Interference Alignment at Finite SNR: General Message Sets

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***Abstract*—Interference alignment has proven to be a powerful tool in characterizing the high SNR behavior of wireless networks. Using this technique, Cadambe and Jafar showed that the sum degrees-of-freedom for the M-user interference channel is $M/2$ rather than 1 as was originally expected. In recent work, we showed that these gains are not limited to the high SNR regime. In fact, for the interference channel, each user can achieve at least half its interference-free capacity at any SNR. The key to this result was a new achievability technique called ergodic interference alignment. In this paper, we extend this technique to include more general message sets. Specifically, we consider the multicast case where each transmitter has a message destined for more than one receiver. We also look at the X channel configuration for 2 receivers where each transmitter has an independent message for each receiver.**

I. Introduction

Significant progress has been made recently towards the capacity of the interference channel. The work of Etkin, Wang, and Tse characterized the capacity region of the two-user Gaussian interference channel to within 1 bit [1]. However, extending this result to the M-user Gaussian interference channel is more involved than one might originally suspect. This is in part due to the possibility of interference alignment [2], [3] which allows the transmitters to exploit the channel to ensure that all of the interference at any one receiver only occupies half of the dimensions, leaving the other half for the desired signal.

One also has to deal with the fact that parallel interference channels are inseparable [4], [5]. That is, one cannot separately code for each channel instance and simply allocate power across these codes since any variation in the channel could be potentially exploited for alignment. By exploiting these properties, Cadambe and Jafar characterized the behavior of the M-user interference channel in the high signal-to-noise ratio (SNR) limit [3]. Specifically, they showed that the sum degrees-of-freedom is $\frac{M}{2}$. Recent work by Motahari *et al.* has shown that this is also achievable for fixed channel matrices as well [6]. High SNR results are also available for more involved message structures such as the X channel framework where each transmitter has an independent message for each receiver [2], [7].

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In recent work, we showed that these high SNR results are also available at finite SNRs for time-varying channels through a new technique called ergodic interference alignment [8]. Essentially, each channel instance in a time-varying interference channel can be paired with a complement to exactly cancel out the undesired signals. Concurrent work by Jeon and Chung proposed a similar scheme for finite field networks [9].

In this paper, we extend our results to more general classes of message sets including the X channel configuration with 2 receivers. We also consider the multicast case where each transmitter has one message and each receiver is interested in more than one of these messages.

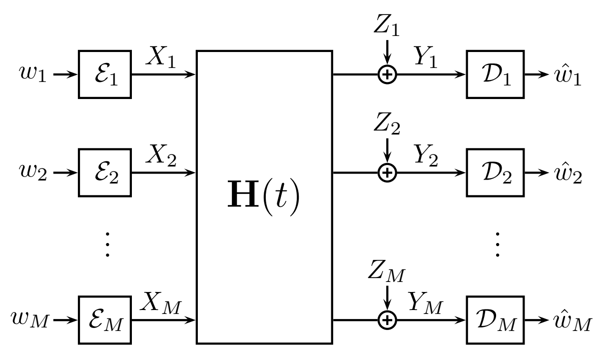

Fig. 1. M-user interference channel.

II. Problem Statement

There are M transmitters and N receivers that communicate across a common wireless medium over T time steps. We will use a fast fading model wherein the channel coefficients change at every time step. At time t, the channel output seen by the n^{th} receiver is given by:

$$y_n[t] = \sum_{m=1}^{M} h_{nm}[t] x_m[t] + z_n[t] \tag{1}$$

where $h_{nm}[t]$ indicates the channel coefficient between the m^{th} transmitter and the n^{th} receiver, $x_m[t]$ represents the channel input from the m^{th} transmitter, and $z_n[t]$ is additive noise. The inputs, outputs, noise, and channel coefficients all take values on the complex field $\mathbb{C}$.

We assume that at each time step each channel coefficient is drawn i.i.d. from a distribution with uniform phase. We also require that channel coefficients are independent of one another (although they may be drawn from different distributions). The uniform phase condition is critical for our achievable scheme as it guarantees that:

$$P(h_{nm} = a) = P(h_{nm} = ae^{jb}) \tag{2}$$

for all values of $a \in \mathbb{C}$ and $b \in [0, 2\pi)$.

The transmitters and receivers have causal knowledge of all coefficients. That is, before time t, each transmitter and receiver is given $h_{nm}[t]$ for all n and m.

