propagation channels is used in [8] for optimal frequency allocation problem. In this work we provide examples of iterative algorithms to achieve interference alignment on wireless interference channels. These algorithms combine elements of all the above-mentioned approaches, especially [14] and [8].

## III. SYSTEM MODEL

Consider the $K$-user MIMO interference channel where the $k^{th}$ transmitter and receiver are equipped with $M^{[k]}$ and $N^{[k]}$ antennas respectively. Note that the antennas could represent symbol extensions in time or frequency as well. However, if the antennas correspond to symbol extensions over orthogonal dimensions (time, frequency slots) then the channel matrices will have a diagonal structure. The channel is defined as:

$$Y^{[k]}(n) = \sum_{l=1}^{K} \mathbf{H}^{[kl]}(n)X^{[l]}(n) + Z^{[k]}(n), \;\; \forall k \in \{1, 2, \cdots, K\}$$

where, at the $n^{th}$ channel use, $Y^{[k]}(n), Z^{[k]}(n)$ are the $N^{[k]} \times 1$ received signal vector and the zero mean unit variance circularly symmetric additive white Gaussian noise vector (AWGN) at receiver $k$, $X^{[l]}(n)$ is the $M^{[k]} \times 1$ signal vector transmitted by transmitter $l$, and $\mathbf{H}^{[kl]}(n)$ is the $N^{[k]} \times M^{[k]}$ matrix of channel coefficients between transmitter $l$ and receiver $k$. The transmit power at transmitter $l$ is $\text{E}[||X^{[l]}||^2] = P^{[l]}$.

For the $K$ user interference channel defined above, we also define a reciprocal channel, where the role of transmitters and receivers are switched. For every variable on the original channel, the corresponding variable on the reciprocal channel is denoted with a left arrow on top. The reciprocal channel is defined as:

$$\overleftarrow{Y}^{[k]}(n) = \sum_{l=1}^{K} \overleftarrow{\mathbf{H}}^{[kl]}(n)\overleftarrow{X}^{[l]}(n) + \overleftarrow{Z}^{[k]}(n), \;\; \forall k \in \{1, 2, \cdots, K\}$$

where, at the $n^{th}$ channel use, $\overleftarrow{Y}^{[k]}(n), \overleftarrow{Z}^{[k]}(n)$ are the $M^{[k]} \times 1$ received signal vector and the zero mean unit variance circularly symmetric additive white Gaussian noise vector (AWGN) at receiver $k$ which is equipped with $M^{[k]}$ antennas,

$\overleftarrow{X}^{[l]}(n)$ is the $N^{[k]} \times 1$ signal vector transmitted by transmitter $l$, and $\overleftarrow{\mathbf{H}}^{[kl]}(n) = \mathbf{H}^{[lk]\dagger}(n)$ is the $M^{[k]} \times N^{[k]}$ matrix of channel coefficients between transmitter $l$ and receiver $k$. The transmit powers at transmitter $l$ on the reciprocal channel is $\mathrm{E}[||\overleftarrow{X}^{[l]}||^2 = \overleftarrow{P}^{[l]}$. The channel use index $n$ is henceforth suppressed to avoid cumbersome notation.

*Notation:* $\mathbf{A}_{\star d}$ represents the $d^{th}$ column of matrix $\mathbf{A}$. $\mathbf{I}_d$ represents the $d \times d$ identity matrix. $\mathrm{Tr}[\mathbf{A}]$ denotes the trace of the matrix $\mathbf{A}$ and $\mathbf{A}^\dagger$ is the conjugate transpose of matrix $\mathbf{A}$.

## IV. INTERFERENCE ALIGNMENT OVER LIMITED DIMENSIONS - AN OPEN PROBLEM

References [1] and [11] present interference alignment schemes that are constructed over symbol extensions of time-varying channels. It is shown that by using *long* symbol extensions the degrees of freedom achieved per dimension approach arbitrarily close to the theoretical outerbound, thereby establishing the degrees of freedom of time-varying interference and $X$ networks. However, the extent to which interference can be aligned over a *limited* number of dimensions remains an open problem. As a consequence, the maximum number of degrees of freedom that can be achieved through alignment of interference signal vectors is not known in general.

