# Parallel Gaussian Interference Channels are Not Always Separable

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**Abstract**

It is known that the capacity of parallel (multi-carrier) Gaussian point-to-point, multiple access and broadcast channels can be achieved by separate encoding for each subchannel (carrier) subject to a power allocation across carriers. In this paper we show that such a separation does not apply to parallel Gaussian interference channels in general. A counter-example is provided in the form of a 3 user interference channel where separate encoding can only achieve a sum capacity of $2\log(1+3\text{SNR})$ while the actual capacity, achieved only by joint-encoding across carriers, is $3\log(1+2\text{SNR}))$. As a byproduct of our analysis, we propose a class of multiple-access-outer bounds on the capacity of the 3 user interference channel.

**Index Terms**

Capacity, Degrees of Freedom, Multiple Access, Interference, Multiuser Detection

## I. Introduction

The study of parallel Gaussian channels is motivated by the frequency-selective or time-varying nature of the wireless channel. With multi-carrier modulation, (assuming no inter-carrier interference (ICI)) a frequency selective channel can be viewed as a set of parallel channels with channel coefficients that vary from one carrier to another but may be assumed constant (flat-fading) over each carrier. Similarly, if inter-symbol-interference (ISI) is absent, the time-varying channel gives rise to parallel channels whose values are fixed during each symbol but vary from one symbol to another. In this paper, we will use the terminology of frequency-selective channels and multi-carrier modulation to refer to parallel Gaussian channels. It is understood that the model is equally applicable to the time-varying channel as well.

It is well known that over the parallel Gaussian point-to-point channel, coding separately over the individual subchannels (carriers) achieves the capacity subject to optimal power allocation. Thus the capacity of the parallel Gaussian point-to-point channel is equal to the sum of the capacities of the point-to-point Gaussian subchannels with corresponding powers chosen through the water-filling algorithm. Similarly, it has also been shown that separate coding over each carrier is optimal for parallel Gaussian multiple access (MAC) and broadcast (BC) channels [1], [2]. The separability of parallel Gaussian point-to-point, MAC and BC is useful because it provides a direct connection between the single-carrier channel models studied extensively in classical information theory and the frequency-selective (or time varying) channels that may be more relevant in practice. Coding schemes designed for the classical (single carrier) models can be applied directly to multi-carrier systems subject to a power allocation across carriers. A key question that remains open is whether such a separation holds for other Gaussian networks, and in particular, if separate encoding is optimal for multi-carrier interference networks.

Much work on multi-carrier interference networks (e.g. in the context of DSL [3]–[10]) has focused on optimal power allocation across carriers under the assumption of *separate coding* over each carrier. For the two user parallel interference channel with strong interference it is shown in [10] that indeed the sum capacity is the sum of the rates that can be achieved by separately encoding over each carrier subject to an overall power optimization. For the case where more than 2 users are present or when the channels are not restricted to the strong interference case, since the capacity of even the single-carrier interference channel is not known, usually the rate optimization is carried out under the practically motivated assumption that all interference is to be treated as noise. Both centralized

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and distributed algorithms, some of which are based on game-theoretic formulations, have been proposed for this "dynamic spectrum management" problem and the optimality and convergence properties of these algorithms have been established under the separate encoding assumption.

Joint encoding of multiple-carriers has been used recently in [11] to characterize the sum capacity per carrier, of the $K$ user multi-carrier Gaussian interference channel. The sum capacity (per carrier) is found to be

$$C(\text{SNR}) = \frac{K}{2}\log(\text{SNR}) + o(\log(\text{SNR})),$$

where SNR[1] represents the signal to noise power ratio. In other words, the $K$ user interference channel has $K/2$ degrees of freedom[2] per orthogonal time and frequency dimension. The key to the capacity characterization is a novel interference alignment scheme introduced in [11]. Interference alignment, as defined in [13], refers to the construction of signals such that they cast overlapping shadows at the receivers where they constitute interference while they remain distinghishable at the receivers where they are desired. The interference alignment constructions proposed in [11] are based on *joint encoding* over multiple frequencies. Due to interference alignment, the joint encoding scheme of [11] outperforms the dynamic spectrum management schemes of [4]–[8] in terms of degrees of freedom [3]. However, it has not been shown that this joint encoding is *necessary* to achieve capacity. Interestingly, another recent work in [14] has provided examples where interference alignment is achieved over a single-carrier interference channel, i.e., with separate encoding. Thus, it remains unclear whether the capacity of multi-carrier interference channels can be achieved by separate encoding over each carrier and a power allocation across carriers. It is this open problem that we address in this paper. Our main contribution, stated in Theorem 1 is an example of two parallel interference channels, each of which by itself has only 1 degree of freedom, but when taken together they allow 3 degrees of freedom. Thus, we show that for the capacity of parallel interference channels, *the whole is more than the sum of its parts.* Moreover, our example shows that even within the restricted class of coding schemes where interference must be treated as noise (e.g., the vast majority of coding schemes considered in the dynamic spectrum management literature), separate coding is strictly sub-optimal compared to joint coding across parallel channels.

