Degrees of Freedom of Wireless Networks with Relays, Feedback, Co-operation and Full Duplex Operation

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Abstract

We find the degrees of freedom of a network with S source nodes, R relay nodes and D destination nodes, with random time-varying/frequency-selective channel coefficients and global channel knowledge at all nodes. We allow full-duplex operation at all nodes, as well as causal noise-free feedback of all received signals to all source and relay nodes. An outer bound to the capacity region of this network is obtained. Combining the outer bound with previous interference alignment based achievability results, we conclude that the techniques of relays, feedback, full-duplex operation and noisy co-operation do not increase the degrees of freedom of interference and X networks. As a second contribution, we show that for a network with K full duplex nodes and $K(K-1)$ independent messages with one message from every node to each of the other $K-1$ nodes, the total degrees of freedom are bounded above and below by $\frac{K(K-1)}{(2K-2)}$ and $\frac{K(K-1)}{(2K-3)}$ respectively.

Index Terms

Capacity, Degrees of Freedom, Interference Alignment, Interference Channel, MIMO, X Channel

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I. Introduction

The recent surge of interest in approximate capacity characterizations of wireless networks has lead to substantial progress on several long standing open problems. The capacity of certain relay networks have been characterized within a constant number of bits [1]. In their seminal paper, Etkin, Tse and Wang [2] found the capacity region of the 2 user interference channel within one bit. The sum capacity of the K-user time-varying/frequency-selective interference channel (see Figure 1) was approximated in [3] as

$$C(\text{SNR}) = \frac{K}{2}\log(\text{SNR}) + o(\log(\text{SNR})) \tag{1}$$

where SNR represents the signal to noise ratio (the total transmit power of all nodes when the local noise power at each receiver is normalized to unity). Equivalently, the K user interference channel has $K/2$ degrees of freedom[1]. Since, by definition, at high SNR the $o(\log(\text{SNR}))$ term is a vanishing fraction of $\log(\text{SNR})$, the accuracy of such a capacity characterization approaches 100% as the SNR approaches infinity. The achievable scheme for the capacity characterization of interference networks is based on the idea of interference alignment [3].

The interference alignment technique of [3] has been generalized in [4] to find a capacity approximation of wireless X networks within $o(\log(\text{SNR}))$. X networks are a generalization of interference networks - unlike an interference network where each transmitter has a message for only its corresponding receiver, in an X network every transmitter has an independent message for every receiver. Reference [4] studied the frequency-selective $S \times D$ X network (Figure 1), i.e., a network with S transmitters, D receivers and SD independent messages - one message for each transmitter-receiver pair. Using an interference alignment based achievable scheme, [4] characterized the sum capacity of the $S \times D$ X network as

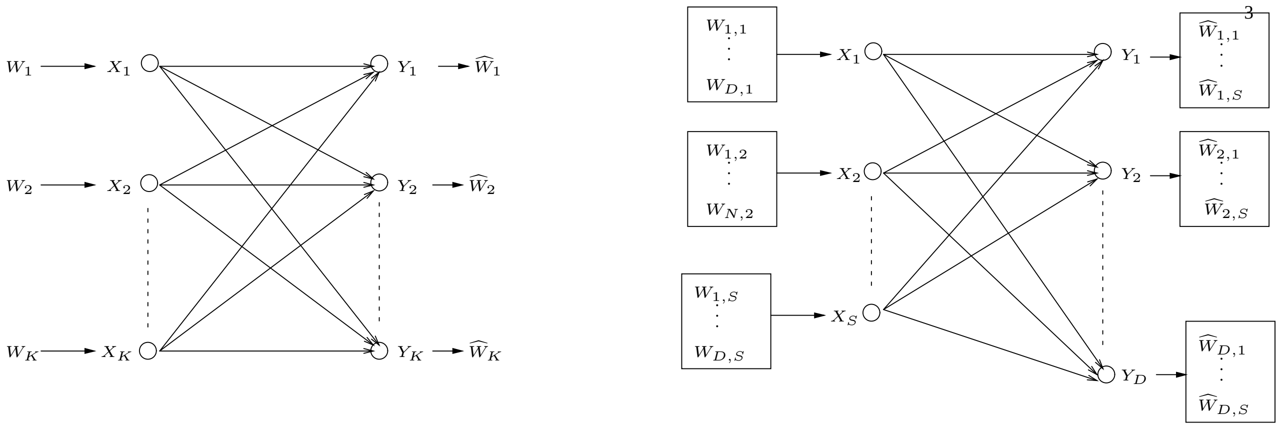
$$C(\text{SNR}) = \frac{SD}{S+D-1}\log(\text{SNR}) + o(\log(\text{SNR})). \tag{2}$$

Equivalently, the frequency selective $S \times D$ X network has $\frac{SD}{S+D-1}$ degrees of freedom.

The degrees of freedom characterizations shed light on the loss of capacity due to the distributed nature of a network. Note that a total of $\min(S, D)$ degrees of freedom can be achieved in a network with S transmitters and D receivers by full cooperation among source nodes as in a vector broadcast channel [5]–[7], full cooperation among destination nodes as in a vector multiple access channel [8], or both as in a $S \times D$ point to point multiple input multiple output (MIMO) channel [9]. Neither the interference network considered in [3] nor the X network considered in [4] allow any mechanism for the source nodes to cooperate among themselves by learning each other's messages or for the destination nodes to cooperate among themselves by sharing their received signals. The price paid for the distributed nature of the network is the loss of $\min(S, D) - \frac{SD}{S+D-1}$ degrees of freedom. A natural question that follows from this observation is whether some of these lost degrees of freedom can be recovered by other means - e.g. by allowing source nodes (as well as destination nodes) to communicate among themselves, by employing relay nodes, by allowing feedback from the destination nodes to source nodes, or even by allowing full duplex operation so that all nodes can transmit and receive simultaneously. In this work we seek to answer these questions for wireless X networks.

Previous work in this direction [10] obtained a degrees of freedom outerbound of $K/2$ for the K user interference network with relays and noisy transmitter/receiver co-operation. Since the results of [3] show that $K/2$ degrees of

[1] If the sum capacity of a network is characterized as $C(\text{SNR}) = d\log(\text{SNR}) + o(\log(\text{SNR}))$, then we say that the network has d degrees of freedom (also known as multiplexing gain or capacity pre-log).



The K user interference channel

$C(\text{SNR}) = \frac{K}{2}\log(\text{SNR}) + o(\log(\text{SNR}))$

The $S \times D$ X channel

$C(\text{SNR}) = \frac{SD}{S+D-1}\log(\text{SNR}) + o(\log(\text{SNR}))$

Fig. 1. Interference and X channels

freedom can be achieved even without relays and cooperation, the conclusion is that relays and transmitter/receiver cooperation cannot increase the degrees of freedom of frequency-selective interference networks. In this paper, we seek a generalization of the results of [10] to X networks.

There are two main results in this work. First, we show that for wireless networks with a set of S source nodes and a disjoint set of D destination nodes and time-varying/frequency-selective channel coefficients, the available degrees of freedom *cannot be increased* by allowing source nodes and/or destination nodes to communicate among themselves over physical channels, or by allowing relays, feedback to source/relay nodes and full duplex operation. Thus, the total degrees of freedom of a wireless network with S source and D destination nodes remains $\frac{SD}{S+D-1}$ with or without cooperation, relays, feedback and full duplex operation. Note that the network considered is essentially an X network with relays, feedback, full-duplex operation and noisy co-operation. Thus the achievability $\frac{SD}{S+D-1}$ degrees of freedom without relays, feedback, full duplex operation and cooperation follows trivially from [4]. The main contribution of this paper is the converse (outerbound) proved in Theorem 1 of Section III. It states that even with relays, feedback, cooperation and full duplex operation it is not possible to achieve more than $\frac{SD}{S+D-1}$ degrees of freedom. The converse is applicable even if feedback is perfect, and/or relays have multiple antennas. While the main result implies that the techniques of relays, feedback, full-duplex operation and co-operation cannot improve the degrees of freedom of X networks, we also show that these techniques cannot improve degrees of freedom of fully connected wireless interference networks (see Corollary 2). Generalizations of the outerbound to networks that are not fully connected and other interesting observations related to the result can be found in Section III-C.