We first review the interference alignment problem and the reciprocity property of interference alignment. Let $d^{[k]} \leq \min(M^{[k]}, N^{[k]}), k \in \{1, 2, \cdots, K\}$ denote the degrees of freedom for user $k$'s message.

*Precoding at Transmitter:* Let $\mathbf{V}^{[k]}$ be an $M^{[k]} \times d^{[k]}$ matrix whose columns are the orthonormal basis of the transmitted signal space of user $k$. Mathematically, the transmitted signal vector of user $k$ is given by:

$$X^{[k]} = \sum_{d=1}^{d^{[k]}} \mathbf{V}_{[\star d]}^{[k]} \overline{X}_d^{[k]} = \mathbf{V}^{[k]} \overline{X}^{[k]}, \quad \overline{X}^{[k]} \sim \mathcal{N}\left(0, \frac{P^{[k]}}{d^{[k]}} \mathbf{I}_{d^{[k]}}\right) \tag{2}$$

Each element of the $d^{[k]} \times 1$ vector $\overline{X}^{[k]}$ represents an independently encoded Gaussian codebook symbol with power $\frac{P^{[k]}}{d^{[k]}}$ that is beamformed with the corresponding vector of $\mathbf{V}^{[k]}$.

*Remark:* For interference alignment or the achievability of the degrees of freedom it suffices if the beamforming vectors are linearly independent. However, we assume the beamforming vectors are orthonormal in the formulation above. Note that this does not affect the feasibility of interference alignment, and it naturally leads to the iterative algorithm to be presented in this paper.

*Interference Suppression at Receiver:* Let $\mathbf{U}^{[k]}$ be an $N^{[k]} \times d^{[k]}$ matrix whose columns are the orthonormal basis of the interference-free desired signal subspace at receiver $k$. The $k^{th}$ receiver filters its received signal to obtain $\overline{Y}^{[k]} = \mathbf{U}^{[k]\dagger} Y^{[k]}$. If interference is aligned into the null space of $\mathbf{U}^{[k]}$ then the following condition must be satisfied:

$$\mathbf{U}^{[k]\dagger} \mathbf{H}^{[kj]} \mathbf{V}^{[j]} = 0, \forall j \neq k \tag{3}$$

$$\mathrm{rank}\left(\mathbf{U}^{[k]\dagger} \mathbf{H}^{[kk]} \mathbf{V}^{[k]}\right) = d_k \tag{4}$$

In other words the desired signals are received through a $d^{[k]} \times d^{[k]}$ full rank channel matrix $\overline{\mathbf{H}}^{[kk]} \triangleq \mathbf{U}^{[k]\dagger} \mathbf{H}^{[kk]} \mathbf{V}^{[k]}$

while the interference is completely eliminated. The effective channel for user $k$ is then expressed as:

$$\overline{Y}^{[k]} = \overline{\mathbf{H}}^{[kk]} \overline{X}^{[k]} + \overline{Z}^{[k]} \tag{5}$$

where $\overline{Z}^{[k]} \sim \mathcal{N}(0, I)$ is the effective $d^{[k]} \times 1$ AWGN vector at receiver $k$. The rate achieved on this channel is:

$$R^{[k]} = \log\left|\mathbf{I}_{d^{[k]}} + \frac{P^{[k]}}{d^{[k]}} \overline{\mathbf{H}}^{[kk]} \overline{\mathbf{H}}^{[kk]\dagger}\right| = d^{[k]} \log(P^{[k]}) + o(\log(P^{[k]}))$$

Thus, $d^{[k]}$ degrees of freedom are achieved by user $k$.