As a byproduct of our analysis we also find a class of outer bounds on the capacity of the 3 user interference channel. These outer bounds share the property that one receiver (possibly aided by a genie and/or noise reduction) is able to decode all messages - so that the multiple-access channel capacity to the genie-aided receiver becomes an outer bound on the sum capacity of the 3 user interference channel. The MAC outer bounds can be viewed as a generalization, to 3 users, of Kramer's genie-aided approach [15] and Carleial's [16] noise-reduction approach. These outer bounds play an important role in identifying singularity conditions for interference channels that have only one degree of freedom. However, the bounds are generally loose in the degrees of freedom sense and tighter bounds at high SNR may be obtained by an application of Carleial's outer bound on each of the 2 user channels contained within the $K$ user interference channel.

## II. Channel Model

Consider the 3 user, memoryless interference channel, with $M$ parallel sub-channels, defined as:

$$\mathbf{Y}^{[m]}(n) = \mathbf{H}^{[m]}\mathbf{X}^{[m]}(n) + \mathbf{Z}^{[m]}(n),\ \ m \in \{1, 2, \cdots, M\} \tag{1}$$

where, during the $n^{th}$ channel use,

$$\begin{aligned}
\mathbf{Y}^{[m]}(n) &= [Y_1^{[m]}(n), Y_2^{[m]}(n), Y_3^{[m]}(n)] &(2)\\
\mathbf{X}^{[m]}(n) &= [X_1^{[m]}(n), X_2^{[m]}(n), X_3^{[m]}(n)] &(3)\\
\mathbf{Z}^{[m]}(n) &= [Z_1^{[m]}(n), Z_2^{[m]}(n), Z_3^{[m]}(n)] &(4)
\end{aligned}$$

are the vectors containing the received symbols, the transmitted symbols and the zero mean unit variance circularly symmetric complex Gaussian AWGN terms, respectively, for users indexed by the subscripts, over sub-channels

[1]For this result, SNR may be interpreted as the transmit power at each transmitter with the additive white Gaussian noise power (AWGN) at each receiver normalized to unity.

[2]Also known as multiplexing-gain (See [12]) or capacity pre-log.

[3]Interestingly, in both cases interference is treated as noise, so no multiuser detection is involved.

that are indexed by the superscripts. For each sub-channel $m \in \{1, 2, \cdots, M\}$, the channel matrix $\mathbf{H}^{[m]}$ is a $3 \times 3$ matrix with elements $H_{k,j}^{[m]}$ representing the channel coefficients from transmitter $j$ to receiver $k$, $j, k \in \{1, 2, 3\}$. Each sub-channel matrix is fixed across channel uses. All channel coefficients, inputs and outputs as well as noise terms are complex[4].

For codewords spanning $N$ channel uses, the transmitted symbols from transmitter $k$, $[X_k^1(n), \cdots, X_k^{[M]}(n)]$, $n = 1, 2, \cdots, N$, are subject to the total transmit power constraint

$$\frac{1}{N} \sum_{n=1}^{N} \sum_{m=1}^{[m]} E[|X_k^{[m]}(n)|^2] \leq P_k \tag{5}$$

To avoid cumbersome notation, the channel-use index $n$ is not explicitly mentioned henceforth.

Transmitter $i$ has message $W_i$ for receiver $i$ for $i = 1, 2, 3$. The messages are independent. The rate of the $i$th user is defined as $R_i = \frac{\log(|W_i|)}{N}$ where $|W_i|$ is the cardinality of the message set corresponding to message $W_i$. A rate vector $\mathbf{R} = (R_1, R_2, R_3)$ is said to be *achievable* if messages $W_i, i = 1, 2, 3$, can be simultaneously encoded at rates $R_i, i = 1, 2, 3$ so that the probability of decoding error can be made arbitrarily small by choosing an appropriately large $N$. The capacity region $C(\mathbf{P})$ is the closure of the set of all achievable rate vectors in the network, where we define $\mathbf{P} = [P_1, P_2, P_3]$ as the vector of transmit powers. The sum capacity $C_\Sigma(\mathbf{P})$ of the network is defined as

$$C_\Sigma(\mathbf{P}) = \max_{\mathbf{R} \in \mathbf{C}(\mathbf{P})} \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{3}} \mathbf{R_i}$$

*Separability:* We say that the 3 user interference network is separable if and only if

$$C_\Sigma(\mathbf{P}) = \max_{\mathbf{P}^{[m]}: \sum_{m=1}^{M} \mathbf{P}^{[m]} = \mathbf{P}} \quad \sum_{m=1}^{M} C_\Sigma^{[m]}(\mathbf{P}^{[m]}) \tag{6}$$

where $C_\Sigma(\mathbf{P}^{[m]})$ is the sum-capacity of the $m^{th}$ sub-channel subject to a transmit power constraint vector $\mathbf{P}^{[m]}$. Thus, parallel interference channels are separable if the sum capacity can be achieved by achieving the stand-alone sum-capacity of each sub-channel, subject to an optimal power allocation across sub-channels.

*Notation:* The notation $Z \sim \mathcal{N}^c(a, b)$ indicates that the random variable $Z$ follows a circularly symmetric complex Gaussian distribution with mean $a$ and variance $b$. For a matrix $H$, $H^T$ refers to the transpose of $H$, Trace$[H]$ is the trace of $H$, det$(H)$ is the determinant of $H$ and $H^\dagger$ refers to the conjugate transpose of $H$.

