Feasibility Conditions for Interference Alignment

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***Abstract*—The degrees of freedom (DoF) of K-user MIMO interference networks with constant channel coefficients are not known in general. Determining the feasibility of a linear interference alignment is a key step toward solving this open problem. Our approach in this paper is to view the alignment problem for interference networks as a multivariate polynomial system and determine its solvability by comparing the number of equations and the number of variables. Consequently, we divide the interference networks into two classes - proper and improper, where interference alignment is and is not achievable, respectively. An interference network is called proper if the cardinality of every subset of equations in the corresponding polynomial system is less than or equal to the number of variables involved in that subset of equations. Otherwise, it is called improper. Our intuition in this paper is that for general channel matrices, proper systems are almost surely feasible and improper systems are almost surely infeasible. We prove the direct link between proper (improper) and feasible (infeasible) systems for some important cases, thus significantly strengthening our intuition. Numerical simulation results also support our intuition.**

I. Introduction

The degrees of freedom (DoF) of wireless interference networks represent the number of interference-free signaling-dimensions in the network. In a network with K transmitters and K receivers and non-degenerate channel conditions, it is well known that K non-interfering spatial signaling dimensions can be created if the transmitters or the receivers are able to jointly process their signals. Cadambe and Jafar [1] recently introduced the idea of interference alignment for K-user wireless interference network with time-varying/frequency-selective channel coefficients, and showed that $K/2$ spatial signaling dimensions are available in spite of the distributed nature of network, which precludes joint processing of signals at transmitters or receivers. While a number of interference alignment solutions have appeared since [1] for different channel settings, many fundamental questions remain unanswered. One such problem is to determine the feasibility of linear interference alignment for K-user MIMO interference networks with constant channel coefficients. It is this open problem that we address in this paper.

†The author is supported in part by The Scientific and Technological Research Council of Turkey (TUBITAK). The author is on leave at University of California, Irvine.

II. Overview of Main Results

The main contribution of this work is that we provide an analytical criteria for determining the feasibility of interference alignment for K-user MIMO interference networks. First, we present some examples to demonstrate the questions that we answer in this paper.

A. Symmetric Systems

Let $(M \times N, d)^K$ denote the K-user MIMO interference network, where every transmitter has M antennas, every receiver has N antennas and each user wishes to achieve d DoF. We call such a system a symmetric system. Consider the following examples.

- $(2 \times 2, 1)^3$ - It is shown in [1] that for the 3-user interference network with 2 antennas at each node, each user can achieve 1 DoF by presenting a closed form solution. However, is there a way to analytically determine the feasibility of this system without finding a closed form solution?
- $(5 \times 5, 2)^4$ - Consider the 4-user interference network with 5 antennas at each user and we wish to achieve 2 DoF per user for a total of 8 DoF. A theoretical solution to this problem is not known but numerical evidence in [2] clearly indicates that a linear interference alignment solution exists. Numerical algorithms are one way to determine the feasibility of linear interference alignment. However, is there a way to theoretically determine the feasibility of alignment without running the numerical simulation?
- Now, consider the three distinct systems -$(6 \times 4, 2)^4$, $(7 \times 3, 2)^4$, and $(8 \times 2, 2)^4$. Are these systems feasible? Clearly, the last one $(8 \times 2, 2)^4$ is feasible because simple transmit zero-forcing is enough to eliminate the interference at every receiver. Is the feasibility of $(8 \times 2, 2)^4$ system related to the feasibility of other systems? We will show that these three systems and the systems $(5 \times 5, 2)^4, (4 \times 6, 2)^4, (3 \times 7, 2)^4,$ and $(2 \times 8, 2)^4$ all belong to the same group, where any system in the group can be obtained by successively transferring an antenna between transmitters and receivers. In more general terms, we will show that the group $(K \times 1, 1)^K, \big((K-1) \times 2, 1\big)^K,$

$\cdots, \left(2 \times (K-1), 1\right)^K, (1 \times K, 1)^K$ is a proper group. The first and last members of a group are easily seen to be proper (because a simple zero-forcing solution exists), thereby also determining the status of the rest of group members. Same argument is valid for grouping improper systems as well.