### A. Feasibility of Alignment

Given the channel matrices $\mathbf{H}^{[kj]}, k, j \in \{1, 2, \cdots, K\}$, we say that the degrees of freedom allocation $(d^{[1]}, d^{[2]}, \cdots, d^{[K]})$ is feasible if there exist transmit precoding matrices $\mathbf{V}^{[k]}$ and receive interference suppression matrices $\mathbf{U}^{[k]}$:

$$\mathbf{V}^{[k]} : M^{[k]} \times d^{[k]}, \qquad \mathbf{V}^{[k]\dagger} \mathbf{V}^{[k]} = \mathbf{I}_{d^{[k]}} \tag{6}$$

$$\mathbf{U}^{[k]} : N^{[k]} \times d^{[k]}, \qquad \mathbf{U}^{[k]\dagger} \mathbf{U}^{[k]} = \mathbf{I}_{d^{[k]}} \tag{7}$$

such that

$$\mathbf{U}^{[k]\dagger} \mathbf{H}^{[kj]} \mathbf{V}^{[j]} = 0, \forall j \neq k \tag{8}$$

$$\mathrm{rank}\left(\mathbf{U}^{[k]\dagger} \mathbf{H}^{[kk]} \mathbf{V}^{[k]}\right) = d_k, \ \forall k \in \{1, 2, \cdots, K\} \tag{9}$$

The solution to the feasibility problem is not known in general. In other words, given a set of randomly generated channel matrices and a degree-of-freedom allocation $(d^{[1]}, d^{[2]}, \cdots, d^{[K]})$, it is not known if one can almost surely find transmit and receive filters that will satisfy the feasibility conditions. The distributed interference algorithm developed in this paper will be useful in (numerically) solving this open problem.

### B. Reciprocity of Alignment

An interesting observation from the problem formulation above is the duality relationship between interference alignment on a given interference channel and its reciprocal channel obtained by switching the direction of communication. Specifically, let $\overleftarrow{\mathbf{V}}^{[k]}, \overleftarrow{\mathbf{U}}^{[k]}$ denote the transmit precoding filters and the receive interference suppression filters on the reciprocal channel. The feasibility conditions on the reciprocal channel are:

$$\overleftarrow{\mathbf{V}}^{[k]} : N^{[k]} \times d^{[k]}, \qquad \overleftarrow{\mathbf{U}}^{[k]\dagger} \overleftarrow{\mathbf{U}}^{[k]} = \mathbf{I}_{d^{[k]}} \tag{10}$$

$$\overleftarrow{\mathbf{U}}^{[k]} : M^{[k]} \times d^{[k]}, \qquad \overleftarrow{\mathbf{U}}^{[k]\dagger} \overleftarrow{\mathbf{U}}^{[k]} = \mathbf{I}_{d^{[k]}} \tag{11}$$

such that

$$\overleftarrow{\mathbf{U}}^{[j]\dagger} \overleftarrow{\mathbf{H}}^{[jk]} \overleftarrow{\mathbf{V}}^{[k]} = 0, \forall j \neq k \tag{12}$$

$$\mathrm{rank}\left(\overleftarrow{\mathbf{U}}^{[k]\dagger} \overleftarrow{\mathbf{H}}^{[kk]} \overleftarrow{\mathbf{V}}^{[k]}\right) = d_k, \ \forall k \in \{1, 2, \cdots, K\} \tag{13}$$

Suppose we set $\overleftarrow{\mathbf{V}}^{[k]} = \mathbf{U}^{[k]}, \overleftarrow{\mathbf{U}}^{[k]} = \mathbf{V}^{[k]}$. Then the feasibility conditions on the reciprocal channel become identical to the original feasibility conditions. Thus, the following observation can be made:

**Reciprocity of Alignment:** *Since the feasibility conditions are identical, if the degrees of freedom allocation $(d^{[1]}, d^{[2]}, \cdots, d^{[K]})$ is feasible on the original interference network then it is also feasible on the reciprocal network (and*

*vice versa). Interference alignment on the reciprocal interference network is simply achieved by choosing the transmit filters and the receive filters on the reciprocal channel as the receive filters and the transmit filters (respectively) of the original channel.*

Reciprocity of alignment is a key property used for distributed interference alignment algorithms, described in the next section.