## III. THE CASE OF ONLY ONE SUB-CHANNEL, $M = 1$

We start with a lemma, applicable to the case of only one-subchannel, $M = 1$. Since this lemma applies only to $M = 1$, we suppress the superscripts to simplify the notation.

Recall that the minor $M_{ij}(\mathbf{H})$ of the matrix $\mathbf{H}$ is the determinant of the $2 \times 2$ matrix that remains after removing from $\mathbf{H}$ its $i^{th}$ row and $j^{th}$ column. Also, the channel is called fully-connected if and only if all channel coefficients are non-zero.

**Lemma 1:** If $\exists (i, j, k) \in \{1, 2, 3\}, i \neq j, j \neq k, k \neq i$ for which $M_{ij}(\mathbf{H}) = 0$, then the sum-capacity of the 3 user fully-connected interference channel with channel matrix $\mathbf{H}$ is bounded as follows:

$$C_\Sigma(\mathbf{P}) \leq \log\left(1 + \frac{|H_{j1}|^2 P_1 + |H_{j2}|^2 P_2 + |H_{j3}|^2 P_3}{\min\left(1, \frac{|H_{ji}|^2}{|H_{ii}|^2}, \frac{|H_{jk}|^2}{|H_{kk}|^2}\right)}\right) \tag{7}$$

The following example illustrates an application of Lemma 1.

*Example:* Consider the channel matrix

$$\mathbf{H} \quad = \quad \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \tag{8}$$

[4]Note that the results of this work extend in a straightforward manner even if all symbols are restricted to take only real values.

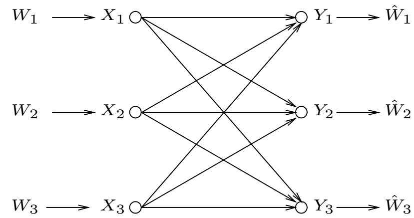


Fig. 1. The 3 user interference channel

This channel matrix satisfies the condition of Lemma 1, i.e. $M_{2,1} = 0$ and therefore, its sum-capacity is bounded by $\log(1+P_1+P_2+P_3)$. Note that this channel has only one degree of freedom (sum-capacity pre-log factor = 1). For this channel, interestingly, the sum-capacity is exactly $\log(1+P_1+P_2+P_3)$, i.e. the bound of Lemma 1 is tight. Achievability is straightforward, because (as we show next in the proof of Lemma 1) this is the multiple-access channel sum-capacity for each of the three multiple-access channels corresponding to receivers $1, 2, 3$ from all three transmitters. In other words, any reliable coding scheme for a multiple access channel to, e.g., receiver 1, is also a reliable coding scheme for receivers 2 and 3, so that they can all decode all three messages.

*Proof of Lemma 1:* As required by the statement of the lemma, let us assume that the channel **H** is fully connected (all elements are non-zero), and the condition of Lemma 1 is satisfied. Without loss of generality, we consider the case $M_{3,1} = 0$, so that

$$\frac{H_{2,3}}{H_{2,2}} = \frac{H_{1,3}}{H_{1,2}} = \gamma, \gamma \neq 0.$$

Consider any coding scheme that achieves a rate tuple $(R_1, R_2, R_3)$, and where the probability of error can be made arbitrarily small by using appropriately long codewords. In particular, suppose $P_e \leq \epsilon/3$ is achievable with codewords of length $N$. We will define a new receiver, called receiver 0, and show that the same coding scheme achieves probability of error $P_e \leq \epsilon$ for the multiple access channel from all transmitters to receiver $0$, i.e., any rate pair achievable on the interference channel is also available on the multiple access channel to receiver $0$. The sum-capacity of the multiple access channel therefore becomes an outerbound to the interference channel sum-capacity.

Define a new receiver, receiver $0$, with received signal $Y_0$, which is a possibly reduced-noise version of the received signal of receiver $1$.

$$\begin{aligned} Y_0 &= H_{11}Y_1 + H_{12}Y_2 + H_{13}Y_3 + Z_0 & (9)\\ Z_0 &\sim \mathcal{N}^c\left(0, \min\left(1, \frac{|H_{12}|^2}{|H_{22}|^2}, \frac{|H_{13}|^2}{|H_{33}|^2}\right)\right) & (10)\end{aligned}$$

Since $Y_1$ is a degraded version of $Y_0$, if receiver 1 can decode his message $W_1$ from his received signal $Y_1$, then receiver 0 can decode $W_1$ as well.

After decoding message $W_1$ successfully, receiver 0 is able to reconstruct transmitter 1's symbols $X_1$,[5] which allows it to create a new signal:

$$\begin{aligned} Y_0' &= \frac{H_{22}}{H_{12}}(Y_0 - H_{11}X_1) + H_{21}X_1 & (11)\\ &= H_{21}X_1 + H_{22}X_2 + H_{23}X_3 + Z_0' & (12)\\ Z_0' &\sim \mathcal{N}^c\left(0, \min\left(\frac{|H_{22}|^2}{|H_{12}|^2}, 1, \frac{|H_{22}|^2|H_{13}|^2}{|H_{12}|^2|H_{33}|^2}\right)\right) & (13)\end{aligned}$$

Since $Y_2$ is a degraded version of $Y_0'$, if receiver 2 can decode his message $W_2$ from his received signal $Y_2$, then receiver 0 can decode $W_2$ as well.