B. Asymmetric Systems

Let $\left(M^{[1]} \times N^{[1]}, d^{[1]}\right) \cdots (M^{[K]} \times N^{[K]}, d^{[K]})$ denote the K-user MIMO interference network, where the k^{th} transmitter and receiver have $M^{[k]}$ and $N^{[k]}$ antennas, respectively and the k^{th} user demands $d^{[k]}$ DoF. We call such a system an asymmetric system. Consider the following examples.

- Consider the simple system $(2 \times 1, 1)^2$, which is clearly feasible. However, now consider the $(2 \times 1, 1)(1 \times 2, 1)$ system, where the same total number of DoF is desired. Although these systems have the same number of total antennas, is the latter system still achievable?
- Consider a feasible system $(2 \times 3, 1)(3 \times 2, 1)$ [3], where a total of 2 DoF is desired. Now, consider the same scheme with increased number of users; that is, the 4-user interference network $(2 \times 3, 1)^2(3 \times 2, 1)^2$, where a total of 4 DoF is desired. Is this system still achievable, where DoF is doubled by simply going from two users to four users?

In this paper, we address all these questions. The basic approach is to consider the linear interference alignment problem as a solvability of multivariate polynomial system. We determine the correct way to count the number of variables and equations for a general MIMO interference alignment problem. Then, based on the number of variables and equations, we classify the system as either proper (the cardinality of every subset of equations is less than or equal to the number of variables involved in that subset of equations) or improper with the intuitive understanding that *proper systems are almost surely feasible and improper systems are almost surely infeasible*. For the cases where each user demands only 1 DoF, we show that the precise feasibility condition is given by a specialized version of Bernshtein's theorem [4] in terms of the mixed volume of the Newton polytopes of the support sets of a multivariate polynomial system. For the cases $(2\times 3, 1)^4$ and $(2\times 3, 1)^2(3\times 2, 1)^2$ $\left((2\times 2, 1)^3(3\times 5, 1)^1\right)$, we obtain nonzero (zero) mixed volumes and therefore, we show the direct link between proper (improper) and feasible (infeasible) systems for these important cases. We omit the proofs for other cases, where each user wishes to achieve 1 DoF since mixed volume computation is #P-complete [4]. On the other hand, for any case, where at least one user wishes to achieve more than 1 DoF an innovative approach in algebraic geometry is needed to prove the solvability of corresponding polynomial system. Our intuition is also supported by numerical results for a wide variety of cases including the specific examples listed above.

III. System Model

We consider the same K-user MIMO interference network as considered in [2]. The received signal at the n^{th} channel use can be written as follows:

$$\mathbf{Y}^{[k]}(n) = \sum_{l=1}^{K} \mathbf{H}^{[kl]}(n)\mathbf{X}^{[l]}(n) + \mathbf{Z}^{[k]}(n),$$

$\forall k \in \mathcal{K} \triangleq \{1, 2, ..., K\}$. Here, $\mathbf{Y}^{[k]}(n)$ and $\mathbf{Z}^{[k]}(n)$ are the $N^{[k]} \times 1$ received signal vector and the zero mean unit variance circularly symmetric additive white Gaussian noise vector (AWGN) at the k^{th} receiver, respectively. $\mathbf{X}^{[l]}(n)$ is the $M^{[l]} \times 1$ signal vector transmitted from the l^{th} transmitter and $\mathbf{H}^{[kl]}(n)$ is the $N^{[k]} \times M^{[l]}$ matrix of channel coefficients between the l^{th} transmitter and the k^{th} receiver. $\mathrm{E}[||\mathbf{X}^{[l]}(n)||^2] = P^{[l]}$ is the transmit power of the l^{th} transmitter. Hereafter, we omit the channel use index n for the sake of simplicity. The DoF for the k^{th} user's message is denoted by $d^{[k]} \le \min(M^{[k]}, N^{[k]})$.