## V. DISTRIBUTED ALGORITHM FOR INTERFERENCE ALIGNMENT

In this section we construct distributed interference alignment algorithms for the interference channel with multiple-antenna nodes and no symbol extensions. Continuing with the system model of Section IV, this implies that the relevant interference alignment feasibility condition is (8) while (9) is automatically satisfied [16]. Basically, (8) requires that at each receiver, all interference is suppressed, leaving as many interference-free dimensions as the degrees of freedom allocated to that receiver.

Since we are interested in distributed algorithms, we start with arbitrary transmit and receive filters $\mathbf{V}^{[k]}, \mathbf{U}^{[k]}$ and iteratively update these filters to approach interference alignment. The quality of alignment is measured by the power in the *leakage interference* at each receiver, i.e. the interference power remaining in the received signal after the receive interference suppression filter is applied. The goal is to achieve interference alignment by progressively reducing the leakage interference. If interference alignment is feasible then eventually leakage interference will be zero.

The total interference leakage at receiver $k$ due to all undesired transmitters ($j \neq k$) is given by:

$$I^{[k\star]} = \mathrm{Tr}\left[\mathbf{U}^{[k]\dagger}\mathbf{Q}^{[k]}\mathbf{U}^{[k]}\right] \qquad (14)$$

where

$$\mathbf{Q}^{[k]} = \sum_{j=1,j\neq k}^{K} \frac{P^{[j]}}{d^{[j]}}\mathbf{H}^{[kj]}\mathbf{V}^{[j]}\mathbf{V}^{[j]\dagger}\mathbf{H}^{[kj]\dagger} \qquad (15)$$

is the interference covariance matrix at receiver $k$. Similarly, in the reciprocal network, the total interference leakage at receiver $j$ due to all undesired transmitters ($k \neq j$) is given by:

$$\overleftarrow{I}^{[j\star]} = \mathrm{Tr}\left[\overleftarrow{\mathbf{U}}^{[j]\dagger}\overleftarrow{\mathbf{Q}}^{[j]}\overleftarrow{\mathbf{U}}^{[j]}\right] \qquad (16)$$

where

$$\overleftarrow{\mathbf{Q}}^{[j]} = \sum_{k=1,k\neq j}^{K} \frac{\overleftarrow{P}^{[k]}}{d^{[k]}}\overleftarrow{\mathbf{H}}^{[jk]}\overleftarrow{\mathbf{V}}^{[k]}\overleftarrow{\mathbf{V}}^{[k]\dagger}\overleftarrow{\mathbf{H}}^{[jk]\dagger} \qquad (17)$$

is the interference covariance matrix at receiver $j$.

The iterative algorithm alternates between the original and reciprocal networks. Within each network only the receivers update their interference suppression filters to minimize their total leakage interference.

*Step I:* In the original network, each receiver solves the following optimization problem.

$$\min_{\mathbf{U}^{[k]}:N^{[k]}\times d^{[k]}, \; \mathbf{U}^{[k]}\mathbf{U}^{[k]\dagger}=\mathbf{I}_{d^{[k]}}} I^{[k\star]} \qquad (18)$$

In other words, receiver $k$ chooses its interference suppression filter $\mathbf{U}^{[k]}$ to minimize the leakage interference due to all undesired transmitters. The $d^{[k]}$ dimensional received signal subspace that contains the least interference is the space spanned by the eigenvectors corresponding to the $d^{[k]}$ smallest eigenvalues of the interference covariance matrix $\mathbf{Q}^{[k]}$. Thus, the $d^{[k]}$ columns of $\mathbf{U}^{[k]}$ are given by:

$$\mathbf{U}^{[k]}_{\star d} = \nu_d[\mathbf{Q}^{[k]}], \qquad d = 1, \cdots, d^{[k]} \qquad (19)$$

where $\nu_d[\mathbf{A}]$ is the eigenvector corresponding to the $d^{th}$ smallest eigenvalue of $\mathbf{A}$.