[5]It is easily shown that there is no loss of generality in the assumption that the mapping from messages to codewords is deterministic. See, e.g. Proposition 1 in [17].

After decoding message $W_1, W_2$, receiver 0 is able to reconstruct both $X_1, X_2$ which allows it to create a new signal:

$$\begin{aligned} Y_0'' &= \frac{H_{33}}{H_{13}}(Y_0 - H_{11}X_1 - H_{12}X_2) + H_{31}X_1 + H_{32}X_2 && (14)\\ &= H_{31}X_1 + H_{32}X_2 + H_{33}X_3 + Z_0'' && (15)\\ Z_0'' &\sim \mathcal{N}^c\left(0, \min\left(\frac{|H_{33}|^2}{|H_{13}|^2}, \frac{|H_{33}|^2|H_{12}|^2}{|H_{13}|^2|H_{22}|^2}, 1\right)\right) && (16)\end{aligned}$$

Since $Y_3$ is a degraded version of $Y_0''$, if receiver 3 can decode his message $W_3$ from his received signal $Y_3$, then receiver 0 can decode $W_3$ as well.

We have established that receiver 0 is able to decode all three messages. The probability of error for the multiple access channel is easily bounded by the union bound, $P_e \leq \epsilon$. The sum-rate, therefore, cannot be more than the sum-capacity of all the multiple access channel from all three transmitters to receiver 0. Since this is true for any reliable coding scheme, this gives us a sum-capacity bound for the interference channel

$$C_\Sigma(\mathbf{P}) \leq \log\left(1 + \frac{H_{11}^2P_1 + H_{12}^2P_2 + H_{13}^2P_3}{\min\left(1, \frac{|H_{12}|^2}{|H_{22}|^2}, \frac{|H_{13}|^2}{|H_{33}|^2}\right)}\right) \tag{17}$$

Thus the proof of Lemma 1 is complete. ■

Lemma 1 has interesting implications for the degrees-of-freedom characterization of constant interference channels. While the degrees of freedom are known for $K$ user interference channels that are time-varying or frequency-selective [11], the degrees of freedom of even a 3 user fully-connected interference channel with *constant* channel coefficients remain unknown in general. Several examples are known [11], [14], [18] of 3 user interference channels that can achieve more than 1 degree of freedom, using various forms of interference alignment. Lemma 1 provides a counterpoint, identifying a class of 3 user interference channels which have only 1 degree of freedom.

An interesting interpretation of the counterexample presented above is the following. Consider a game that is played between two players. The players will pick the channel coefficient values for a (single-carrier) 3 user interference channel. Player 1 intends to maximize the number of degrees of freedom of the channel. Player 2 wants to minimize the number of degrees of freedom of the channel. In this game, player 1 moves first and player 2 moves second. During his turn, player 1 is allowed to select the values of all the channel coefficients. Player 2 can only change the value of 1 channel coefficient after the values have been chosen by player 1. Which channel coefficient player 2 is allowed to change is also decided by player 1. There is a constraint that all channel co-efficients must be non-zero. First, consider the constant interference channel. Note that [14] has already shown that there exist 3 user channels with close to $3/2$ degrees of freedom. Therefore, in absence of player 2, player 1 can design a channel that will achieve close to $3/2$ degrees of freedom. However, if player 2 can control any one of the channel co-efficients, he can use the result of Lemma 1 to win the game by reducing the number of degrees of freedom to unity. For example, if player 2 has control of $h_{1,2}$, he can choose the channel co-efficient to be equal to $\frac{h_{1,3}h_{2,2}}{h_{2,3}}$ to ensure that the channel has only 1 degree of freedom. Thus, in a constant single-carrier channel, player 2 wins the game.

Now, suppose the channel coefficients vary with time, i.e., we have a parallel Gaussian channel. At each time instant the players take turns to design the channel coefficients according to the rules described above. Corresponding to each sub-channel, player 2 has control of one of the channel co-efficients. In this case, player 2 can kill the degrees of freedom of the individual subchannels by using Lemma 1. However, player 1 still wins the game since $3/2$ degrees of freedom are achievable through the interference alignment scheme of [11] which codes across all parallel channels. Thus, in the time-varying case, player 1 wins the game.

## IV. Parallel Interference Channels, $M > 1$

The following theorem states the main result of this paper.

**Theorem 1:** Parallel interference channels are not always separable, i.e.,

$$C_\Sigma(\mathbf{P}) \geq \max_{\mathbf{P}^{[m]}:\sum_{m=1}^{M}\mathbf{P}^{[m]}=\mathbf{P}} \sum_{m=1}^{M} C_\Sigma^{[m]}(\mathbf{P}^{[m]}) \tag{18}$$

and the inequality cannot be replaced with an equality in general.