As defined earlier, $(M \times N, d)^K$ denotes the K-user symmetric MIMO interference network, where each transmitter and receiver has M and N antennas, respectively and each user demands d DoF and therefore, the total DoF demand is Kd. In general, let $\Pi_{k=1}^{K}\left(M^{[k]} \times N^{[k]}, d^{[k]}\right) = \left(M^{[1]} \times N^{[1]}, d^{[1]}\right) \cdots (M^{[K]} \times N^{[K]}, d^{[K]})$ denote the K-user MIMO interference network, where the k^{th} transmitter and receiver have $M^{[k]}$ and $N^{[k]}$ antennas, respectively and the k^{th} user demands $d^{[k]}$ DoF.

IV. Linear Interference Alignment Scheme

In interference alignment precoding, the transmitted signal from the k^{th} user is $\mathbf{X}^{[k]} = \mathbf{V}^{[k]}\tilde{\mathbf{X}}^{[k]}$, where $\tilde{\mathbf{X}}^{[k]}$ is a $d^{[k]} \times 1$ vector that denotes the $d^{[k]}$ independently encoded streams transmitted from the k^{th} user. The $M^{[k]} \times d^{[k]}$ precoding filters $\mathbf{V}^{[k]}$ are designed to maximize the overlap of interference signal subspaces at each receiver while ensuring that the desired signal vectors at each receiver are linearly independent of the interference subspace. Therefore, each receiver can zero-force all the interference signals without zero-forcing any of the desired signals. The zero-forcing filters at the receiver are denoted by $\mathbf{U}^{[k]}$. In [2], it is shown that an interference alignment solution requires the simultaneous satisfiability of the following conditions:

$$\mathbf{U}^{[k]\dagger}\mathbf{H}^{[kj]}\mathbf{V}^{[j]} = 0, \forall j \neq k \quad \text{and} \tag{1}$$

$$\text{rank}\left(\mathbf{U}^{[k]\dagger}\mathbf{H}^{[kk]}\mathbf{V}^{[k]}\right) = d^{[k]}, \quad \forall k \in \{1, 2, ..., K\}, \tag{2}$$

where † denotes the conjugate transpose operator. Very importantly, [2] explains how the condition (2) is automatically satisfied almost surely if the channel matrices do not have any special structure, $\text{rank}(\mathbf{U}^{[k]}) = \text{rank}(\mathbf{V}^{[k]}) = d^{[k]} \le \min(M^{[k]}, N^{[k]})$ and $\mathbf{U}^{[k]}, \mathbf{V}^{[k]}$ are designed to satisfy (1), which is independent of all direct channels $\mathbf{H}^{[kk]}$.

For the cases, where it is difficult to theoretically determine the feasibility of interference alignment, it can be numerically determined by using an iterative algorithm proposed in [2]. In this paper, we develop an analytical criteria to determine the feasibility of interference alignment. Our approach is to count the number of equations and variables in (1).

V. Proper System

While the formal definition appears later, put simply that the $\Pi_{k=1}^{K}\left(M^{[k]} \times N^{[k]}, d^{[k]}\right)$ system is proper if and only if the cardinality of every subset of equations in the corresponding polynomial system obtained from (1) is less than or equal to the number of variables involved in that subset of equations. Otherwise, the system is improper. The reason for this classification is the following intuition that forms the basis of our approach in this paper:

Key Insight: The interference alignment is almost surely feasible for proper systems and almost surely infeasible for improper systems.

The insight is supported by extensive simulations (some of which are presented in this paper). Next, we explicitly define the condition for proper systems. Let us start with the total number of equations N_e and the total number of variables N_v in the polynomial system (1).

A. Counting the Number of Equations N_e and Variables N_v

To obtain N_e and N_v, we rewrite the condition in (1) as follows:

$$\mathbf{u}_m^{[k]\dagger}\mathbf{H}^{[kj]}\mathbf{v}_n^{[j]} = 0, \quad j \neq k,\ j,k \in \{1,2,...,K\} \qquad (3)$$
$$\forall n \in \{1,2,...,d^{[j]}\} \text{ and } \forall m \in \{1,2,...,d^{[k]}\}$$

where $\mathbf{v}_n^{[j]}$ and $\mathbf{u}_m^{[k]}$ are the transmit and receive beamforming vectors (columns of precoding and interference suppression filters, respectively).