*Step II:* The second step is identical to the first step, but performed in the reciprocal network. Consider the reciprocal network obtained by reversing the roles of the transmitters and the receivers. The transmit precoding matrices in the reciprocal network, $\overleftarrow{\mathbf{V}}^{[k]}$, are the receive interference suppression matrices $\mathbf{U}^{[k]}$ from the original network that were determined in Step I. Each receiver in the reciprocal network solves the following optimization problem.

$$\min_{\overleftarrow{\mathbf{U}}^{[j]}:M^{[j]}\times d^{[j]}, \; \overleftarrow{\mathbf{U}}^{[j]}\overleftarrow{\mathbf{U}}^{[j]\dagger}=\mathbf{I}_{d^{[j]}}} \overleftarrow{I}^{[j\star]} \qquad (20)$$

Similar to Step I, the $d^{[j]}$ columns of $\overleftarrow{\mathbf{U}}^{[j]}$ are given by:

$$\overleftarrow{\mathbf{U}}^{[j]}_{\star d} = \nu_d[\overleftarrow{\mathbf{Q}}^{[j]}], \qquad d = 1, \cdots, d^{[j]} \qquad (21)$$

The receive interference suppression filters in the reciprocal network are then used as the transmit precoding matrices in the original network, and the algorithm returns to Step I. The iterations continue in this manner until the algorithm converges. A pictorial representation is shown in Fig. 2.

### A. Proof of Convergence

We now show that the algorithm must converge. The proof also highlights the intuition behind the algorithm. We define a metric called the weighted leakage interference (WLI) as:

$$
\begin{aligned}
I_w &= \sum_{k=1}^{K}\sum_{j=1,j\neq k}^{K} \frac{\overleftarrow{P}^{[k]}}{d^{[k]}} I^{[kj]} \\
&= \sum_{k=1}^{K}\sum_{j=1,j\neq k}^{K} \frac{\overleftarrow{P}^{[k]}}{d^{[k]}}\frac{P^{[j]}}{d^{[j]}}\mathrm{Tr}\left[\mathbf{U}^{[k]\dagger}\mathbf{H}^{[kj]}\mathbf{V}^{[j]}\mathbf{V}^{[j]\dagger}\mathbf{H}^{[kj]\dagger}\mathbf{U}^{[k]}\right]
\end{aligned}
$$

We show that each step in the algorithm reduces the value of WLI. Since WLI is bounded below by zero, this implies that the algorithm must converge. Note that an interference alignment solution corresponds to WLI= 0. The WLI associated with receiver $k$ is $\frac{\overleftarrow{P}^{[k]}}{d^{[k]}}I^{[k\star]}$. Therefore the value of $\mathbf{U}^{[k]}$ computed in Step I to minimize $I^{[k\star]}$ also minimizes $I_w^{[k\star]}$. Since $I_w = \sum_{k=1}^{K} I_w^{[k\star]}$, we have

$$
\begin{aligned}
\min_{\mathbf{U}^{[1]},\mathbf{U}^{[2]},\cdots,\mathbf{U}^{[K]}} I_w &= \min_{\mathbf{U}^{[1]},\mathbf{U}^{[2]},\cdots,\mathbf{U}^{[K]}} \sum_{k=1}^{K} I_w^{[k\star]} \\
&= \sum_{k=1}^{K}\left[\min_{\mathbf{U}^{[k]}} I_w^{[k\star]}\right] = \sum_{k=1}^{K}\frac{\overleftarrow{P}^{[k]}}{d^{[k]}}\left[\min_{\mathbf{U}^{[k]}} I^{[k\star]}\right]
\end{aligned}
$$