The results of Theorem 1 and its corollaries are limited to networks where source nodes are disjoint from destination nodes, i.e., they are applicable to networks where, if a node is a source node for a message, then it cannot be the destination node for any message in the network (and vice versa). A second contribution of this work is to extend to converse of Theorem 1 and the interference alignment based achievable scheme of [3] to networks where every node can behave as both a source for some message and a destination for another message. The most general case is where every node may have an independent message for every other node. For this case, we show that for a network with K full duplex nodes and $K(K-1)$ independent messages with one message from every node to each of the other $K-1$ nodes, the total degrees of freedom are bounded above and below by $K(K-1)/(2K-2)$ and $K(K-1)/(2K-3)$ respectively.

II. System Model for an $S \times R \times D$ node X Network

Consider an $S \times R \times D$ node network, i.e., a network with $S+R+D$ nodes where nodes $1, 2, \cdots, S$ are sources, nodes $S+1, S+2, \cdots, S+R$ are relays, and nodes $S+R+1, S+R+2, \cdots, S+R+D$ are destination nodes (see Figure 2). Following the definition of an X network [4], for all $j \in \{1, 2, \cdots, S\}$ and for all $i \in \{S+R+1, S+R+2, \cdots, S+R+D\}$, there is an independent message $W_{i,j}$ to be communicated from source node j to destination node i.

Full duplex operation is assumed so that all nodes are capable of transmitting and receiving simultaneously. The input and output signals of the $S \times R \times D$ node network are related as:

$$Y_i(n) = \sum_{j=1}^{S+R+D} H_{i,j}(n) X_j(n) + Z_i(n), \quad i \in \{1, 2, \cdots, S+R+D\}, n \in \mathbb{N} \tag{3}$$

where, corresponding to the n^{th} use of the channel, $X_j(n)$ is the symbol transmitted by node j, $Y_i(n)$ is the symbol received by node i, $H_{i,j}(n)$ is the channel gain from node j to node i and $Z_i(n)$ is the zero mean unit variance additive white Gaussian noise (AWGN) at node i. We use the following notation,

$$X_i^n \triangleq \{X_i(1), X_i(2), \cdots, X_i(n)\} \tag{4}$$

Similar notation is used for output signals and the additive noise terms as well.

The channel coefficients $H_{i,j}(n)$, $\forall i, j \in \{1, 2, \cdots, S+D+R\}$ are known *apriori*[2] to all nodes. All channel coefficients take non-zero values and the network is fully-connected. The AWGN terms $Z_i(n)$ have unit variance and are independent identically distributed (i.i.d.) in time and across nodes.

Perfect (noise-free) and causal feedback of all received signals is available to all source and relay nodes, but not to the destination nodes. For codewords spanning N channel uses, the encoding functions are as follows,

$$X_i(n) = \begin{cases} f_{i,n}\left(W_{S+R+1,i}, W_{S+R+2,i}, \cdots, W_{S+R+D,i}, Y_1^{n-1}, Y_2^{n-1}, \cdots, Y_{S+R+D}^{n-1}\right), & i \in \{1, 2, \cdots, S\} \\ f_{i,n}\left(Y_1^{n-1}, Y_2^{n-1}, \cdots, Y_{S+R+D}^{n-1}\right), & i \in \{S+1, S+2, \cdots, S+R\} \\ f_{i,n}\left(Y_i^{n-1}\right), & i \in \{S+R+1, \cdots, S+R+D\} \end{cases}$$

for $n = 1, 2, \cdots, N$. In other words, the signal transmitted from a source node at time n is completely determined by all the messages originating at that source node and the received signals of *all* the nodes upto time $n-1$ (causality condition). The signal transmitted by a destination node at time n can only depend on all the received signals at *that* node upto time $n-1$. This is because the destination nodes do not receive feedback of other nodes' received signals. The signal transmitted from a relay node can only depend on the received signals of *all* the nodes upto time $n-1$.

The decoding functions are as follows,

$$\hat{W}_{i,j} = g_{i,j}\left(Y_i^N\right), i \in \{S+R+1, S+R+2, \cdots, S+R+D\}, j \in \{1, 2, \cdots, S\}$$

Thus, a destination node can only use its own received signal to decode all its desired messages. The probability of error is the probability that there is at least one message $W_{i,j}$ that is not decoded correctly, i.e. $\hat{W}_{i,j} \neq W_{i,j}$ for some (i, j).

[2] Thus, we also show that non-causal channel knowledge does not increase the degrees of freedom.

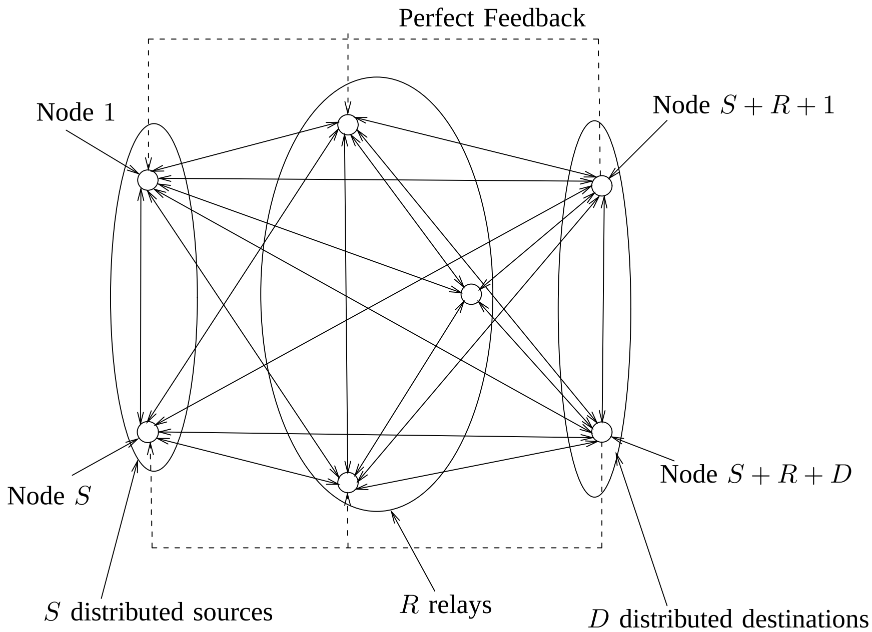


Fig. 2. The $S \times R \times D$ network

The total power across all transmitters is assumed to be SNR per channel use. We denote the size of the message set by $|W_{i,j}(\text{SNR})|$. Let $R_{i,j}(\text{SNR}) = \frac{\log|W_{i,j}(\text{SNR})|}{N}$ denote the rate of the codeword encoding the message $W_{i,j}$, where the codewords span N slots. A rate-matrix$[(R_{i,j}(\text{SNR}))]$ is said to be *achievable* if messages $W_{i,j}$ can be encoded at rates $R_{i,j}(\text{SNR})$ so that the probability of error can be made arbitrarily small simultaneously for all messages by choosing appropriately long N.