N_e is directly obtained from (3) as follows:

$$N_e = \sum_{\substack{k,j\in\mathcal{K}\\ k\neq j}} d^{[k]}d^{[j]}.$$

However, calculating the number of variables N_v is less straightforward. In particular, we have to be careful not to count any superfluous variables that do not help with interference alignment.

At the k^{th} transmitter, the number of $M^{[k]} \times 1$ transmit beamforming vectors to be designed is $d^{[k]}$ $\left(\mathbf{v}_n^{[k]},\ \forall n \in \{1,2,...,d^{[k]}\}\right)$. Therefore, at first sight, it may seem that the precoding filter of the k^{th} transmitter, $\mathbf{V}^{[k]}$, has $d^{[k]}M^{[k]}$ variables. However, as we argue next, we can eliminate $(d^{[k]})^2$ of these variables without loss of generality.

The $d^{[k]}$ linearly independent columns of transmit precoding matrix $\mathbf{V}^{[k]}$ span the transmitted signal space

$$\begin{aligned}\mathcal{T}^{[k]} &= \text{span}(\mathbf{V}^{[k]})\\ &= \{\mathbf{v} : \exists \mathbf{a} \in \mathbb{C}^{d^{[k]}\times 1},\ \mathbf{v} = \mathbf{V}^{[k]}\mathbf{a}\}.\end{aligned}$$

Thus, the columns of $\mathbf{V}^{[k]}$ are the basis for the transmitted signal space. However, the basis representation is not unique for a given subspace. In particular, consider any full rank $d^{[k]} \times d^{[k]}$ matrix $\mathbf{B}$. Then, continuing from the last step of the above equations,

$$\begin{aligned}\mathcal{T}^{[k]} &= \{\mathbf{v} : \exists \mathbf{a} \in \mathbb{C}^{d^{[k]}\times 1},\ \mathbf{v} = \mathbf{V}^{[k]}\mathbf{B}^{-1}\mathbf{B}\mathbf{a}\}\\ &= \text{span}(\mathbf{V}^{[k]}\mathbf{B}^{-1}).\end{aligned}$$

Thus, post-multiplication of the transmit precoding matrix with any invertible matrix on the right does not change the transmitted signal subspace. Suppose that we choose $\mathbf{B}$ to be the $d^{[k]} \times d^{[k]}$ matrix that is obtained by deleting the bottom $M^{[k]} - d^{[k]}$ rows of $\mathbf{V}^{[k]}$. Then, we have $\mathbf{V}^{[k]}\mathbf{B}^{-1} = \tilde{\mathbf{V}}^{[k]}$, which is a $M^{[k]} \times d^{[k]}$ matrix with the following structure:

$$\tilde{\mathbf{V}}^{[k]} = \left[\begin{array}{ccccc} & & \mathbf{I}_{d^{[k]}} & & \\ \bar{\mathbf{v}}_1^{[k]} & \bar{\mathbf{v}}_2^{[k]} & \bar{\mathbf{v}}_3^{[k]} & \ldots & \bar{\mathbf{v}}_{d^{[k]}}^{[k]} \end{array}\right]$$

where $\mathbf{I}_{d^{[k]}}$ is the $d^{[k]} \times d^{[k]}$ identity matrix and $\bar{\mathbf{v}}_n^{[k]}, \forall n \in \{1,2,...,d^{[k]}\}$ are $\left(M^{[k]} - d^{[k]}\right) \times 1$ vectors. It is easy to argue that there is no other basis representation for the transmitted signal space with fewer variables.