In other words, given the values of $\mathbf{V}^{[j]}, j \in \{1, 2, \cdots, K\}$, Step I minimizes the value of $I_w$ over all possible choices of
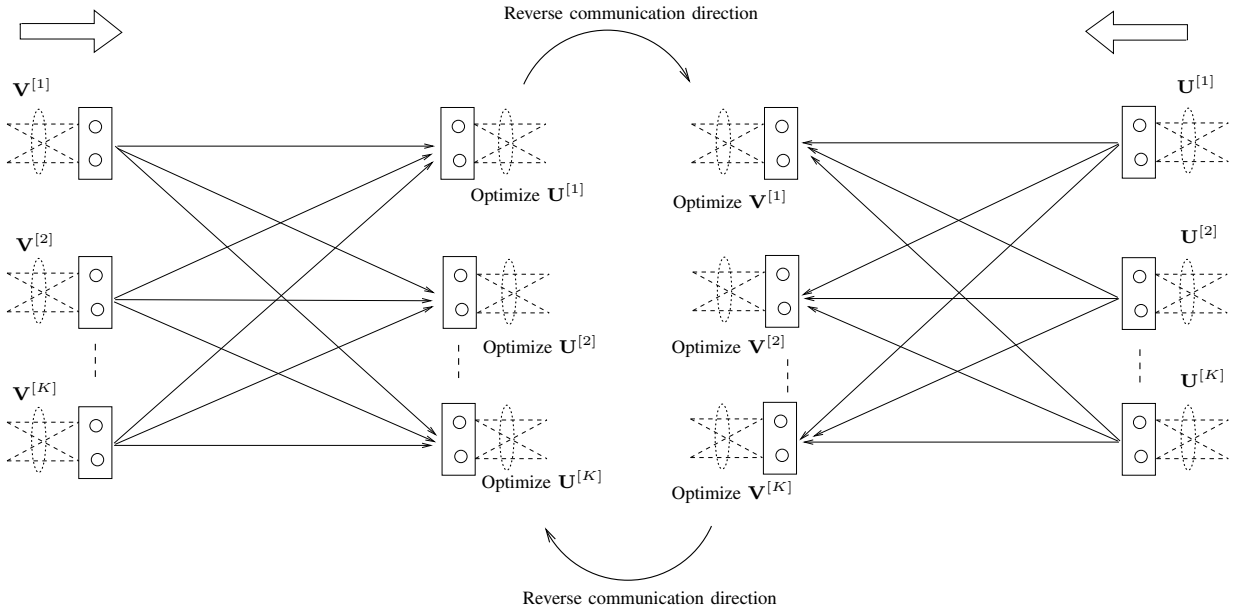
Fig. 2. Pictorial representation of the iterative interference alignment algorithm where the receive directions are optimized to minimize interference power at the receivers. Each link (arrow) represents a MIMO channel.

$\mathbf{U}^{[k]}, k \in \{1, 2, \cdots, K\}$. In particular, Step I can only reduce the value of $I_w$.

The weighted leakage interference associated with transmitter $j$ is

$$
\begin{aligned}
I_w^{[\star j]} & = \frac{P^{[j]}}{d^{[j]}} \sum_{k \neq j} \frac{\overleftarrow{P}^{[k]}}{d^{[k]}} \mathrm{Tr}\left[ \mathbf{U}^{[k]\dagger} \mathbf{H}^{[kj]} \mathbf{V}^{[j]} \mathbf{V}^{[j]\dagger} \mathbf{H}^{[kj]\dagger} \mathbf{U}^{[k]} \right] \\
& = \frac{P^{[j]}}{d^{[j]}} \sum_{k \neq j} \frac{\overleftarrow{P}^{[k]}}{d^{[k]}} \mathrm{Tr}\left[ \overleftarrow{\mathbf{V}}^{[k]\dagger} \overleftarrow{\mathbf{H}}^{[jk]\dagger} \overleftarrow{\mathbf{U}}^{[j]} \overleftarrow{\mathbf{U}}^{[j]\dagger} \overleftarrow{\mathbf{H}}^{[jk]} \overleftarrow{\mathbf{V}}^{[k]} \right] \\
& = \frac{P^{[j]}}{d^{[j]}} \mathrm{Tr}\left[ \overleftarrow{\mathbf{U}}^{[j]\dagger} \overleftarrow{\mathbf{Q}}^{[j]} \overleftarrow{\mathbf{U}}^{[j]} \right]
\end{aligned}
$$