The channel inputs are subject to the usual power constraint:

$$\frac{1}{T}\sum_{t=1}^{T} |x_m[t]|^2 \leq P_m \tag{3}$$

for some $P_m \in \mathbb{R}_+$.

We also assume that each additive noise term is i.i.d. across time and drawn from a circularly symmetric complex Gaussian distribution with variance σ_n^2, $z_n[t] \sim \mathcal{CN}(0, \sigma_n^2)$.

Our goal is to reliably communicate messages from transmitters to receivers at the highest rates possible. In the most general case, each transmitter would have a message intended for each subset of receivers. In this paper, we will only consider a few limiting cases of this general model. Each message will be drawn independently and uniformly from the set $\{1, 2, \ldots, 2^{n\tilde{R}}\}$ for some choice of $\tilde{R}$. Each transmitter has one or more such messages which it maps into T channel inputs with an encoding function $\mathcal{E}_m$. Each receiver, after observing T channel outputs attempts to recover its desired messages with a decoding function $\mathcal{D}_n$.

Definition 1: The rate R is achievable for a given message if for all $\epsilon > 0$ and T large enough, there exist fixed encoders and decoders, such that a message drawn at rate $\tilde{R} > R - \epsilon$ can be recovered at all intended destinations with total probability of error not exceeding ϵ.

We will use boldface lowercase letters to denote column vectors and boldface uppercase letters to denote matrices. Let $\mathbf{h}^*$ denote the Hermitian (or conjugate) transpose of $\mathbf{h}$.

We now describe the special cases of message sets.

A. Standard Message Set

In the classical interference channel model, there are M transmitter-receiver pairs that wish to exchange messages. Let w_m denote the message from the m^{th} transmitter to the m^{th} receiver and let R_m denote its rate. See Figure 1.

B. Multicast Message Set

Now consider the case where there are M transmitters, each with a single message w_m with rate R_m, and N receivers that want exactly L messages each. For simplicity, we will assume that all messages are requested by the same number of receivers. (Note that this implicitly assumes that $\frac{NL}{M}$ is an integer.) Denote the subset of receivers that want message m by $\mathcal{S}_m$. In Figure 2, we provide a block diagram of a case with $M = 4$ transmitters, $N = 4$ receivers, and message requests $\mathcal{S}_1 = \{1, 2\}, \mathcal{S}_2 = \{2, 3\}, \mathcal{S}_3 = \{3, 4\}$, and $\mathcal{S}_4 = \{4, 1\}$.

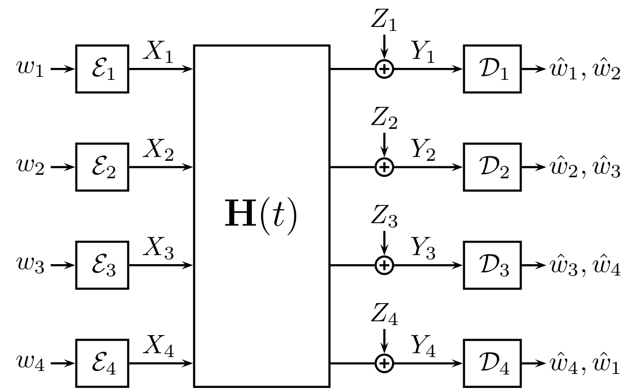

Fig. 2. Multicast message set for $M = 4$ transmitters and $N = 4$ receivers.

C. X Message Set

In this special case, each transmitter has an independent message for each receiver. Let $w_{m\ell}$ denote the message sent from the m^{th} transmitter to the ℓ^{th} receiver where ℓ takes values from 1 to N. Each message has rate $R_{m\ell}$. In Figure 3, we give a block diagram of an X message set for $M = 2$ transmitters and $N = 2$ receivers.

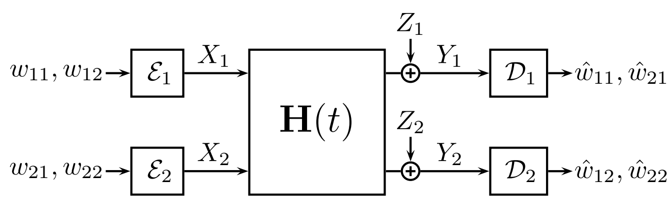

Fig. 3. X message set for $M = 2$ transmitters and $N = 2$ receivers.

III. Channel Quantization

Our scheme relies on matching up time indices based on the phase and magnitude of the channel coefficients. In order to ensure that most channel coefficients are matched, we need strong typicality and for this we need the channel coefficients to take values on a finite set. We will accomplish this by quantizing the channel coefficients with the resolution determined by our desired gap to the target rate. By taking finer and finer quantizations, we can achieve the target rate in the limit.