Therefore, by eliminating all superfluous variables for the interference alignment problem, the number of variables to be designed for the precoding filter of the k^{th} transmitter, $\tilde{\mathbf{V}}^{[k]}$, is $d^{[k]}\left(M^{[k]} - d^{[k]}\right)$. Likewise, the actual number of variables to be designed for the interference suppression filter of the k^{th} receiver, $\tilde{\mathbf{U}}^{[k]}$, is $d^{[k]}\left(N^{[k]} - d^{[k]}\right)$. As a result, the total number of variables in the network to be designed is:

$$N_v = \sum_{k=1}^{K} d^{[k]}\left(M^{[k]} + N^{[k]} - 2d^{[k]}\right).$$

B. Proper System Characterization

To formalize the definition of a proper system, we first introduce some notation. We use the notation E_{mn}^{kj} to represent the equation

$$\mathbf{u}_m^{[k]\dagger}\mathbf{H}^{[kj]}\mathbf{v}_n^{[j]} = 0.$$

The set of variables involved in an equation E is indicated by the function $\text{var}(E)$. Clearly

$$|\text{var}(E_{mn}^{kj})| = (M^{[j]} - d^{[j]}) + (N^{[k]} - d^{[k]}),$$

where $|\cdot|$ is the cardinality of a set.

Using this notation, we denote the set of N_e equations as follows:

$$\begin{aligned}\mathcal{E} \;=\; \{E_{mn}^{kj} |\ & j,k \in \mathcal{K}, k \neq j,\\ & m \in \{1,\cdots,d^{[k]}\}, n \in \{1,\cdots,d^{[j]}\}\}.\end{aligned}$$

This leads us to the formal definition of a proper system.

Definition 1. A $\Pi_{k=1}^{K}(M^{[k]} \times N^{[k]}, d^{[k]})$ system is proper if and only if

$$\forall S \subset \mathcal{E}, |S| \leq \left|\bigcup_{E\in S} \text{var}(E)\right|. \qquad (4)$$

In other words, for all subsets of equations, the number of variables involved must be at least as large as the number of equations in that subset.

Note that the above definition can be computationally cumbersome because we have to test all subsets of equations. Luckily, in most cases, the system can be easily checked whether it is proper or improper by using simple inequalities for both symmetric and asymmetric systems. We first start with symmetric systems.

C. Symmetric Systems $(M \times N, d)^K$

For symmetric systems, simply comparing the total number of equations and the total number of variables suffices to determine whether the system is proper or improper.

Theorem 1. *A symmetric system $(M \times N, d)^K$ is proper if and only if*

$$N_v \geq N_e \Rightarrow M + N - (K+1)d \geq 0.$$

Proof: Because of the symmetry, each equation involves the same number of variables and any deficiency in the number of variables shows up in the comparison of the total number of variables versus the total number of equations. Plugging in the values of N_v and N_e computed earlier, we have the result of Theorem 1. ∎

Example 1. Consider the $(2 \times 3, 1)^4$ system. For this system, $M + N - (K+1)d = 2 + 3 - (5) = 0$ so that this system is proper.

Example 2. Consider the $(5 \times 5, 2)^4$ system. For this system, $M + N - (K+1)d = 5 + 5 - 10 = 0$ so that this system is proper.

Remark 1. Theorem 1 implies that for every user to achieve d DoF in a K-user symmetric network, it suffices to have a total of $M + N \geq (K+1)d$ antennas between the transmitter and receiver of a user. The antennas can be distributed among the transmitter and receiver arbitrarily as long as the symmetric nature of the system is preserved. In particular, to achieve K DoF in a K-user symmetric network (1 DoF per user, $d^{[i]} = 1$), we only need a total of $K+1$ antennas between the transmitter and receiver of a user.

Example 3. Consider a 4-user symmetric network, where we wish to achieve 4 DoF. Then, 5 antennas between the transmitter and receiver of a user would suffice, e.g., $(2 \times 3, 1)^4$.

The following corollary shows the limitations of linear interference alignment over constant MIMO channels (with no symbol extensions).