Therefore the value of $\overleftarrow{\mathbf{U}}^{[j]}$ computed in Step II to minimize $\overleftarrow{I}^{[j\star]}$ also minimizes $I_w^{[\star j]}$. Since $I_w = \sum_{j=1}^{K} I_w^{[\star j]}$, it is easily seen that Step II can also only reduce the value of $I_w$. Since the value of $I_w$ is monotonically reduced after every iteration, convergence of the algorithm is guaranteed although convergence to global minimum is not guaranteed due to the non-convex nature of the optimization problem. The following observations summarize the intuition behind the iterative algorithm.

1) Dimensions along which a receiver sees the least interference from other nodes are also the dimensions along which it causes the least interference to other nodes in the reciprocal network where it functions as a transmitter.
2) The weighted leakage interference is unchanged in the original and reciprocal networks if the transmit and receive filters are switched.

### B. Max-SINR Algorithm

The algorithm presented above seeks perfect interference alignment. In particular it seeks to create an interference-free subspace of the required number of dimensions, that is designated as the *desired* signal subspace. However, note

that interference alignment makes no attempt to maximize the desired signal power within the desired signal subspace. In fact the algorithm described above does not depend at all on the direct channels $\mathbf{H}^{[kk]}$ through which the desired signal arrives at the intended receiver. Therefore, while the interference is eliminated within the desired space, no coherent combining gain (array gain) for the desired signal is obtained with interference alignment. While this is optimal as all signal powers approach infinity, it is not optimal in general at intermediate SNR values. Therefore other algorithms may be designed which will perform better than the interference alignment algorithm at intermediate SNR values. We consider one such natural extension of the interference alignment algorithm where the receive filters $\mathbf{U}^{[k]}$ and $\overleftarrow{\mathbf{U}}^{[k]}$ are chosen to maximize SINR at the receivers instead of only minimizing the leakage interference. The details of the algorithm are available in the full paper [16].

### VI. NUMERICAL RESULTS

Consider the 3 user interference channel where each node is equipped with 2 antennas and all channel coefficients are i.i.d. zero mean unit variance circularly symmetric complex Gaussian. As shown in Fig. 1 with interference alignment each user achieves 1 degree of freedom. In Fig. 3 we compare the performance of the iterative interference alignment algorithm for this channel with the theoretical interference alignment which assumes global channel knowledge. It can be seen that the algorithm performs very close to the theoretical case and more importantly it provides significant benefits over the orthogonal case. The sum rate for the orthogonal scheme is calculated assuming equal time sharing for the users, and with power $3P$ per node (the transmit power per node in the iterative algorithm is $P$). The modified interference alignment scheme where the receive-combining vectors maximize the SINR provides a considerable performance gain at low and intermediate $P$. We also plot the performance of interference
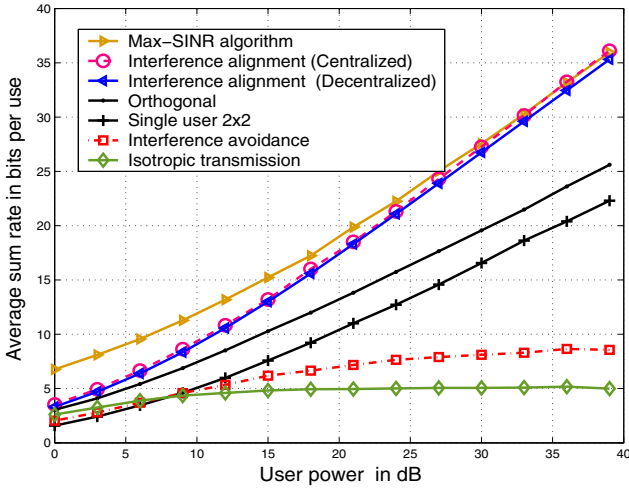
Fig. 3. Performance of the decentralized interference alignment algorithm for the three user two antenna case.