Let $\mathcal{C}(\text{SNR})$ represent the capacity region of the $S \times R \times D$ node network, i.e., it represents the set of all achievable rate-matrices $[(R_{i,j}(\text{SNR}))]$. Analogous to the capacity region, the degrees of freedom region of the $S \times R \times D$ node network is defined as

$$\mathcal{D} = \Bigg\{ [(d_{i,j})] \in \mathbb{R}_+^{SD} : \forall [(\alpha_{i,j})] \in \mathbb{R}_+^{SD}$$
$$\sum_{i=S+R+1}^{S+R+D} \sum_{j=1}^{S} \alpha_{i,j} d_{i,j} \quad \leq \quad \lim \sup_{\text{SNR}\to\infty} \left[\sup_{[(R_{i,j}(\text{SNR}))] \in C(\text{SNR})} \sum_{i=S+R+1}^{S+R+D} \sum_{j=1}^{S} (\alpha_{i,j} R_{i,j}(\text{SNR})) \frac{1}{\log(\text{SNR})} \right] \Bigg\}$$

Note that the above equation means that in a capacity optimal achievable scheme, the achieved rate $R_{i,j}(\text{SNR})$ can be approximated as $d_{i,j}\log(\text{SNR}) + o(\log(\text{SNR}))$. Equivalently, the degrees of freedom region of a network approximates its capacity region within $o(\log(\text{SNR}))$.

III. Degrees of Freedom of the $S \times R \times D$ node X network

Theorem 1: Let

$$\mathcal{D}^{out} \triangleq \Bigg\{ [(d_{i,j})] : \forall (u,v) \in \{1, 2 \cdots S\} \times \{S+R+1, S+R+2, \cdots, S+R+D\}$$
$$\sum_{q=S+R+1}^{S+R+D} d_{q,u} + \sum_{p=1}^{S} d_{v,p} - d_{v,u} \quad \leq 1 \quad \Bigg\}$$

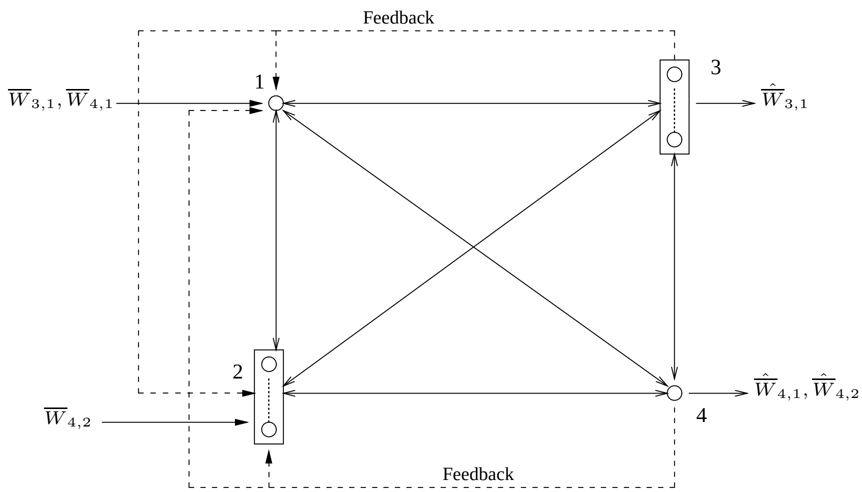


Fig. 3. 4 node X network

Then $\mathcal{D} \subseteq \mathcal{D}^{out}$ where $\mathcal{D}$ represents the degrees of freedom region of the $S \times R \times D$ node X network. Furthermore, the total number of degrees of freedom of the $S \times R \times D$ network can be upperbounded as follows

$$\max_{[(d_{i,j})] \in \mathcal{D}} \sum_{j=1}^{S} \sum_{i=S+R+1}^{S+R+D} d_{i,j} \quad \leq \quad \frac{SD}{S+D-1}$$

Equivalently, the sum capacity $C(\text{SNR})$ of this network can be bounded as

$$C(\text{SNR}) \leq \frac{SD}{S+D-1} \log(\text{SNR}) + o(\log(\text{SNR}))$$

The proof of the above theorem can be divided into two stages. In the first stage, we first construct a 4 node MIMO X network whose capacity is larger than the capacity of the original $S \times R \times D$ network. In the second stage, we upperbound the capacity of this 4 node X network using Fano's inequality. We now proceed to the proof.

A. First Stage of Proof - Construction of a 4 node X network

To prove the theorem, all we need to show is that for any $(u,v) \in \{1, 2, \ldots, S\} \times \{S+R+1, S+R+2, \ldots, S+R+D\}$

$$\sum_{i=S+R+1}^{S+R+D} d_{i,u} + \sum_{j=1}^{S} d_{v,j} - d_{v,u} \leq 1$$

We therefore need to show that for all messages that either originate at node u or are intended for node v, the total number of degrees of freedom cannot be more than one. Summing all inequalities of the above form over all $(u,v) \in \{1, 2, \ldots S\} \times \{S+R+1 \ldots S+R+D\}$, the bound on the *total* number of degrees of freedom can be obtained.

For convenience, we will show the inequality for $(u,v) = (1, S+R+D)$. By symmetry, the inequality extends to all desired values of u, v. We therefore intend to show that

$$\sum_{i=S+R+1}^{S+R+D} d_{i,1} + \sum_{j=1}^{S} d_{S+R+D,j} - d_{S+R+D,1} \leq 1$$

To show this, we first eliminate all the messages that are not associated with either source node 1 or destination node $S+R+D$, i.e., we set $W_{i,j} = \phi, (i-(S+R+D))(j-1) \neq 0$. Since we are only seeking an outerbound on the rates of a subset of messages, eliminating a message can not hurt the rates of the remaining messages [11]. Now, we transform the original $S \times R \times D$ network with single antenna nodes into a $2 \times 0 \times 2$ node network, i.e., an X network with 2 source nodes, zero relay nodes and 2 destination nodes where one source and one destination have multiple antennas (see Figure 3). This is done by allowing full cooperation between the $S-1$ source nodes $2, \cdots, S$ and the R relay nodes $S+1, S+2, \cdots, S+R$ so that they effectively become one transmitter with $S+R-1$ antennas. Similarly, destination nodes $S+R+1, S+R+2, \cdots, S+R+D-1$ are also allowed to perfectly cooperate so that they form one receiver with $D-1$ antennas. Again, note that allowing the nodes to cooperate cannot reduce the degrees of freedom region and therefore does not contradict our outerbound argument. We represent the resulting 4 node X network (Figure 3) by the following input-output equations.

$$\overline{Y}_i(n) = \sum_{j=1}^{4} \overline{H}_{i,j}(n)\overline{X}_j(n) + \overline{Z}_i(n), \qquad i \in \{1,2,3,4\} \tag{5}$$

where

$$\begin{aligned}
\overline{Y}_1(n) &= Y_1(n) \\
\overline{Y}_2(n) &= [Y_2(n) \;\; Y_3(n) \;\; \cdots \;\; Y_{S+R}(n)]^T \\
\overline{Y}_3(n) &= [Y_{S+R+1}(n) \;\; Y_{S+R+2}(n) \;\; \cdots \;\; Y_{S+R+D-1}(n)]^T \\
\overline{Y}_4(n) &= Y_{S+R+D}(n)
\end{aligned}$$

Thus, nodes 2 and 3 act as multiple antenna nodes with $S+R-1$ and $D-1$ antennas respectively. $\overline{X}_i(n), \overline{Z}_i(n)$ are also defined in a corresponding manner for $i \in \{1,2,3,4\}$. The definition of the channel coefficients $\overline{H}_{i,j}(n)$ is clear from equations (3) and (5), and from Figures 2 and 3. Multiple messages that have the same source and the same destination are combined in the 4 node X network as follows:

$$\overline{W}_{3,1} = [W_{S+R+1,1} \;\; W_{S+R+2,1} \;\; \cdots \;\; W_{S+R+D-1,1}] \tag{6}$$

$$\overline{W}_{3,2} = \phi \tag{7}$$

$$\overline{W}_{4,1} = W_{S+R+D,1} \tag{8}$$

$$\overline{W}_{4,2} = [W_{S+R+D,2} \;\; W_{S+R+D,3} \;\; \cdots \;\; W_{S+R+D,S}] \tag{9}$$