*Proof of Theorem 1:* The proof is through a specific example. Consider two parallel interference channels, i.e., $M = 2$,

$$\mathbf{H}^1 = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \tag{19}$$

$$\mathbf{H}^2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \tag{20}$$

and symmetric power constraints $\frac{P_1}{M} = \frac{P_2}{M} = \frac{P_3}{M} = \text{SNR}$.

*Separate Coding Capacity:* From Lemma 1, we find that the sum-capacity of sub-channel $\mathbf{H}^1$ under power constraint $\mathbf{P}^1 = [\text{SNR}, \text{SNR}, \text{SNR}]$ is $\log(1 + 3\text{SNR})$. This is easily seen because Lemma 1 provides the converse, while achievability follows by an orthogonal time division scheme, i.e., each user transmits for a fraction $1/3$ of the total time, with power 3SNR. Similarly, the sum-capacity of sub-channel $\mathbf{H}^2$ under power constraint $\mathbf{P}^2 = [\text{SNR}, \text{SNR}, \text{SNR}]$ is $\log(1+3\text{SNR})$. The optimality of uniform power allocation follows from the symmetry of the channels and we have the maximum rate achievable with separate coding

$$\max_{\mathbf{P}^{[m]}:\sum_{m=1}^{M}\mathbf{P}^{[m]}=\mathbf{P}} \sum_{m=1}^{M} C_{\Sigma}^{[m]}(\mathbf{P}^{[m]}) = 2\log(1 + 3\text{SNR}) \tag{21}$$

On the other hand, we show that the sum-capacity of these parallel channels, taken together, is $3\log(1+2\text{SNR}) > 2\log(1+3\text{SNR})$. This is achievable only by joint coding, thus establishing the inseparability of parallel interference channels.

*Joint Coding Capacity - Achievability:* The joint coding scheme that achieves the capacity $3\log(1 + 2\text{SNR})$ is particularly simple. Each transmitter repeats its transmitted symbol over the two sub-channels,

$$X_k^{[1]} = X_k^{[2]} = X_k$$

and each receiver simply adds the outputs of the two sub-channels

$$Y_k^{[1]} + Y_k^{[2]} = Y_k$$

Because of the special structure of these channel matrices,

$$\mathbf{H}^1 + \mathbf{H}^2 = 2\mathbf{I}, \tag{22}$$

all interference is cancelled, and the resulting interference channel is described as:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = 2\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \tag{23}$$

where

$$Z_k^{[1]} + Z_k^{[2]} = Z_k \sim \mathcal{N}^c(0, 2) \tag{24}$$

Desired signals add in phase so that the received signal power is quadrupled while the independent noise terms add to double the noise power. Thus the SNR is effectively doubled and each user achieves a rate $\log(1 + 2\text{SNR})$ for a sum-rate of $3\log(1 + 2\text{SNR})$.

*Joint Coding Capacity - Converse:* While the achievability of $3\log(1 + 2\text{SNR})$ is enough to prove the inseparability of parallel interference channels, we show that $3\log(1+2\text{SNR})$ is in fact the capacity of the parallel channels $\mathbf{H}^{[1]}, \mathbf{H}^{[2]}$. For the converse argument, we bound the sum-rate of two users at a time. For example, suppose we wish to find an outer bound on the sum-rate of users 1 and 2. We eliminate user 3, which cannot hurt users 1 and 2, to obtain the two user parallel interference channel.

$$\text{Receiver 1:}\quad \begin{bmatrix} Y_1^{[1]} \\ Y_1^{[2]} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_1^{[1]} \\ X_1^{[2]} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} X_2^{[1]} \\ X_2^{[2]} \end{bmatrix} + \begin{bmatrix} Z_1^{[1]} \\ Z_1^{[2]} \end{bmatrix} \tag{25}$$

$$\text{Receiver 2:}\quad \begin{bmatrix} Y_2^{[1]} \\ Y_2^{[2]} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_1^{[1]} \\ X_1^{[2]} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_2^{[1]} \\ X_2^{[2]} \end{bmatrix} + \begin{bmatrix} Z_2^{[1]} \\ Z_2^{[2]} \end{bmatrix} \tag{26}$$

On this channel we establish a multiple-access bound as follows. Consider any reliable coding scheme that achieves rates $R_1, R_2$. Since the coding scheme is reliable, receiver 1 can decode his message and reconstruct $X_1^{[1]}, X_1^{[2]}$, which allows receiver 1 to generate the following equivalent signal.