Fig. 4. Percentage of interference power in desired signal space as a function of the total number of data streams in the network.

avoidance algorithm which is a selfish approach. The isotropic transmission case refers to the case where each transmitter sends 2 streams of equal power without regard to the channel information.

While the iterative interference alignment algorithm is useful for circumventing the need for global channel knowledge, it can also be used to check theoretical feasibility of interference alignment for a given number of streams per user. Let $(d^{[1]}, d^{[2]}, \cdots d^{[K]})$ denote the number of transmit streams of the users. For perfect interference alignment $\sum_{j=1}^{d^{[k]}} \lambda_j[\mathbf{Q}^{[k]}] = 0$ at receiver $k$ where $\lambda_j[\mathbf{A}]$ denotes the $j$th smallest eigenvalue of $\mathbf{A}$. Using the algorithm, we plot in Fig. 4, the percentage of interference in the desired signal space versus the total number of transmit streams in the network. The fraction of interference in the desired signal space of receiver $k$ is defined as

$$p_k = \frac{\sum_{j=1}^{d^{[k]}} \lambda_j[\mathbf{Q}^{[k]}]}{\text{Tr}[\mathbf{Q}^{[k]}]}.$$

When interference alignment is feasible the fraction of interference in desired signal space will be zero (within numerical errors). Fig. 4 suggests that interference alignment is feasible on the four user interference channel with 5 antennas at each node when each transmitter sends two streams. With increase in the number of streams $p_k$ increases which is an indication that interference alignment is not possible. Although the upperbound on the degrees of freedom for this network is 10, approaching the upperbound may entail channel extensions. The plot suggests that 8 degrees of freedom ($d^{[1]}=d^{[2]}=d^{[3]}=d^{[4]}=2$) can be achieved without channel extension. Similarly for the 4 antenna case, the plot indicates that interference alignment is possible for up to a total of 6 streams in the 4 user interference network with only 4 antennas at each node.

## VII. CONCLUSION

Distributed interference alignment algorithms are investigated. Interference alignment is found to be achievable through iterative algorithms based on network reciprocity and the "minimize interference to others" approach. Numerical comparisons to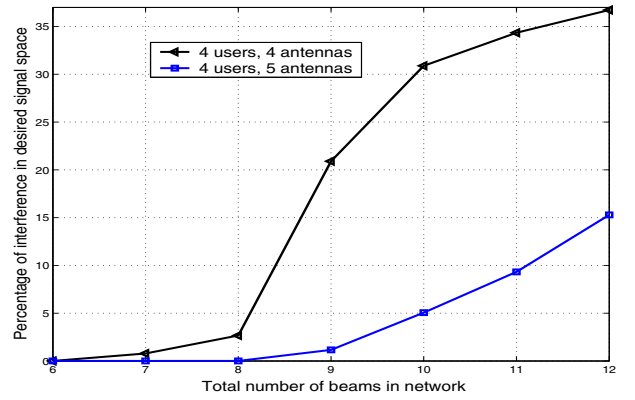 orthogonal schemes, simultaneous transmission schemes and selfish interference avoidance schemes show that the benefits of distributed interference alignment algorithm are significant and close to the theoretical predictions. As mitigating interference is the fundamental problem of wireless networks, the 'do no harm' approach based algorithms have enormous applications in wireless networks.

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