Over this X network, the encoding functions are as follows:

$$\overline{X}_1(n) = \overline{f}_{1,n}\left(\overline{W}_{3,1}, \overline{W}_{4,1}, \overline{Y}_1^{n-1}, \overline{Y}_2^{n-1}, \overline{Y}_3^{n-1}, \overline{Y}_4^{n-1}\right) \tag{10}$$

$$\overline{X}_2(n) = \overline{f}_{2,n}\left(\overline{W}_{4,2}, \overline{Y}_1^{n-1}, \overline{Y}_2^{n-1}, \overline{Y}_3^{n-1}, \overline{Y}_4^{n-1}\right) \tag{11}$$

$$\overline{X}_i(n) = \overline{f}_{i,n}\left(\overline{Y}_i^{n-1}\right), i = 3,4 \tag{12}$$

and the decoding functions are the following:

$$\hat{\overline{W}}_{i,j} = \overline{g}_{i,j}\left(\overline{Y}_i^{N}\right), (i,j) \in \{(3,1),(4,1),(4,2)\} \tag{13}$$

The rates and the degrees of freedom region of this network are defined in a manner similar to the $S \times R \times D$ network. This completes the construction of the 4 node X network.

B. Second stage of Proof - Capacity bound on the 4 node X network

As discussed above, we intend to show

$$\sum_{i=S+1+R}^{S+R+D} d_{i,1} + \sum_{j=1}^{S} d_{S+R+D,j} - d_{S+R+D,1} \leq 1$$

Using equation (6)-(9), we can re-write the above outerbound in terms of degrees of freedom of the 4 node X network as

$$\overline{d}_{3,1} + \overline{d}_{4,1} + \overline{d}_{4,2} \leq 1 \tag{14}$$

where $\overline{d}_{i,j}$ represents the number of degrees of freedom corresponding to message $\overline{W}_{i,j}$. Therefore, to complete the proof, we need to show (14).

The converse argument is as follows. Consider any achievable coding scheme in the 4 node X network. Let a genie provide the messages $\overline{W}_{4,1}, \overline{W}_{4,2}$ and $\overline{Y}_2^N, \overline{Y}_4^N$ to node 3, where the codewords span N symbols. Next we find outerbounds on the rates in the genie supported 4 node X network. Using Fano's inequality, for any $\epsilon_N > 0$, we can bound the rates of messages corresponding to receivers 3 and 4 as follows.

$$\begin{aligned}
& N\left(\overline{R}_{4,1}(\text{SNR}) + \overline{R}_{4,2}(\text{SNR}) + \overline{R}_{3,1}(\text{SNR}) - \epsilon_N\right) \\
& \leq I(\overline{W}_{4,1}, \overline{W}_{4,2}; \overline{Y}_4^N) + I(\overline{W}_{3,1}; \overline{W}_{4,1}, \overline{W}_{4,2}, \overline{Y}_3^N, \overline{Y}_2^N, \overline{Y}_4^N) && (15) \\
& \leq I(\overline{W}_{4,1}, \overline{W}_{4,2}; \overline{Y}_4^N) + I(\overline{W}_{3,1}; \overline{Y}_3^N, \overline{Y}_2^N, \overline{Y}_4^N | \overline{W}_{4,1}, \overline{W}_{4,2}) && (16) \\
& \leq h(\overline{Y}_4^N) - h(\overline{Y}_4^N | \overline{W}_{4,1}, \overline{W}_{4,2}) + h(\overline{Y}_3^N, \overline{Y}_2^N, \overline{Y}_4^N | \overline{W}_{4,1}, \overline{W}_{4,2}) - h(\overline{Y}_3^N, \overline{Y}_2^N, \overline{Y}_4^N | \overline{W}_{4,1}, \overline{W}_{4,2}, \overline{W}_{3,1}) && (17) \\
& \leq N\left(\log \text{SNR} + o\left(\log \text{SNR}\right)\right) + \underbrace{h(\overline{Y}_2^N, \overline{Y}_3^N | \overline{Y}_4^N, \overline{W}_{4,1}, \overline{W}_{4,2})}_{T_1} - \underbrace{h(\overline{Y}_2^N, \overline{Y}_3^N, \overline{Y}_4^N | \overline{W}_{4,1}, \overline{W}_{4,2}, \overline{W}_{3,1})}_{T_2} && (18)
\end{aligned}$$

In (16), we have used the fact that $\overline{W}_{4,1}, \overline{W}_{4,2}$ are independent of $\overline{W}_{3,1}$. To obtain (18) from (17), we have used the fact that Gaussian variables maximize entropy for the first term of (17) and the chain rule to combine the second and the third terms of (17). We now simplify T_1, T_2 as follows.

$$\begin{aligned}
T_2 &= h(\overline{Y}_2^N, \overline{Y}_3^N \overline{Y}_4^N | \overline{W}_{3,1}, \overline{W}_{4,1}, \overline{W}_{4,2}) && (19) \\
&= \sum_{n=1}^{N} h(\overline{Y}_2(n), \overline{Y}_3(n), \overline{Y}_4(n) | \overline{W}_{3,1}, \overline{W}_{4,1}, \overline{W}_{4,2}, \overline{Y}_2^{n-1}, \overline{Y}_3^{n-1}, \overline{Y}_4^{n-1}) && (20) \\
&\geq \sum_{n=1}^{N} h(\overline{Y}_2(n), \overline{Y}_3(n), \overline{Y}_4(n) | \overline{W}_{3,1}, \overline{W}_{4,1}, \overline{W}_{4,2}, \overline{Y}_2^{n-1}, \overline{Y}_3^{n-1}, \overline{Y}_4^{n-1}, \overline{X}_1(n), \overline{X}_2(n), \overline{X}_3(n), \overline{X}_4(n)) && (21) \\
&= \sum_{n=1}^{N} h(\overline{Z}_2(n), \overline{Z}_3(n), \overline{Z}_4(n) | \overline{W}_{3,1}, \overline{W}_{4,1}, \overline{W}_{4,2}, \overline{Y}_2^{n-1}, \overline{Y}_3^{n-1}, \overline{Y}_4^{n-1}, \overline{X}_1(n), \overline{X}_2(n), \overline{X}_3(n), \overline{X}_4(n)) && (22) \\
&= \sum_{n=1}^{N} h(\overline{Z}_2(n), \overline{Z}_3(n), \overline{Z}_4(n)) && (23) \\
\Rightarrow T_2 &\geq N\mathcal{O}(1) && (24)
\end{aligned}$$

In (21), we have used the fact that, conditioning reduces entropy. In (22), we have used the fact that nodes have global channel knowledge and therefore, the effect of $\overline{X}_i(n), i = 1, 2, 3, 4$ can be canceled from $\overline{Y}_j(n), j = 2, 3, 4$. In (23), we have used the independence of the noise terms at symbol n, and the inputs $\overline{X}_i(n), 1, 2, 3, 4$, outputs $\overline{Y}_i^{n-1}, i = 1, 2, 3, 4$ and messages $\overline{W}_{3,1}, \overline{W}_{4,1}, \overline{W}_{4,2}$.