$$\begin{bmatrix} \overline{Y}_1^{[1]} \\ \overline{Y}_1^{[2]} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left( \begin{bmatrix} Y_1^{[1]} \\ Y_1^{[2]} \end{bmatrix} - \begin{bmatrix} X_1^{[1]} \\ X_1^{[2]} \end{bmatrix} \right) + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_1^{[1]} \\ X_1^{[2]} \end{bmatrix} \tag{27}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_1^{[1]} \\ X_1^{[2]} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_2^{[1]} \\ X_2^{[2]} \end{bmatrix} + \begin{bmatrix} Z_1^{[1]} \\ -Z_1^{[2]} \end{bmatrix} \tag{28}$$

which is statistically equivalent to the received signal (26) of receiver 2. Since receiver 2 is able to decode $W_2$ from his received signal, receiver 1 can decode $W_2$ as well. Now, since receiver 1 is able to decode both messages $W_1, W_2$, the sum-rate of the interference channel is bounded above by the sum-capacity of the multiple access channel from transmitters $1, 2$ to receiver 1. Thus, we have the following outer bound on $R_1 + R_2$,

$$R_1 + R_2 \leq \max \log \det \left( \left[ \mathbf{I} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{Q}_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{Q}_2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right] \right) \tag{29}$$

$$\text{s.t.}\quad \mathrm{E}\left( \begin{bmatrix} X_1^{[1]} \\ X_1^{[2]} \end{bmatrix} \begin{bmatrix} X_1^{[1]} \\ X_1^{[2]} \end{bmatrix}^{\dagger} \right) \triangleq \mathbf{Q}_1 \geq 0 \tag{30}$$

$$\mathrm{E}\left( \begin{bmatrix} X_2^{[1]} \\ X_2^{[2]} \end{bmatrix} \begin{bmatrix} X_2^{[1]} \\ X_2^{[2]} \end{bmatrix}^{\dagger} \right) \triangleq \mathbf{Q}_2 \geq 0 \tag{31}$$

$$\text{Trace}[\mathbf{Q}_1] \leq 2\text{SNR} \tag{32}$$

$$\text{Trace}[\mathbf{Q}_2] \leq 2\text{SNR} \tag{33}$$

Because of the concavity of the $\log\det(\cdot)$ function, it is maximized by $\mathbf{Q}_1 = \mathbf{Q}_2 = (\text{SNR})\mathbf{I}$, which results in the outer bound:

$$R_1 + R_2 \leq 2\log(1 + 2\text{SNR}) \tag{34}$$

Similarly, we find the outer bounds $R_1 + R_3 \leq 2\log(1 + 2\text{SNR})$, $R_2 + R_3 \leq 2\log(1 + 2\text{SNR})$. Adding all the bounds, we have the sum-capacity bound

$$C_\Sigma \leq 3\log(1 + 2\text{SNR}). \tag{35}$$

Since this is also achievable as described earlier, this is the capacity of the parallel interference channels in (19), (20). ■

Thus, unlike point-to-point, multiple-access and broadcast channels, in general separate coding *does not suffice* to achieve the capacity of parallel interference channels. Figure 2 illustrates the suboptimality of separate coding over each carrier in comparison with the interference alignment based joint coding scheme for the channel described in equations (19)-(20). Note that no such example can be constructed for the parallel Gaussian point to point, multiple access and broadcast channels because in all those cases separate coding over each carrier is capacity-optimal for *any* channel realization.

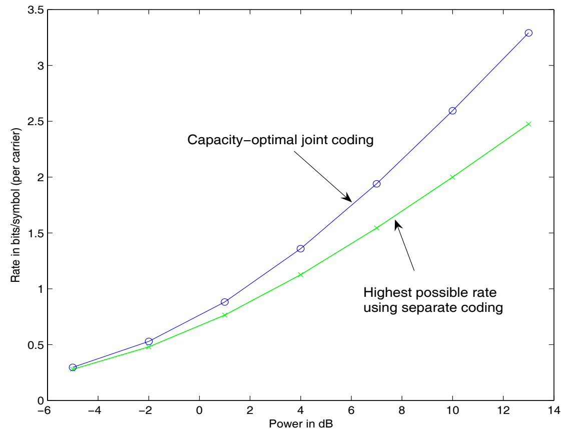


Fig. 2. A comparison of the joint coding capacity versus separate coding capacity on the parallel 3 user interference channel

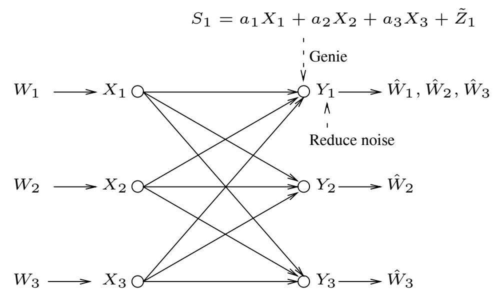


Fig. 3. Multiple access outer bound for the classical 3 user interference channel

## V. MULTIPLE ACCESS OUTER BOUNDS FOR THE CAPACITY OF 3 USER INTERFERENCE CHANNEL

In this section, we provide an interesting application of the result of Lemma 1 in the form of a class of outer bounds for the classical (single-carrier) 3 user interference channel. The outer bound argument goes as follows. Consider any achievable coding scheme. Using this coding scheme, receiver 1 can decode $W_1$. Our aim is to enhance receiver 1 with enough information so that it can decode $W_2$ and $W_3$ as well (see Figure 3). Then the capacity region of the multiple access channel(MAC) formed by the three transmitters and the (enhanced) receiver 1 forms an outer bound for the capacity region of the interference channel. The improvements to receiver 1 are described in the following steps