Corollary 1. *The DoF of a proper $(M \times N, d)^K$ system, which is normalized by a single user's DoF in the absence of interference, is upper bounded by:*

$$\frac{dK}{\min(M,N)} \leq 1 + \frac{\max(M,N)}{\min(M,N)} - \frac{d}{\min(M,N)}.$$

Proof: The proof is straightforward from the condition of Theorem 1. ∎

Remark 2. For the case $M = N$, note that the DoF of a proper system is no more than twice the DoF achieved by each user in the absence of interference. Note that for diagonal (time-varying) channels, it is shown in [1] that the DoF of a K-user network is $K/2$ times the number of DoF achieved by each user in the absence of interference. This result shows that the diagonal structure of the channel matrix is very helpful. Going from the case of no structure (general MIMO channels) to diagonal structure, the ratio of total DoF to the single user DoF increases from a maximum value of 2 to $K/2$.

The following corollary identifies the groups of symmetric systems, which are either all proper or all improper.

Corollary 2. *If $(M \times N, d)^K$ system is proper (improper) then so is the $\left((M+1) \times (N-1), d\right)^K$ system as long as $d \leq \min(M, N-1)$. Similarly, if the $(M \times N, d)^K$ system is proper (improper) then so is the $\left((M-1) \times (N+1), d\right)^K$ system as long as $d \leq \min(M-1, N)$.*

Proof: Since the condition in Theorem 1 depends only on $M + N$, it is clear that we can transfer transmit and receive antennas without affecting the proper (or improper) status of the system. ∎

Example 4. Consider the $(2 \times 8, 2)^4$ symmetric system. Again, the DoF of this interference network can be trivially obtained by zero-forcing at each receiver. By successively transferring an antenna from each receiver to transmitter, an equivalent $(5 \times 5, 2)^4$ symmetric system seen in Example 2 is obtained, which is also proper. Thus, the systems $(8 \times 2, 2)^4$, $(7 \times 3, 2)^4$, $(6 \times 4, 2)^4$, $(5 \times 5, 2)^4$, $(4 \times 6, 2)^4$, $(3 \times 7, 2)^4$, and $(2 \times 8, 2)^4$ are in the same group and are all proper.

D. Asymmetric Systems $\Pi_{k=1}^{K}\left(M^{[k]} \times N^{[k]}, d^{[k]}\right)$

For asymmetric systems, if the system is improper, simply comparing the total number of equations and the total number of variables may suffice.

Theorem 2. *An asymmetric system $\Pi_{k=1}^{K}(M^{[k]} \times N^{[k]}, d^{[k]})$ is improper if*

$$N_v < N_e \Leftrightarrow \sum_{k=1}^{K} d^{[k]}\left(M^{[k]} + N^{[k]} - 2d^{[k]}\right) < \sum_{\substack{k,j \in \mathcal{K} \\ k \neq j}}^{K} d^{[k]} d^{[j]}. \tag{5}$$

Example 5. Consider the $(5 \times 5, 3)(5 \times 5, 2)^3$ system. There are 60 equations in total and therefore, there are $2^{60} - 1$ subsets of equations. Testing each of them could be very challenging. However, since the total number of variables $N_v = 48$ is less than the number of equations, the system is easily seen to be improper.

Note that we can sometimes identify the bottleneck equations in the system by checking the equations with the fewest number of variables, i.e., the equations involving the fewest number of transmitter and receiver antennas.

Example 6. Consider the simple system $(2 \times 1, 1)^2$, which is clearly feasible (proper) because simple zero-forcing is enough for achievability. However, now consider the $(2 \times 1, 1)(1 \times 2, 1)$ system, which also has the same total number of equations N_e and variables N_v as the $(2 \times 1, 1)^2$ system. Thus, only comparing N_v and N_e would mislead one to believe that this system is proper. However, suppose that we only check the subset of equations between the transmitter 2 and receiver; that is, $S = \{E_{11}^{12}\}$ so that $|S| = 1$ and $\text{var}(E_{11}^{12}) = 0$. Thus, this system has an equation with zero variables, which makes the system improper and therefore, infeasible.