$$T_1 = h(\overline{Y}_2^N, \overline{Y}_3^N | \overline{W}_{4,1}, \overline{W}_{4,2}, \overline{Y}_4^N) \tag{25}$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} h(\overline{Y}_2(n), \overline{Y}_3(n) | \overline{W}_{4,1}, \overline{W}_{4,2}, \overline{Y}_4^N, \overline{Y}_3^{n-1}, \overline{Y}_2^{n-1}) && (26)\\
&= \sum_{n=1}^{N} h(\overline{Y}_2(n), \overline{Y}_3(n) | \overline{W}_{4,1}, \overline{W}_{4,2}, \overline{Y}_4^N, \overline{Y}_3^{n-1}, \overline{Y}_2^{n-1}, X_2(n), \overline{X}_3(n), \overline{X}_4(n)) && (27)\\
&\leq \sum_{n=1}^{N} h(\overline{H}_{21}\overline{X}_1(n) + \overline{Z}_2(n), \overline{H}_{31}\overline{X}_1(n) + \overline{Z}_2(n) | \overline{H}_{41}\overline{X}_1(n) + \overline{Z}_4(n)) && (28)\\
&\leq \sum_{n=1}^{N} h(\overline{H}_{21}\overline{X}_1(n) + \overline{Z}_2(n) | \overline{H}_{41}\overline{X}_1(n) + \overline{Z}_4(n)) + h(\overline{H}_{31}\overline{X}_1(n) + \overline{Z}_2(n) | \overline{H}_{41}\overline{X}_1(n) + \overline{Z}_4(n)) && (29)\\
&\leq No(\log(\text{SNR})) && (30)
\end{aligned}
$$

(27) follows because, given $\overline{W}_{4,1}, \overline{W}_{4,2}, \overline{Y}_4^N, \overline{Y}_3^{n-1}, \overline{Y}_2^{n-1}$, the node has information of $X_i(n), i = 2, 3, 4$ because of encoding functions (10)-(12). In (28), we have canceled the effect of $\overline{X}_2(n), \overline{X}_3(n), \overline{X}_4(n)$, from $Y_i(n), i = 2, 3, 4$, and then used the fact that conditioning reduces entropy. (30) can be shown using the fact that the Gaussian distribution maximizes conditional entropy, similar to Lemma 1 in [12]. Therefore, using (24), (30) in (17) we can write

$$\overline{R}_{3,1}(\text{SNR}) + \overline{R}_{4,1}(\text{SNR}) + \overline{R}_{4,2}(\text{SNR}) - 2\epsilon_N \quad \leq \quad \log(\text{SNR}) + o(\log(\text{SNR})) \tag{31}$$

This implies that the total number of degrees of freedom of the 4 node X network described is upper-bounded by 1 so that we can write

$$\max_{\mathcal{D}^X} \overline{d}_{3,1} + \overline{d}_{4,2} + \overline{d}_{4,3} \leq 1$$

■

Corollary 1: Consider the fully connected $S \times R \times D$ network where all the channel coefficients are time-varying/frequency-selective with values drawn randomly from a continuous distribution with support bounded below by a non-zero constant. Then, the capacity of the network maybe approximated as

$$C(\text{SNR}) = \frac{SD}{S + D - 1} \log(\text{SNR}) + o(\log(\text{SNR}))$$

Proof: The converse follows from Theorem 1. Achievability simply follows from the interference alignment based achievable scheme of [4] over the X channel formed by the S source nodes and D destination nodes. ■

Corollary 2: Consider a fully connected K user interference network with R relays, where all the channel coefficients are time-varying/frequency-selective with values drawn randomly from a continuous distribution with support bounded below by a non-zero constant. Let all nodes be full-duplex allowing noisy transmitter/receiver co-operation. Also, let the source and relay nodes receive perfect feedback from all other nodes. Then, this interference network has $\frac{K}{2}$ degrees of freedom.

Proof: Achievability follows trivially from [3]. The converse is shown here. Now, note that the network considered is essentially the $K \times R \times K$ network with certain messages set to null. In the $K \times R \times K$ network nodes $1, 2 \ldots K$ are the source nodes, $K + 1, \ldots K + R$ are relay nodes and the nodes $K + R + 1 \ldots 2K + R$ are destination nodes. There are only K messages in the network with $W_{i,j} = \phi, i \neq (j + R + K)$. Now, writing bounds of Theorem 1 for the non-null messages, we get

$$d_{u+R+K,u} + d_{v+R+K,v} \leq 1, u \neq v, u, v \in \{1, 2, \ldots K\}$$

Summing all bounds of the above form, the total number of degrees of freedom can be bounded by $K/2$. ■

C. Observations

1) The achievable schemes in Corollary 1 and Corollary 2 do not use relays, feedback or co-operation. Therefore, the implication of the result is that relays, feedback, full-duplex operation and noisy co-operation do not improve the degrees of freedom of frequency-selective interference and X networks.
2) The outerbound of Theorem 1 applies to fully connected wireless networks that are not necessarily frequency-selective or time-varying. Furthermore, the outerbound is valid even if nodes do not have global channel knowledge. This is because global channel knowledge can only increase the capacity of a network and therefore, the outerbound still holds. For similar reasons, the outerbound is valid if the feedback channel is noisy, i.e. not perfect. The frequency-selective/time-varying nature of the channel and global channel knowledge are only required for the achievable schemes used in the corollaries of the theorem.
3) The bounds of Theorem 1 are applicable even if some or all relays have multiple antennas. This is because the converse starts by allowing full co-operation between all relay nodes to effectively form a MIMO relay node. Further, the outerbound is independent on the number of antennas in this effective MIMO relay node. Therefore, the converse argument stands even if certain (or all) relays have multiple antennas.
4) Note that the second stage of the proof of Theorem 1 is valid even if the 4 node X network is not fully connected as long as $\overline{H}_{4,1}$ is non-zero. If $\overline{H}_{4,1}$ is equal to zero, the argument fails because the upper-bound of $o(\log(\text{SNR}))$ in (30) is no longer valid. All other inequalities hold for arbitrary channel co-efficients. This implies that the converse technique of Theorem 1 can also be used to bound the degrees of freedom regions of networks that are not fully connected. For example, in a $3 \times 0 \times 3$ network where the channel gain between source node 1 and destination node 6 is zero and all other channel gains are non-zero, we can write

$$\sum_{q=S+R+1}^{S+R+D} d_{q,u} + \sum_{p=1}^{S} d_{v,p} - d_{v,u} \leq 1$$

where $(u,v) \in \{1,2,3\} \times \{4,5,6\} \backslash \{(1,6)\}$ with $\backslash$ used to indicate the difference between two sets. Note that although the converse technique is generalizable to networks that are not fully connected, the bound on the sum capacity and the corollaries of Theorem 1 do not hold. This is because if a network is not fully connected, only some of the bounds of Theorem 1 are valid. In fact, it is easy to see that a relay can improve total degrees of freedom if a network is not fully connected. For example, in the classical Gaussian relay channel with a single source, a single destination and a relay, if the source is not connected to the destination, then the presence of the relay trivially increases the degrees of freedom.

IV. K USER FULL DUPLEX NETWORK

In this section, we derive bounds on the degrees of freedom of the K-user full-duplex network (see Figure 4 (a)). The K user full duplex network is a fully connected network with K full-duplex nodes $1, 2, \ldots K$. In this network there exists a message from every node to every other node in the network so that there are a total of $K(K-1)$ messages in the system. The message from node j to node $i \neq j$ is denoted by $W_{j,i}$. Let $H_{i,j}(n)$ represent the channel gain between nodes i and j corresponding to the nth symbol. The channel gains satisfy the reciprocity, i.e., $H_{i,j}(n) = H_{j,i}(n)$ and $H_{i,i} = 0$. As usual, all nodes have apriori knowledge of all channel gains. The input-output relations in this channel are represented by

$$Y_i(n) = \sum_{j=1}^{K} H_{i,j}(n) X_j(n) + Z_i(n), \qquad i \in \{1, 2 \ldots K\} \tag{32}$$

where $Y_i(n), X_i(n), Z_i(n)$ represent, respectively, the received symbol, the transmitted symbol and the AWGN term at node i. For codewords of length N, the encoding functions in this network are defined as

$$X_i(n) = f_{i,n}(W_{1,i}, W_{2,i}, \ldots, W_{i-1,i}, W_{i+1,i}, \ldots W_{K,i}, Y_i^{n-1}) \tag{33}$$

and the decoding functions are defined as

$$\hat{W}_{j,i} = g_{j,i}(Y_j^N, W_{1,j}, W_{2,j}, \ldots, W_{j-1,j}, W_{j+1,j}, \ldots W_{K,j}), \forall i \neq j. \tag{34}$$

The main result of this section is an approximation of the capacity of the K user full duplex network as follows.