1) *To help receiver* 1 *decode* $W_2$ *:* Let a genie provide receiver 1 with $S_1 = a_1X_1+a_2X_2+a_3X_3+\tilde{Z}_1$ where $\tilde{Z}_1$ is an AWGN term independent of $X_i, i = 1, 2, 3$. Note that this side information effectively acts as an additional antenna at receiver 1. The noise term $\tilde{Z}_1$ can possibly be correlated with other noise variables $Z_i, i = 1, 2, 3$. Now, receiver 1 can linearly combine its received signal with its side information to form $U_1 = \alpha Y_1 + \beta S_1$ to form another (noisy) linear combination of the codewords $X_i, i = 1, 2, 3$. $\alpha$ and $\beta$ can be chosen such that the co-efficients of $X_1$ and $X_2$ in $U_1$ satisfy the conditions of Theorem 1. Note that if these channel co-efficients already satisfy the condition of 1, then side information of $S_1$ is not needed. Now, the proof of Theorem 1 implies that by sufficiently reducing the noise at receiver 1, we can ensure that receiver 1 decodes $W_2$ as well. Thus, with the aid of a genie and possibly reducing the noise, we have ensured that receiver 1 can decode $W_2$. Note that neither the genie information, nor the reduction of noise reduce the capacity of this channel and therefore do not affect the outer bound argument.
2) *To help receiver* 1 *decode* $W_3$ *:* Receiver 1, enhanced as described in the previous step, can now decode $W_1$

and $W_2$. We can now choose $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ such that

$$\begin{aligned} V_1 &= \bar{\alpha} X_1 + \bar{\beta} X_2 + \bar{\gamma} Y_1^{'} \\ &= H_{31} X_1 + H_{32} X_2 + H_{33} X_3 + \bar{\gamma} Z_1^{'}. \end{aligned}$$

Note that receiver 1 can form $V_1$. We use $Y_1^{'}$ and $Z_1^{'}$ above rather than $Y_1$ and $Z_1$ since the previous step involves reducing the noise at receiver 1. Statistically, $V_1$ differs from $Y_3$ only in the variance of the noise term. Therefore, by further reducing the noise if required, receiver 1 can also decode $W_3$. As in the previous step, it is important to note that the reduction of noise does not affect the outer bound argument

Steps 1 and 2 above imply that the capacity region of the 3 user Gaussian interference channel is outer-bounded by the capacity region of the single-input-multiple-output (SIMO) Gaussian MAC which receives $S_1$ on one antenna and a reduced-noise version of $Y_1$ on the other. This class of bounds can be optimized over $a_i, i = 1, 2, 3$ and the statistics of $\tilde{Z}_1$. Further, similar outer bounds can be found by enhancing receiver 2 or receiver 3 rather than receiver 1. Note that, since a MAC with two antennas has 2 degrees of freedom, this class of outer bounds is loose from the perspective of degrees of freedom. Using Carleial's outer bounds on each of the two user interference channels contained within the 3 user interference channel produces a degrees of freedom outer bound of $3/2$ (See [11], [19]).

*Example:* Consider the *perfectly symmetric* 3 user interference channel where $H_{ii} = 1, \forall i = 1, 2, 3$ and $H_{i,j} = h, \forall i \neq j, i, j \in \{1, 2, 3\}$. Suppose $h$ is real and greater than 1. Also, let the transmit power at each transmitter be equal to SNR. Since the channel does not satisfy the conditions of Theorem 1, a genie provides receiver 1 with information of $S_1 = a_1 X_1 + (1-h) X_2 + X_3 + \tilde{Z}_1$ where $\tilde{Z}_1$ is an i.i.d AWGN term correlated with $Z_1$ such that $E\left[|Z_1 + \tilde{Z}_1|^2\right] = 1$. Note that since we started with an achievable coding scheme, receiver 1 can decode $W_1$ using information from $Y_1$. Receiver 1 can subtract the effect of $X_1$ from $S_1$ and $Y_1$ and obtain $\tilde{S}_1 = (1-h) X_2 + \tilde{Z}_1$ and $\tilde{Y}_1 = hX_2 + hX_3 + Z_1$. Receiver 1 can now decode $X_2$ from $hX_1 + \tilde{Y}_1 + \tilde{S}_1$ since it is of the form $hX_1 + X_2 + hX_3 + Z_2^{'}$ where $Z_2^{'}$ is a AWGN term with unit variance. Now that receiver 1 is aware of $X_1$ and $X_2$, it can add appropriate terms to $Y_1$ to form $V_1 = h(hX_1 + hX_2 + X_3) + Z_1$. Since $h > 1$, $Y_3$ is a degraded version of $V_1$ which implies that receiver 1 can decode $W_3$ as well. Thus, all rates achievable in this interference channel, are achievable in the single-input-multiple-output (SIMO) multiple access channel with 3 single antenna nodes respectively transmitting $X_1, X_2, X_3$ and a two-antenna node receiving $Y_1$ along the first antenna and $S_1$ along the second. Thus, the capacity region of this multiple access channel is an outer-bound for the capacity of the interference channel. Furthermore, parameters $a_1$ and $\tilde{Z}_1$ are parameters which can be used for optimization. So, for example, we can bound the sum-capacity $C_\Sigma$ of the 3 user interference channel by