First, we will threshold the channel coefficients by throwing out any time indices that contain a channel coefficient magnitude larger than h_{MAX}. This threshold is chosen such that the probability that one or more channel coefficients violate it in one time instant is τ.

Each channel coefficient is quantized as follows. The complex plane (up to radius h_{MAX}) is divided up into κ disjoint rings of equal width. These rings are further subdivided into equal segments based on η angles spaced equally between

0 and 2π where η. See Figure REF for a diagram. Each segment is a quantization cell for the channel coefficients. The parameters κ and η are chosen such that the maximum distance between any two points in any segment is ν where $\nu > 0$ will be specified later. We also assign all channel coefficients with magnitude larger than h_{MAX} to an erasure symbol.

Lemma 1: Given $h_k \in \mathbb{C}$ satisfying $|h_k| < h_{\text{MAX}}$ for $k = 1, 2, \dots, K$, let $\hat{h}_k$ be any other element of the quantization cell of h_k. For any $a_k \in \mathbb{C}$, the following upper bound holds:

$$\left|\sum_{k=1}^{K} a_k \hat{h}_k\right| \leq \left|\sum_{k=1}^{K} a_k h_k\right| + \nu \sum_{k=1}^{K} |a_k|. \tag{4}$$

Furthermore, if a_k is chosen such that $\left|\sum_{k=1}^{K} a_k h_k\right| > \nu \sum_{k=1}^{K} |a_k|$, then the following lower bound holds:

$$\left|\sum_{k=1}^{K} a_k \hat{h}_k\right| \geq \left|\sum_{k=1}^{K} a_k h_k\right| - \nu \sum_{k=1}^{K} |a_k|. \tag{5}$$

Proof: First, write each $\hat{h}_k = h_k + e_k$ where $|e_k| < \nu$. Now, we have by the triangle inequality:

$$\begin{aligned}\left|\sum_{k=1}^{K} a_k \hat{h}_k\right| = \left|\sum_{k=1}^{K} a_k (h_k + e_k)\right| &\leq \left|\sum_{k=1}^{K} a_k h_k\right| + \left|\sum_{k=1}^{K} a_k e_k\right| \\ &\leq \left|\sum_{k=1}^{K} a_k h_k\right| + \nu \sum_{k=1}^{K} |a_k|.\end{aligned}$$

Similarly, by the reverse triangle inequality, we have that:

$$\left|\sum_{k=1}^{K} a_k (h_k + e_k)\right| \geq \left|\sum_{k=1}^{K} a_k h_k\right| - \left|\sum_{k=1}^{K} a_k e_k\right| \tag{6}$$

$$\geq \left|\sum_{k=1}^{K} a_k h_k\right| - \nu \sum_{k=1}^{K} |a_k|. \tag{7}$$

■

The above lemma will allow us to show that for ν small enough, matching up channel coefficients based on their quantization cells has a negligible effect on the overall rate.

For the remainder of this paper, we will treat all channel coefficients as if they are quantized. Thus, we can treat them as if drawn from a discrete set where the probability of each quantization cell is given by the total probability of all channel coefficients in that cell. By construction, all quantization cells at a given radius have the same probability.

We now recall the notion of strong typicality for sequences of discrete random variables. Let $\mathbf{H} = \{h_{nm}\}$ be the matrix of channel coefficients which takes values in the set $\mathcal{H}$ and let $\mathbf{H}^{[T]}$ denote the sequence of such matrices over T channel uses. Let $N(\mathbf{H}|\mathbf{H}^{[T]})$ denote the number of times the channel matrix $\mathbf{H}$ occurs in the sequence $\mathbf{H}^{[T]}$.

Definition 2: A sequence of channel matrices, $\mathbf{H}^{[T]}$, is γ-typical if:

$$\left|\frac{1}{T} N(\mathbf{H}|\mathbf{H}^{[T]}) - P(\mathbf{H})\right| \leq \gamma \quad \forall \mathbf{H} \in \mathcal{H} \tag{8}$$

where $P(\mathbf{H})$ is the probability of channel $\mathbf{H} \in \mathcal{H}$ under the channel model. Let A_γ^T denote the set of all γ-typical channel matrix sequences.