Theorem 2: The capacity $C(\text{SNR})$ of the K user full-duplex network is bounded as follows.

$$\begin{aligned} C(\text{SNR}) &\geq \frac{K(K-1)}{2K-2}\log(\text{SNR}) + o(\log(\text{SNR})) \\ C(\text{SNR}) &\leq \frac{K(K-1)}{2K-3}\log(\text{SNR}) + o(\log(\text{SNR})) \end{aligned}$$

Equivalently,

$$\frac{K}{2} = \frac{K(K-1)}{2K-2} \leq d_{fd} \leq \frac{K(K-1)}{2K-3}$$

where d_{fd} represents the total number of degrees of freedom of the K user full-duplex network.

In order to prove the above theorem, we need the lemma below which transforms the K user full duplex network to a network whose source nodes are disjoint from destination nodes.

Lemma 1: The K user full-duplex network is equivalent to a network with K half-duplex source nodes and K half-duplex destination nodes with the following properties (also see Figure 4)

1) The input-output relations are described as

$$\widetilde{Y}_i(n) = \sum_{j=1}^{K} \widetilde{H}_{i,j}(n)\widetilde{X}_j(n) + \widetilde{Z}_i(n), \qquad i \in \{1, 2 \ldots K\}$$

where $\forall i, j \in \{1, 2, \ldots K\}$

$$\widetilde{H}_{i,j} = \left\{ \begin{array}{cc} H_{i,j} & i \neq j, \\ H_{i,j} = 0 & i = j \end{array} \right\}$$

Note that this implies $\widetilde{H}_{i,j} = \widetilde{H}_{j,i}, \forall i \neq j$

2) There are $K(K-1)$ messages in the system, denoted by $\widetilde{W}_{j,i}, i \neq j$. These messages are denoted by

$$\widetilde{W}_{j,i} = W_{j,i}, \forall i \neq j, i, j \in \{1, 2, \ldots K\}$$

3) Encoding function of the form

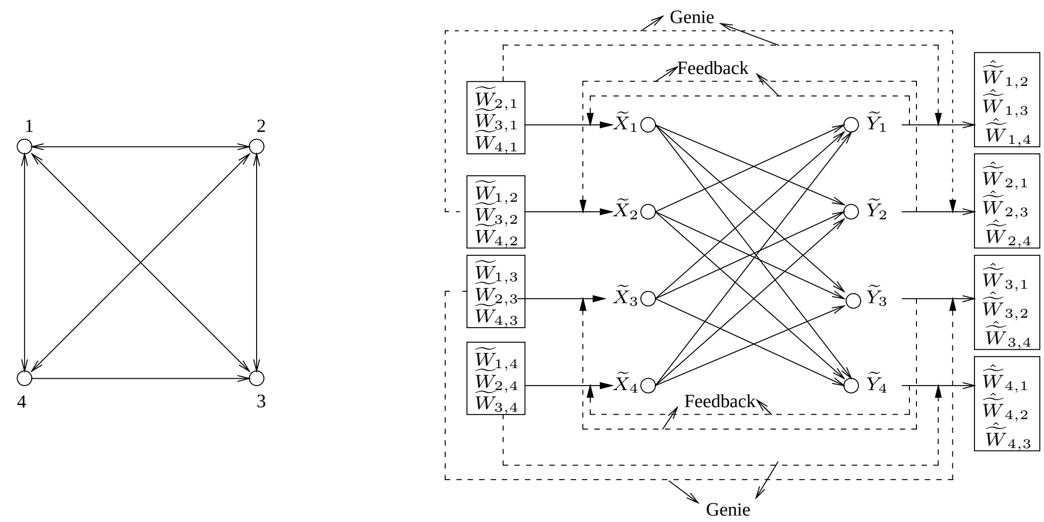
$$\widetilde{X}_i(n) = \widetilde{f}_{i,n}(\widetilde{Y}_i^{n-1}, \widetilde{W}_{1,i}, \widetilde{W}_{2,i} \ldots \widetilde{W}_{i-1,i}, \widetilde{W}_{i+1,i} \ldots \widetilde{W}_{K,i}) \tag{35}$$

4) Decoding function of the form

$$\hat{\widetilde{W}}_{j,i} = \widetilde{g}_{j,i}(\widetilde{Y}_j^N, \widetilde{W}_{1,j}, \widetilde{W}_{2,j} \ldots \widetilde{W}_{j-1,j}, \widetilde{W}_{j+1,j} \ldots \widetilde{W}_{K,j}), j \neq i \tag{36}$$

Note that the encoding and decoding functions imply that

- A genie provides receiver j with apriori knowledge of all messages at source j i.e. $\widetilde{W}_{i,j}, \forall i = \{1, 2, \ldots K\} - \{j\}$
- There is perfect feedback from destination K to source K.



(a) **(b)**

Fig. 4. (a) the K user full duplex network for $K = 4$ (b) An network whose capacity is identical to the 4 user full duplex network

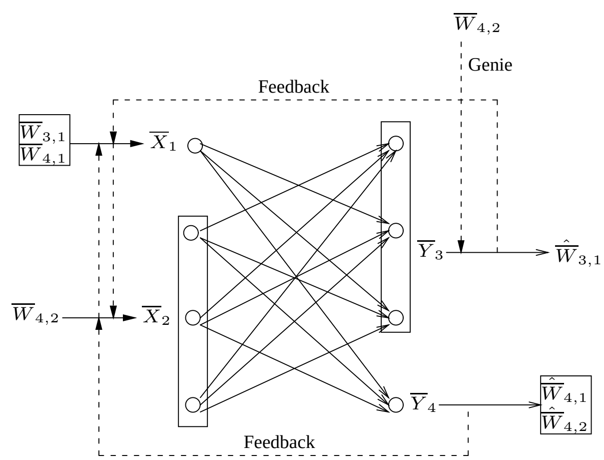


Fig. 5. 4 node network used in the outerbound of Theorem 2

Proof: By comparing encoding equations (33), (35) and decoding equations (34), (36), the lemma can easily be proved, i.e., it can be verified that any encoding scheme that can be implemented on the K user full duplex network, can also be implemented on network described in the above lemma and vice-versa. ■

Note that we have transformed the K user network to an equivalent network whose source and destination nodes are disjoint. Now, we extend the achievability and converse of Theorem 1 to the network described in the lemma to show the required result. We place the converse argument in the next section. The achievable scheme used to show the innerbound is placed in Appendix I.

We now proceed to prove the outerbound of Theorem 2.

A. Proof of Outer-bound of Theorem 2

Note that the outerbound of Theorem 2 is equivalent to the following statement

$$d_{fd} \leq \frac{K(K-1)}{2K-3}$$

where d_{fd} represents the number of degrees of freedom of the K user full duplex network.