Example 7. Several interesting cases emerge from applying the condition (5). For example, consider the 2-user interference network $(2 \times 3, 1)(3 \times 2, 1)$, where a total of 2 DoF is desired. It is easily checked that this system is proper and the achievable scheme is described in [3]. Now, consider the 4-user interference network, which consists of two sets of these networks, all interfering with each other $(2 \times 3, 1)^2(3 \times 2, 1)^2$, where a total of 4 DoF is desired. By using (5), it is easily verified that this is a proper system. Surprisingly, by simply going from two users to four users, DoF is doubled in this case.

VI. Numerical Results

We tested numerous interference alignment problems, both symmetric and asymmetric, and especially including each of the examples presented in this paper by using the numerical algorithm in [2]. In every case so far, we have found the results to be consistent with the guiding intuition of this work; that is, proper systems are almost surely feasible and improper systems are not.

In this section, we provide numerical results for a few interesting and representative cases. The results are in terms of the interference percentage, which is defined in [2]. i.e., the fraction of the interference power that is existent in the dimensions reserved for the desired signal.

In Fig. 1, the interference percentages versus the total number of beams are shown. The total number of beams starts from expected total DoF of each network. Therefore, after the first point on the x-axis, where excess total DoF is demanded the interference percentage of each network is not zero. The nonzero interference percentage indicates that interference alignment is not possible for the demanded total DoF.

Therefore, by observing zero interference percentages on the DoF point in Fig. 1, we show that the numerical results are consistent with our statements in Section V that these networks are proper and thus, feasible.

VII. The Direct Links Between Proper (Improper) and Feasible (Infeasible) Systems

Our approach in this paper is to view the alignment problem for an interference network as a multivariate polynomial system (1) and determine its solvability. Our intuition in this paper is that if the system is proper, the corresponding polynomial system is almost surely solvable; that is, the system is feasible. To rigorously prove that a polynomial system is solvable, specialized versions of Bezout's (more widely known) and Bernshtein's theorems can be used, which are both interested in the number of solutions of a multivariate polynomial system. Both theorems provide the exact number of solutions under specific conditions. For both theorems, the coefficients must be independent random variables (can also be called generic choices of coefficients as seen in Mathematics terminology, which is explained in the journal version of this

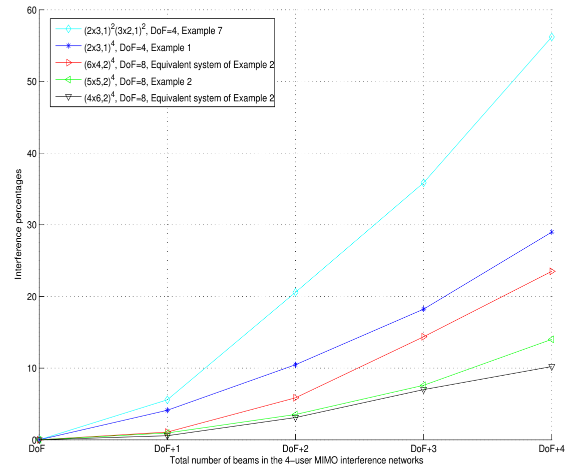

Fig. 1. Interference percentages as a function of the total number of beams in the networks (DoF: Expected total degrees of freedom. DoF+i (i=1,2,3,4): Excess total degrees of freedom).

paper [5]). However, for Bezout's and Bernshtein's theorems, the polynomial system must be dense and sparse, respectively, which we explain next.

Let $\deg(f_i)$ denote the degree of the polynomial f_i. e.g.,

$$f_1 = x^3y + xy + y, \ \deg(f_1) = 4.$$

For a dense polynomial system, every polynomial f_i in the system must have all combinations of monomials up to $\deg(f_i)$. e.g.,

$$f_2 = x^4 + y^4 + x^3y + xy^3 + x^2y^2 + \cdots + 1, \ \deg(f_2) = 4.$$

Therefore, f_1 and f_2 are sparse and dense polynomials, respectively.