Lemma 2 (Csiszar-Körner 2.12): For any i.i.d. sequence of channel matrices, $\mathbf{H}^{[T]}$, the probability of the set of all γ-typical sequences, A_γ^T, is lower bounded by:

$$P(A_\gamma^T) \geq 1 - \frac{|\mathcal{H}|}{4T\gamma^2} \tag{9}$$

For a proof, see [10]. Due to the channel quantization, the size of $\mathcal{H}$ is $|\mathcal{H}| = (\kappa\eta + 1)^{MN}$.

We will only work with sequences of channel matrices that are γ-typical and declare errors on the rest. This ensures that nearly all time indices can be matched up appropriately.

IV. Equation Coefficients

The key to our achievability proof is creating equations of the transmitted codewords at the receivers which can be solved for the desired messages. Essentially, at well-chosen time indices, all encoders retransmit symbols that were sent at an earlier time. This has the effect of giving the decoders equations with the symbols as the variables and the coefficients given by the channel.

First, we assume that all channel coefficients are quantized as described in Section III. In order to ensure that all channel coefficients can be appropriately matched, we only consider matchings between individual coefficients of the same magnitude. Since the phase of each coefficient is assumed to be uniform, all equations will have the same probability.

The goal is to specify a set of K equations such that the receiver can recover its desired messages. Each receiver is free to choose its own equations and repeat transmissions only occur when all receivers see the appropriate equations. These equations are fully specified by phase shifts $\phi_{nm}^{(k)}$ (with $\phi_{nm}^{(1)} = 1$ by default). For ease of analysis, these are restricted to take values on the set $\{e^{jb} : b = 0, \frac{2\pi}{\eta}, \frac{4\pi}{\eta}, \dots, \frac{2\pi(\eta-1)}{\eta}\}$ so that they are in correspondence with the quantization cells. We write the phase shifts at each receiver for the k^{th} equation in matrix form below:

$$\mathbf{\Phi}^{(k)} = \begin{bmatrix} \phi_{11}^{(k)} & \phi_{12}^{(k)} & \cdots & \phi_{1M}^{(k)} \\ \phi_{21}^{(k)} & \phi_{22}^{(k)} & \cdots & \phi_{2M}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1}^{(k)} & \phi_{N2}^{(k)} & \cdots & \phi_{NM}^{(k)} \end{bmatrix} \tag{10}$$

We now show how to match up channel matrices based on these phase shifts. Let $\mathbf{A} \odot \mathbf{B} \triangleq \{a_{nm}b_{nm}\}$ denote the Hadamard product of $\mathbf{A}$ and $\mathbf{B}$. We divide up the T channel uses into K intervals of length T/K. Using Lemma 2, we have that for T large enough, all K intervals will be γ-typical with probability at least $(1 - \frac{\epsilon}{2})$. By Definition 2, this means that the number of occurrences of each possible channel matrix in each interval is bounded as follows:

$$\frac{T}{K}\left(P(\mathbf{H}) - \gamma\right) \leq N(\mathbf{H}|\mathbf{H}^{[T/K]}) \leq \frac{T}{K}\left(P(\mathbf{H}) + \gamma\right) \tag{11}$$

for all $\mathbf{H} \in \mathcal{H}$.

Each encoder uses a length $T_{\mathcal{C}}$ codebook $\mathcal{C}_m$ with rate $\tilde{R}_m$ generated i.i.d. from a circularly symmetric Gaussian distribution with variance $P_M - \epsilon$.

Assume that the intervals are γ-typical. Each matrix will occur at least $\frac{T}{K}\left(P(\mathbf{H}) - \gamma\right)$ in each interval. During the first time interval, each encoder transmits a new symbol from its codeword at each time step t unless:

1) The channel matrix $\mathbf{H}[t]$ contains one or more elements with magnitude larger than h_{MAX}.
2) The channel matrix $\mathbf{H}[t]$ does not violate the threshold but has already occurred at least $\frac{T}{K}\left(P(\mathbf{H}) - \gamma\right)$ times.

The number of useable time slots is equal to:

$$T_{\mathcal{C}} = \frac{T}{K} \sum_{\mathbf{H}:|h_{nm}|<h_{\text{MAX}}} \left(P(\mathbf{H}) - \gamma\right) \tag{12}$$

$$= \frac{T}{K}\left(1 - \tau - (\kappa\eta)^{MN}\gamma\right) \tag{13}$$

and we set this to be the length of the codebooks.