If $\mathcal{D}_{fd}^{[K]}$ is the degrees of freedom region of the K user full duplex network, we show that

$$\sum_{j\in\{1,2,3,\ldots p-1,p+1,\ldots K\}} d_{j,p} + \sum_{i\in\{1,2,3,\ldots q-1,q+1,\ldots K\}} d_{q,i} - d_{q,p} \leq 1, \forall p \neq q$$

for all $\left(d_{i,j}\right) \in \mathcal{D}_{fd}^{[K]}$. Summing inequalities of the above form over all $(p,q), p \neq q$ gives the desired outerbound. It is enough to show the inequality for $p=1$ and $q=K$. The inequality extends to all other values of (p,q) by symmetry. Therefore, we intend to show

$$\sum_{j\in\{2,3,\ldots K\}} d_{j,1} + \sum_{i\in\{1,2,3,\ldots K-1\}} d_{i,K} - d_{K,1} \leq 1$$

To show the above inequality, we first set $\widetilde{W}_{i,j} = \phi, (i-K)(j-1) \neq 0$. With these messages set to null, there are no messages intended for destination node 1 and therefore, it can only help the capacity of the network through feedback of the received symbol to node 1. Therefore, we can delete side information of messages $\widetilde{W}_{j,1}, j = 2, 3 \ldots K$ at destination 1 without affecting the converse argument. The rest of the proof is very similar to the proof of the Theorem 1 and we only highlight the differences here. Similar to the first stage of the proof in Section III-A, the network of Lemma 1 is converted to a 4 node X network of Figure 5. This is done by allowing destination nodes $1, 2 \ldots K-1$ to co-operate with each other and source nodes $2, \ldots K$ to co-operate with each other. As usual, since co-operation does not reduce capacity, this argument does not affect the converse argument.

The network is thus transformed to a 4 node X network (Figure 5) with input-output relations described by

$$\overline{Y}_i(n) = \sum_{j=1}^{2} \overline{H}_{i,j}(n)\overline{X}_j(n) + \overline{Z}_i(n), \qquad i \in \{3,4\} \tag{37}$$

where

$$\overline{X}_1(n) = Y_p(n) \tag{38}$$

$$\overline{X}_2(n) = [X_1(n) \;\; X_2(n) \;\; \cdots \;\; X_{p-1}(n) \;\; X_{p+1}(n) \;\; \cdots \;\; X_K(n)]^T \tag{39}$$

$$\overline{Y}_4(n) = Y_q(n) \tag{40}$$

$$\overline{Y}_3(n) = [Y_1(n) \;\; Y_2(n) \;\; \cdots \;\; Y_{q-1}(n) \;\; Y_{q+1}(n) \;\; \cdots \;\; Y_K(n)]^T \tag{41}$$

Nodes 2 and 3 act as multiple antenna nodes, each with $K-1$ antennas. $\overline{X}_i(n), \overline{Z}_j(n)$ are also defined in a corresponding manner for $i = 1, 2, j = 3, 4$. The definition of the channel coefficients $\overline{H}_{i,j}(n)$ is clear from equations (37) and (32), and from Figures 4, 5 and 3. Note that $\overline{H}_{4,1} = H_{K,1} \neq 0$ since we have $p \neq q$. The messages in this 4 node X network are defined as follows

$$\overline{W}_{3,1} = \left[\widetilde{W}_{2,1} \;\; \widetilde{W}_{3,1} \;\; \cdots \;\; \widetilde{W}_{K-1,1}\right] \tag{42}$$

$$\overline{W}_{3,2} = \phi \tag{43}$$

$$\overline{W}_{4,1} = \widetilde{W}_{K,1} \tag{44}$$

$$\overline{W}_{4,2} = \left[\widetilde{W}_{K,2} \quad \widetilde{W}_{q,3} \quad \cdots \quad \widetilde{W}_{K,K-1}\right] \tag{45}$$

The encoding and decoding functions, for codewords of length K over this 4 node X network are defined as

$$\begin{aligned}
\overline{X}_1(n) &= \overline{f}_{1,n}(\overline{W}_{3,1}, \overline{W}_{4,1}, \overline{Y}_3^{n-1}, \overline{Y}_4^{n-1}) \\
\overline{X}_2(n) &= \overline{f}_{2,n}(\overline{W}_{4,2}, \overline{Y}_3^{n-1}, \overline{Y}_4^{n-1}) \\
\hat{\overline{W}}_{4,i} &= \overline{g}_{4,i}(\overline{Y}_4^N), i = 1, 2 \\
\hat{\overline{W}}_{3,1} &= \overline{g}_{3,1}(\overline{Y}_3^N, \overline{W}_{4,2})
\end{aligned}$$

We allow multi-antenna node 3 to have apriori knowledge of message $\overline{W}_{4,2}$ through a genie. We also allow perfect feedback from destination nodes $3, 4$ to source nodes $1, 2$. Note that the side information through feedback and genie in the 4 node X network constructed is stronger than the information at the corresponding nodes in the original K user network of Figure 4. Since we are only providing an outerbound on the degrees of freedom region, the argument is not affected. Now, over this network, we claim that the converse shown in the second stage of the proof of Theorem 1 holds. The 4 node X network differs from the network of Section III-A (Figure 3) in two aspects:

1) In the X network considered in this section, node 3 has information of message $W_{4,2}$ apriori. In the 4 node X network of Figure 3, node 3 does not have this side information.
2) The network constructed here is not fully connected since certain channel co-efficients are equal to zero. However $\overline{H}_{4,1}$ is non-zero.

1) does not affect the converse argument of Theorem 1 because the converse for the 4 node X network begins with the genie providing information of $W_{4,2}$ and $W_{4,1}$ to node 3. 2) does not affect the converse argument because, as noted in Section III-C, the bound in (14) holds as long as $\overline{H}_{4,1} \neq 0$. Therefore, the bound of (14) holds for the 4 node network in consideration here i.e. the network defined by equations (37) and we can write

$$\begin{aligned}
\overline{d}_{3,1} + \overline{d}_{4,1} + \overline{d}_{4,2} &\leq 1 \\
\Rightarrow \sum_{i=2}^{K} d_{i,1} + \sum_{j=1}^{K-1} d_{K,j} - d_{K,1} &\leq 1
\end{aligned}$$

where $\overline{d}_{i,j}$ represents the number of degrees of freedom corresponding to message $W_{i,j}$. The desired result follows from the final equation above. ■

Remark : Full duplex operation can increase the degrees of freedom if the same node can be the source for one message and the destination for another message. For example, in a network of $2K$ users, assuming half-duplex operation, the optimal arrangement is to operate as a $K \times 0 \times K$ network which has $d_{hd} = \frac{K^2}{2K-1}$ degrees of freedom. However, with full-duplex operation, the lower bound of Theorem 2 implies that $d_{fd} \geq K > d_{hd}$ where d_{fd} represents the degrees of freedom of the network where the nodes are full-duplex. Thus, full-duplex operation can increase the degrees of freedom when source nodes are not disjoint from destination nodes. However, note that Theorem 1 and its first corollary imply that full-duplex operation does not increase degrees of freedom of a network whose source nodes are disjoint from destination nodes.

V. Conclusion

We characterize the capacity, within $o(\log(\text{SNR}))$ of a fully connected network with S source nodes, R relays and D destination nodes with full duplex operation and feedback. We also provide bounds on capacity approximations within $o(\log(\text{SNR}))$ of the K user fully connected network in which there is a message from every node to every other node. The lower and upper bound provided are tight if K is large. Apart from the small gap between the bounds of the K user fully connected network, this work effectively solves the degrees of freedom problem for a fairly large class of wireless networks with time-varying/frequency-selective channel gains. A major implication of our result is that the techniques of relays, perfect feedback to source nodes, noisy co-operation and full duplex operation do not increase the degrees of freedom of fully connected frequency-selective interference and X networks. An important limitation of our results is the assumption of time-varying and/or frequency selective channel gains for the achievability schemes based on interference alignment. However, the outerbounds of Theorem 1 and Theorem 2 are fairly general and hold for all fully connected networks whether the channel coefficients are time-varying or constants. Finally, it must be noted that the results of this work assume that all source and destination nodes have only a single antenna. The impact of relays, feedback, full-duplex operation and co-operation for networks with multi-antenna nodes in the general framework of this correspondence remains to be studied.