For interference networks, the polynomial systems are sparse in nature. However, for interference networks, where at least one user wishes to achieve more than 1 DoF ($d^{[i]} > 1$), the coefficients are not generic because the channel matrix $\mathbf{H}^{[kj]}$ of that user reoccurs more than once in the polynomial system. In other words, there are dependent random coefficients in the polynomial system. In this case, Bernshtein's theorem provides only an upper bound for the number of solutions. Therefore, we cannot use Bernshtein's theorem for the cases with $d^{[i]} > 1$ to prove that the polynomial system is almost surely solvable. On the other hand, for interference networks, where every user wishes to achieve 1 DoF ($d^{[i]} = 1$), the coefficients in the polynomial system are generic. Therefore, we can use Bernshtein's theorem for these cases, which states that the mixed volume of the Newton polytopes of the support sets of a multivariate polynomial system is exactly equal to the number of solutions of that system. Bezout's and Bernshtein's theorems are explained in the journal version of this paper.

Next, we show that some proper (improper) systems with $d^{[i]} = 1$ do (not) satisfy the feasibility condition (1) almost surely by using Bernshtein's theorem. We use the softwares mentioned in [4] in order to rigorously find the mixed volumes of some important cases.

Example 8. For the systems $(2 \times 3, 1)^4$ and $(2 \times 3, 1)^2(3 \times 2, 1)^2$, the mixed volumes are 9 and 8, respectively. In other words, these polynomial systems with independent random coefficients are solvable almost surely since each has nonzero mixed volume, which is equal to the exact number of common solutions.

Example 9. Now, consider the system $(2 \times 2, 1)^3(3 \times 5, 1)$, which is infeasible according to the simulation result. Since the subset of equations, which is obtained by shutting down the fourth receiver has 9 equations and 8 variables, this system is improper. The mixed volume of this system is 0. In other words, the corresponding polynomial system with independent random coefficients is not solvable almost surely.

Note once again that we only provide mixed volumes of only some cases since mixed volume computation is #P-complete [4].

For the cases with $d^{[i]} > 1$, where the system has dependent random coefficients, an innovative approach in algebraic geometry is required in order to find a Bernshtein's equivalent theorem, which would provide the exact number of solutions. Moreover, the structure of polynomial system is important for its solvability. As mentioned in Remark 2, for diagonal (time-varying) channels, the DoF of a K-user network is $K/2$ times the number of DoF achieved by each user ($d^{[i]} > 1$) in the absence of interference [1]. Note that for the corresponding polynomial system, $N_e > N_v$. Although interference networks with diagonal channels are improper (because $N_e > N_v$), the interference alignment is feasible. However, our intuition is still not violated; that is, for K-user MIMO interference networks with *constant* channel coefficients, improper systems are almost surely infeasible.

VIII. Conclusion

We propose an analytical method to determine the feasibility of linear interference alignment over constant MIMO channels that have no structure (i.e., no symbol extensions). Our approach to determine the feasibility of a network is to count the number of equations and variables in the corresponding polynomial system. We define a system as proper if the cardinality of every subset of equations in the corresponding polynomial system is less than or equal to the number of variables involved in that subset of equations. Otherwise, we define the system as improper.

The guiding intuition in this paper is that proper systems are almost surely feasible while improper systems are almost surely infeasible. We use Bernshtein's theorem to prove the direct link between proper (improper) and feasible (infeasible) systems for some important cases with 1 DoF per user.

Note that while it is not stated explicitly, the feasibility conditions for linear alignment (and the definition of proper systems) also include the general outer bounds on the DoF that follow from [3], i.e.,

$$d^{[i]} \leq \min(M^{[i]}, N^{[i]}) \text{ and}$$

$$\begin{aligned} d^{[i]} + d^{[j]} \leq \quad & \min\big(M^{[i]} + M^{[j]}, N^{[i]} + N^{[j]}, \\ & \max(M^{[i]}, N^{[j]}), \max(M^{[j]}, N^{[i]})\big), \end{aligned}$$

for all $i, j \in \mathcal{K}$.

While we do not focus on these outer bounds in this paper, they must always be checked first to determine infeasibility. This is especially important to keep in mind because these bounds are not explicitly implied by the definition of a proper system as stated in this paper.

Acknowledgment

The first author would like to thank V. R. Cadambe at University of California, Irvine for valuable discussions on interference alignment.

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