We then match up used time slots from the first interval with time slots in the remaining $K-1$ intervals. During the k^{th} time interval, when the channel matrix $\mathbf{\Phi}^{(k)} \odot \mathbf{H}$ occurs, it is matched with the first unmatched time slot from the first interval that had channel matrix $\mathbf{H}$. The encoders retransmit the symbols from the first interval for all matched time slots. Since are intervals are assumed to be γ-typical, all $\frac{T}{K}\left(P(\mathbf{H}) - \gamma\right)$ time indices for each matrix from the first interval can be successfully matched.

After T time steps, receiver n has access to equations of the form:

$$y_n^{(1)} = \sum_{m=1}^{M} h_{nm} x_m + z_n^{(1)} \tag{14}$$

$$y_n^{(2)} = \sum_{m=1}^{M} \phi_{nm}^{(2)} h_{nm} x_m + z_n^{(2)} \tag{15}$$

$$\vdots \tag{16}$$

$$y_n^{(K)} = \sum_{m=1}^{M} \phi_{nm}^{(K)} h_{nm} x_m + z_n^{(K)} \tag{17}$$

where x_m are the symbols from a single index in the chosen codewords, h_{nm} are fixed channel coefficients (up to the quantization cells), and $z_n^{(k)}$ are the noise terms from the matched time indices.

Given these equations, the receiver attempts to recover the symbols from its desired by applying linear transformations. For each desired symbol x_ℓ, the receiver forms an estimate:

$$u_{n\ell} = \sum_{k=1}^{K} a_{n\ell}^{(k)} y_n^{(k)} \tag{18}$$

$$= \sum_{m=1}^{M} h_{nm} x_m \sum_{k=1}^{K} a_{n\ell}^{(k)} \phi_{nm}^{(k)} + \sum_{k=1}^{K} a_{n\ell}^{(k)} z_n^{(k)} \tag{19}$$

for some choice of $a_{n\ell}^{(k)} \in \mathbb{C}$.

Let $\delta[\ell]$ be the Kronecker delta function. The following lemma establishes a worst-case signal-to-interference-and-noise ratio (SINR) for the channel between x_ℓ and u_ℓ.

Lemma 3: Assume that $\phi_{nm}^{(k)}$ and $a_{n\ell}^{(k)}$ are chosen such that $\sum_{k=1}^{K} a_{n\ell}^{(k)} \phi_{nm}^{(k)} = \beta\delta[\ell - m]$ for some $\beta > 0$. Then, the AWGN channel between symbol x_ℓ and estimate $u_{n\ell}$ has an SINR that is lower bounded by:

$$\mathsf{SINR} \geq \frac{P_\ell\left(\beta|h_{n\ell}| - \nu\sum_{k=1}^{K}|a_{n\ell}^{(k)}|\right)^2}{\nu^2\left(\sum_{k=1}^{K}|a_{n\ell}^{(k)}|\right)^2\sum_{m\neq\ell}P_m + \sigma_n^2\sum_{k=1}^{K}|a_{n\ell}^{(k)}|^2}.$$

Furthermore, as ν goes to zero, we have that:

$$\lim_{\nu\downarrow 0}\mathsf{SINR} \geq \frac{P_\ell\beta^2|h_{n\ell}|^2}{\sigma_n^2\sum_{k=1}^{K}|a_{n\ell}^{(k)}|^2}. \tag{20}$$

Proof: First, we lower bound the signal power which is slightly diminished due to channel quantization. By Lemma 1, the signal power is lower bounded as follows:

$$P_\ell\left|\sum_{k=1}^{K} a_{n\ell}^{(k)} h_{n\ell}\phi_{n\ell}^{(k)}\right|^2 \geq P_\ell\left(\beta|h_{n\ell}| - \nu\sum_{k=1}^{K}|a_{n\ell}^{(k)}|\right)^2. \tag{21}$$

Now, we upper bound the power of the remaining interference due to quantization. Again, by Lemma 1, the power of each interferer $m \neq \ell$ at receiver n is upper bounded as follows:

$$P_m\left|\sum_{k=1}^{K} a_{nm}^{(k)} h_{nm}\phi_{nm}^{(k)}\right|^2 \leq P_m\left(0 + \nu\sum_{k=1}^{K}|a_{n\ell}^{(k)}|\right)^2. \tag{22}$$

Finally, the noise terms $z_n^{(k)}$ are each weighted by $a_{n\ell}^{(k)}$ in $u_{n\ell}$. Since the noise is i.i.d. across time, we get that $\sigma_n^2\sum_{k=1}^{K}|a_{n\ell}^{(k)}|^2$ as the power of the sum of the noise. ■