From a practical perspective, the fact that relays, feedback, etc. do not increase degrees of freedom of wireless X networks does not necessarily discourage the application of these techniques in real communication scenarios. By nature of the degrees of freedom approximation, the capacity characterization is an asymptotic result valid only at high SNR (in theory, as SNR tends to infinity). Our result does not preclude huge benefits in terms of capacity of these networks at low or mid-range SNR from co-operation induced by full-duplex relays, causal feedback or noisy channels. Further, since the achievable scheme of this paper requires global channel knowledge and frequency-selectivity, relays and the other factors may potentially have positive degrees of freedom benefits when only local channel knowledge is present or if channel gains are not frequency-selective (See, for example, [13]). Finally, as observed in Section III-C and towards the end of Section IV, these techniques can improve degrees of freedom in scenarios precluded by the $S \times R \times D$ network, such as when the network is not fully connected or when feedback is provided to a decoding node, or when source nodes can be destination nodes as well.

APPENDIX I
PROOF OF INNERBOUND OF THEOREM 2 : ACHIEVABLE SCHEME

Note that the innerbound of theorem 2 is equivalent to

$$d_{fd} \geq \frac{K(K-1)}{2K-2} = \frac{K}{2}$$

where d_{fd} represents the number of degrees of freedom of the K user full duplex network.

The achievability proof is based on interference alignment over the channel described in lemma 1 (Figure 4(b)). Since many of the details are identical to [4], we focus here on the unique aspects of this proof.

Let $\Gamma = (K-1)(K-2)$. We show that $K(K-1)n^{\Gamma}$ degrees of freedom are achievable over a

$$\mu_n = (K-1)\left((n+1)^{\Gamma} + n^{\Gamma}\right)$$

symbol extension of the channel for any $n \in \mathbb{N}$ thus implying the desired result. Over the extended channel, the scheme achieves n^{Γ} degrees of freedom for each of the $K(K-1)$ messages $\widetilde{W}_{i,j}, j \neq i$. The signal vector in the extended channel at the j^{th} user's receiver can be expressed as

$$\mathbf{Y}_j(\kappa) = \sum_{i=1}^{M} \mathbf{H}_{j,i}(\kappa)\mathbf{X}_i(\kappa) + \mathbf{Z}_j$$

where $\mathbf{X}_i$ is a $\mu_n \times 1$ column vector representing the μ_n symbol extension of the transmitted symbol X_i, i.e

$$\mathbf{X}_i(\kappa) \triangleq \begin{bmatrix} \widetilde{X}_i(\mu_n\kappa+1) \\ \widetilde{X}_i(\mu_n\kappa+2) \\ \vdots \\ \widetilde{X}_i(\mu_n(\kappa+1)) \end{bmatrix}$$

Similarly $\mathbf{Y}_i$ and $\mathbf{Z}_i$ represent μ_n symbol extensions of the $\widetilde{Y}_i$ and $\widetilde{Z}_i$ respectively. $\mathbf{H}_{i,j}$ is a diagonal $\mu_n \times \mu_n$ matrix representing the μ_n symbol extension of the channel. Similar to the interference alignment based achievable schemes of the interference and X channels, the message $\widetilde{W}_{i,j}$ is encoded at transmitter j as n^Γ independent streams so that $\mathbf{X}_j$ is

$$\mathbf{X}_j(\kappa) = \sum_{i=\{1,2\ldots K\}-\{j\}} \sum_{m=1}^{(n+1)^\Gamma} x_{i,j}^{[m]}(\kappa)\mathbf{v}_{i,j}^{[m]}(\kappa) = \sum_{i=\{1,2,\ldots K\}-\{j\}} \mathbf{V}_{i,j}(\kappa)\mathbf{x}_{i,j}(\kappa)$$

The received signal at the k^{th} receiver can then be written as

$$\mathbf{Y}_k(\kappa) = \sum_{i=1}^{M} \mathbf{H}_{k,i}(\kappa)\big(\sum_{j=1}^{N} \mathbf{V}_{j,i}(\kappa)\mathbf{x}_{j,i}(\kappa)\big) + \mathbf{Z}_k(\kappa)$$

We now need to ensure that at receiver j, the $(K-1)(K-2)$ interfering spaces $\mathbf{V}_{k,i}, k \neq i, k \neq j, i \neq j$ lie in a $(K-1)(n+1)^\Gamma$ dimensional space so that $(K-1)n^\Gamma$ desired spaces $\mathbf{V}_{j,i}, i \in \{1, 2\ldots\} - \{j\}$ can be decoded free of interference from a μ_n dimensional space. To do this, we first set

$$\mathbf{V}_{j,i} = \mathbf{V}_j, \forall i \neq j$$

Then, we design $\mathbf{V}_j, j = 1, 2\ldots K$ so that they satisfy the interference alignment equations below.

$$\mathbf{H}_{i,j}\mathbf{V}_k \prec \mathbf{I}_k, \forall\{(i,j,k) : i \neq k, k \neq j, j \neq i\} \tag{46}$$

such that rank$(\mathbf{I}_k) = (n+1)^\Gamma$ where $\mathbf{P} \prec \mathbf{Q}$ implies that the span of the column vectors of $\mathbf{P}$ lies in the vector space spanned by the column vectors of $\mathbf{Q}$.. Note that for a fixed k, there are $\Gamma = (K-1)(K-2)$ relations of the above form. We first generate $\mu_n \times 1$ column vectors $\mathbf{w}_k, k = 1, 2\ldots K$ so that all the entries of $\mathbf{w}_k$ are drawn from any continuous distribution independently from each other and independently from all other entries in $\mathbf{w}_l, l \neq k$. The rest of the proof is similar to the achievable scheme for the X channel presented in [4]. It is easy to observe that the dimension of the interfering space at receiver k space is equal to the dimension of the space spanned by all column vectors of matrices $\mathbf{I}_j, j \neq k$ which is equal to $(K-1)(n+1)^\Gamma$. The only difference from the model in [4] is that here, we have $\mathbf{H}_{i,j} = \mathbf{H}_{j,i}$ whereas, in [4] the matrix $\mathbf{H}_{j,i}$ is independent from $\mathbf{H}_{i,j}$. However this difference does not affect the construction of vectors satisfying the desired interference alignment relations (46). The difference does not affect the argument that at any receiver, the signal space is linearly independent with the interference space since the argument only depends on $\mathbf{w}_k$ being independent of $\mathbf{w}_l$ for $l \neq k$. The only condition that needs to be verified is that all the desired streams of at receiver k are linearly independent of each other. In other words, all that needs to be shown is that the column vectors of

$$\begin{aligned} \mathbf{D}_k &= [\mathbf{H}_{k,1}\mathbf{V}_{k,1} \;\; \mathbf{H}_{k,2}\mathbf{V}_{k,2} \;\; \ldots \;\; \mathbf{H}_{k,k-1}\mathbf{V}_{k,k-1} \;\; \mathbf{H}_{k,k+1}\mathbf{V}_{k,k+1} \;\; \mathbf{H}_{k,K}\mathbf{V}_{k,K}] \\ &= [\mathbf{H}_{k,1}\mathbf{V}_k \;\; \mathbf{H}_{k,2}\mathbf{V}_k \;\; \ldots \;\; \mathbf{H}_k\mathbf{V}_k \;\; \mathbf{H}_{k,k+1}\mathbf{V}_k \;\; \mathbf{H}_{k,K}\mathbf{V}_k] \end{aligned}$$

are linearly independent. The linear independence follows from the fact that the construction of $\mathbf{V}_k$ satisfying the relations of (46) is independent of both $\mathbf{H}_{k,i}$ and $\mathbf{H}_{i,k}$ for $i \neq k$. Again, the reader is referred to the achievable scheme in [4] for a formal proof of the same.

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