The requirements on $a_{n\ell}^{(k)}$ and $\phi_{nm}^{(k)}$ in Lemma 3 can be restated as a matrix condition. Let $\mathbf{A}_n$ and $\mathbf{\Phi}_n$ be defined as

$$\mathbf{A}_n = \begin{bmatrix} a_{n1}^{(1)} & a_{n1}^{(2)} & \cdots & a_{n1}^{(K)} \\ a_{n2}^{(1)} & a_{n2}^{(2)} & \cdots & a_{n2}^{(K)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{nM}^{(1)} & a_{nM}^{(2)} & \cdots & a_{nM}^{(K)} \end{bmatrix} \tag{23}$$

$$\mathbf{\Phi}_n = \begin{bmatrix} \phi_{n1}^{(1)} & \phi_{n2}^{(1)} & \cdots & \phi_{nM}^{(1)} \\ \phi_{n1}^{(2)} & \phi_{n2}^{(2)} & \cdots & \phi_{nM}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1}^{(K)} & \phi_{n2}^{(K)} & \cdots & \phi_{nM}^{(K)} \end{bmatrix}. \tag{24}$$

Assume that receiver n wants messages with indices $\ell_1, \ell_2, \ldots, \ell_I$. Then the following condition is equivalent to $\sum_{k=1}^{K} a_{n\ell}^{(k)}\phi_{nm}^{(k)} = \beta\delta[\ell_i - m]$ for $i = 1, 2, \ldots, I$:

$$\mathbf{A}_n\mathbf{\Phi}_n = \beta\begin{bmatrix} \delta[\ell_1 - 1] & \delta[\ell_1 - 2] & \cdots & \delta[\ell_1 - M] \\ \delta[\ell_2 - 1] & \delta[\ell_2 - 2] & \cdots & \delta[\ell_2 - M] \\ \vdots & \vdots & \ddots & \vdots \\ \delta[\ell_I - 1] & \delta[\ell_I - 2] & \cdots & \delta[\ell_I - M] \end{bmatrix}$$

V. Standard Message Set

This setting is often referred to as simply the "M-user interference channel." In previous work, we developed an interference alignment scheme that allows each user to achieve slightly more than half its interference-free rate at any SNR [8]. For completeness, we repeat this theorem and its proof below.

Theorem 1: The following rates are achievable for M users with the standard message set:

$$R_m = \frac{1}{2}E\left[\log\left(1+2|h_{mm}|^2\frac{P_m}{\sigma_m^2}\right)\right] \tag{25}$$

Proof: Receiver m will use $K=2$ equations to recover its desired messages w_m. The phase shifts are given by $\phi_{mm}^{(2)}=1$ and $\phi_{nm}^{(2)}=-1$ for $n\neq m$. The messages are recovered using $a_{mm}^{(1)}=a_{mm}^{(2)}=1$. It follows that $\sum_{k=1}^{2}a_{mm}^{(k)}\phi_{nm}^{(k)}=2\delta[n-m]$. Essentially, we subtract out everything except the desired message using the second equation. From Lemma 3, receiver m can recover the channel input X_m with SINR_m no worse than:

$$\mathsf{SINR}_m \geq \frac{P_m\left(2|h_{mm}|-2\nu\right)^2}{4\nu^2\sum_{\ell\neq m}P_\ell+2\sigma_m^2}.$$

By choosing ν,γ and τ small enough and T large enough, we can guarantee that SINR_m is such that we can find a good code with probability of error at most $\frac{\epsilon}{2}$ and rate at least

$$\frac{1}{2}E\left[\log\left(1+2|h_{mm}|^2\frac{P_m}{\sigma_m^2}\right)\right]-\epsilon. \tag{26}$$

Recall also that with probability $\frac{\epsilon}{2}$ the channel is not γ-typical. Since the total probability of error is less than ϵ, we get the desired result. ■

Remark 1: It is also possible to optimize the power allocation over time. By water-filling over the channel realizations, higher rates can certainly be achieved. However, in this paper we only focus on the rates possible through equal power allocation. See [11] for a study of power allocation for fast fading 2-user interference channels.

VI. Multicast Message Set

Recall that in this setting there are M transmitters with one message each and N receivers. Also, $\mathcal{S}_m$ denotes the index set of all receivers that want message m with $|\mathcal{S}_m|=L$. Let $\omega_K \triangleq e^{j2\pi/K}$ denote the K^{th} root of unity and let $\mathbf{W}_K$ be the size K discrete Fourier transform (DFT) matrix:

$$\mathbf{W}_K = \begin{bmatrix} \omega_K^0 & \omega_K^0 & \omega_K^0 & \cdots & \omega_K^0 \\ \omega_K^0 & \omega_K^1 & \omega_K^2 & \cdots & \omega_K^{K-1} \\ \omega_K^0 & \omega_K^2 & \omega_K^4 & \cdots & \omega_K^{2(K-1)} \\ \vdots & \vdots & \ddots & \vdots & \\ \omega_K^0 & \omega_K^{K-1} & \omega_K^{2(K-1)} & \cdots & \omega_K^{(K-1)^2} \end{bmatrix}. \tag{27}$$

Recall that the inverse DFT matrix has the following form:

$$\mathbf{W}_K^{-1} = \frac{1}{K}\begin{bmatrix} \omega_K^0 & \omega_K^0 & \cdots & \omega_K^0 \\ \omega_K^0 & \omega_K^{-1} & \cdots & \omega_K^{-(K-1)} \\ \omega_K^0 & \omega_K^{-2} & \cdots & \omega_K^{-2(K-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_K^0 & \omega_K^{-(K-1)} & \cdots & \omega_K^{-(K-1)^2} \end{bmatrix}. \tag{28}$$

Theorem 2: For the multicast message set, the following rates are achievable:

$$R_m = \min_{n\in\mathcal{S}_m}\frac{1}{L+1}E\left[\log\left(1+(L+1)|h_{nm}|^2\frac{P_m}{\sigma_n}\right)\right] \tag{29}$$

Proof: Without loss of generality, assume that receiver n is interested in messages from transmitters $1,2,\ldots,L$. (Otherwise, just reindex the transmitters.) To recover these messages, the receiver needs $L+1$ equations: L for the messages and one for the interference. The phase coefficients are chosen from the DFT matrix of size $\mathbf{W}_{L+1}$:

$$\phi_{n\ell}^{(k)} = \begin{cases} \exp\left(\frac{j2\pi(\ell-1)(k-1)}{L+1}\right) & \ell=1,2,\ldots,L \\ \exp\left(\frac{j2\pi L(k-1)}{L+1}\right) & \ell=L+1,L+2,\ldots,M \end{cases}$$

and the recovery coefficients are chosen from the inverse DFT matrix $\mathbf{W}_{L+1}$ scaled by $L+1$:

$$a_{n\ell}^{(k)} = \exp\left(\frac{-j2\pi(\ell-1)(k-1)}{L+1}\right) \text{ for } \ell=1,2,\ldots,L$$

This immediately gives that $\mathbf{A}_n\mathbf{\Phi}_n=(L+1)\mathbf{I}$. We can now apply Lemma 3 to show that the resulting channel from each transmitter has $\mathsf{SINR}_{n\ell}$ no worse than:

$$\mathsf{SINR}_{n\ell} \geq \frac{P_\ell\left((L+1)|h_{n\ell}|-(L+1)\nu\right)^2}{(L+1)^2\nu^2\sum_{\ell\neq m}P_\ell+(L+1)\sigma_n^2}.$$

Since the message from transmitter ℓ is multicast to several receivers, the rate is governed by the worst channel:

$$\mathsf{SINR}_\ell = \min_{n\in\mathcal{S}_\ell}\mathsf{SINR}_{n\ell} \tag{30}$$

By choosing ν,γ and τ small enough and T large enough, we can guarantee that SINR_ℓ is such that we can find a good code to all receivers with probability of error at most $\frac{\epsilon}{2}$ and rate at least

$$\min_{n\in\mathcal{S}_\ell}\frac{1}{L+1}E\left[\log\left(1+(L+1)|h_{n\ell}|^2\frac{P_\ell}{\sigma_n^2}\right)\right]-\epsilon. \tag{31}$$

Recall also that with probability $\frac{\epsilon}{2}$ the channel is not γ-typical. Since the total probability of error is less than ϵ, we get the desired result. ■

Note that if we simply extended the scheme from Theorem 1, to cancel out the interference from each desired message one-by-one, we do not achieve the same rates. Specifically, if we have

$$\mathbf{\Phi}_1 = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 1 & -1 & \cdots & -1 \end{bmatrix} \tag{32}$$

then we only achieve a rate of

$$R_\ell = \min_{n\in\ell} \frac{1}{L+1} E\left[\log\left(1+2|h_{n\ell}|^2\frac{P_\ell}{\sigma_n^2}\right)\right]. \quad (33)$$

VII. X Message Set

Each transmitter m has an independent message $w_{m\ell}$ for each receiver ℓ . Unlike the previous two cases, we cannot hope for the channel to generate an independent coefficient for every message. Transmitters must artificially separate their messages by premultiplying them by phases. This leaves us with fewer variables to work with to align the interference at every receiver. As a result, we are currently only able to provide a scheme for the case with $N = 2$ receivers.

For simplicity, we assume each transmitter splits its power equally between its messages w_{m1} and w_{m2}. The phase rotations at the transmitter for the k^{th} equation are given by $\theta_{m1}^{(k)}$ and $\theta_{m2}^{(k)}$. This results in the following channel input:

$$X_m^{(k)} = \theta_{m1}^{(k)} X_{m1} + \theta_{m2}^{(k)} X_{m2} \quad (34)$$

It is also convenient to represent these phases in matrix form:

$$\mathbf{\Theta}_n = \begin{bmatrix} \theta_{1n}^{(1)} & \theta_{2n}^{(1)} & \cdots & \theta_{Mn}^{(1)} \\ \theta_{1n}^{(2)} & \theta_{2n}^{(2)} & \cdots & \theta_{Mn}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{1n}^{(K)} & \theta_{2n}^{(K)} & \cdots & \theta_{Mn}^{(K)} \end{bmatrix}. \quad (35)$$

We are also free to choose the phases provided by the channel $\phi_{nm}^{(k)}$. The receivers see equations of all transmitted messages of the following form:

$$y_1^{(k)} = \sum_{m=1}^{M} \phi_{1m}^{(k)} h_{nm} (\theta_{m1}^{(k)} x_{m1} + \theta_{m2}^{(k)} x_{m2}) + z_1^{(k)} \quad (36)$$

$$y_2^{(k)} = \sum_{m=1}^{M} \phi_{2m}^{(k)} h_{nm} (\theta_{m1}^{(k)} x_{m1} + \theta_{m2}^{(k)} x_{m2}) + z_2^{(k)} \quad (37)$$

We can represent all of the phases seen at each receiver in a single matrix by ordering the messages as follows $w_{11}, w_{21}, \ldots, w_{M1}, w_{12}, w_{22}, \ldots, w_{M2}$. The matrix of phases is

$$\mathbf{B}_n = [\mathbf{\Theta}_1 \odot \mathbf{\Phi}_n \quad \mathbf{\Theta}_2 \odot \mathbf{\Phi}_n] \quad (38)$$

The key is to choose all of the phases such that the left half of $\mathbf{B}_n$ is full rank at receiver 1 and composed of identical columns at receiver 2 while at the same time ensuring the right half is full rank at receiver 2 and composed of identical columns at receiver 1. This is indeed possible as shown by the following theorem.

Theorem 3: For the X message set with $N = 2$ receivers, the following rates are achievable:

$$R_{nm} = \frac{1}{M+1} E\left[\log\left(1 + \frac{M+1}{2}|h_{nm}|^2 \frac{P_m}{\sigma_n^2}\right)\right] \quad (39)$$

Proof: We will show that it is possible to design the phases so that both receivers see a DFT matrix with independent columns for the desired messages and the same column for undesired messages. Choose the phases as follows:

$$\theta_{m1}^{(k)} = \exp\left(\frac{j2\pi(m-1)k}{M+1}\right) \quad (40)$$

$$\theta_{m2}^{(k)} = \exp\left(\frac{j2\pi Mk}{M+1}\right) \quad (41)$$

$$\phi_{1m}^{(k)} = 1 \quad (42)$$

$$\phi_{2m}^{(k)} = \exp\left(\frac{-j2\pi(m-1)k}{M+1}\right) \quad (43)$$

Let $\alpha = \exp(j2\pi/(M+1))$. We get that $\mathbf{B}_1$ is equal to:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \cdots & \alpha^{(M-1)} & \alpha^M & \alpha^M & \cdots & \alpha^M \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^M & \cdots & \alpha^{(M-1)M} & \alpha^{M^2} & \alpha^{M^2} & \cdots & \alpha^{M^2} \end{bmatrix}$$

and $\mathbf{B}_2$ is equal to:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & \alpha^M & \alpha^{M-1} & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & \alpha^{M^2} & \alpha^{M(M-1)} & \cdots & \alpha^M \end{bmatrix}$$

Now, receivers 1 and 2 can treat this as a multicast problem and choose $\mathbf{A}_1$ to be the first M columns of the size $M+1$ inverse DFT matrix and $\mathbf{A}_2$ to be the last M columns. Following the remaining steps in the proof of Theorem 2 yields the desired result. ∎

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