$$C_\Sigma \leq \min_{\substack{(a_1, \tilde{Z}_1) \\ E\left[(Z_1 + \tilde{Z}_1)^2\right] \leq 1 \\ \tilde{Z}_1 \sim \mathcal{N}^c(0, \sigma^2)}} \log\left(\frac{\det\left(\mathbf{K_z} + \text{SNR}\mathbf{H}\mathbf{H}^\dagger\right)}{\det\left(\mathbf{K_z}\right)}\right)$$

where $\mathbf{K_z}$ indicates the covariance matrix corresponding to noise vector $[Z_1 \quad \tilde{Z}_1]^T$ and

$$\mathbf{H} = \begin{bmatrix} 1 & h & h \\ a_1 & (1-h) & 0 \end{bmatrix}$$

## VI. Discussion

We showed that in general, independent coding over the various channel states of the parallel Gaussian interference channel is not capacity optimal. The key is that even though interference alignment may not be possible over each carrier, it may still be accomplished by coding across carriers. Combined with the interference alignment schemes used in [11] which rely on joint coding, the results of this paper indicate that the benefits of joint-coding are not limited to the specific example used to prove Theorem 1.

However, since our examples rely on interference alignment which is only known to be relevant for interference channels with 3 or more users, we have not shown that the 2 user parallel Gaussian interference channel is inseparable. Prior to this work, it was claimed in [20] that two user one-sided parallel Gaussian interference channels (also known as $Z$ interference channels) are separable. Following our work, Sankar et. al. [21] have shown that parallel $Z$ channels are, in fact, *not* separable, which also establishes the inseparability of parallel two user interference channel. It is interesting to note that in the two user interference channel, interference alignment is not relevant, because there is only one interferer seen by each receiver. The inseparability result for the 2 user interference channel therefore, highlights other interesting aspects of joint coding - e.g., a strong interference link in one subchannel can compensate for a weak interference link in another subchannel, so that joint encoding puts the overall channel into a strong interference regime. The distinct nature of the 2 and 3 user inseparability results is also evident in that for the 2 user interference channel joint coding does not have a *degree-of-freedom* advantage (i.e. the number of degrees of freedom remains equal to 1, with or without joint coding). Moreover, the 3 user example in this paper shows that joint coding, even with single user receivers (i.e. treating interference as noise), can beat the best separate coding scheme that utilizes optimal multiuser detection. To the best of our knowledge this is not the case for the 2 user interference channel where decoding part of the interference appears necessary for the inseparability result. Interestingly, it has been shown that separate coding is optimal if all sub-channels are in the very weak (noisy) interference regime [22].

The inseparability of parallel interference channels has been shown recently to have interesting implications for the capacity of ergodic fading interference channels [23]. From a practical perspective, it prompts a closer look at joint versus separate coding schemes in parallel Gaussian interference channels.

## REFERENCES


[1] D. Tse and S. Hanly, "Multiaccess fading channels. i. polymatroid structure, optimal resource allocation and throughput capacities," *IEEE Transactions on Information Theory*, vol. Vol.44, no. 7, 1998.

[2] D. Tse, "Optimal power allocation over parallel Gaussian broadcast channels," *Information Theory. 1997. Proceedings., 1997 IEEE International Symposium on*, 29 Jun-4 Jul 1997.

[3] S. T. Chung and J. Cioffi, "The capacity region of frequency-selective gaussian interference channels under strong interference," *Communications, IEEE Transactions on*, vol. 55, no. 9, pp. 1812–1821, Sept. 2007.

[4] G. Scutari, D. Palomar, and S. Barbarossa, "Asynchronous iterative water-filling for gaussian frequency-selective interference channels: A unified framework," *Signal Processing Advances in Wireless Communications, 2006. SPAWC '06. IEEE 7th Workshop on*, pp. 1–5, 2-5 July 2006.

[5] K.-K. Leung, C. W. Sung, and K. Shum, "Iterative waterfilling for parallel gaussian interference channels," *Communications, 2006. ICC '06. IEEE International Conference on*, vol. 12, June 2006.

[6] J.-S. Pang, G. Scutari, F. Facchinei, and C. Wang, "Distributed power allocation with rate constraints in gaussian frequency-selective interference channels," *arxiv.org*, 2007. http://arxiv.org/abs/cs/0702162.

[7] W. Yu and J. Cioffi, "Competitive equilibrium in the Gaussian interference channel," *Information Theory, 2000. Proceedings. IEEE International Symposium on*, 2000.

[8] W. Yu, G. Ginis, and J. Cioffi, "Distributed multiuser power control for digital subscriber lines," *Selected Areas in Communications, IEEE Journal on*, vol. 20, no. 5, pp. 1105–1115, Jun 2002.

[9] W. Yu and R. Lui, "Dual methods for nonconvex spectrum optimization of multicarrier systems," *IEEE Transactions on Communications*, vol. 54, pp. 1310–1322, July 2006.

[10] S. T. Chung, "Transmission schemes for frequency selective gaussian interference channels," 2004. Ph.D Thesis.

[11] V. R. Cadambe and S. A. Jafar, "Interference alignment and degrees of freedom region for the K user interference channel," *IEEE Transactions on Information Theory*, vol. 54, pp. 3425–3441, Aug 2008.

[12] L. Zheng and D. N. Tse, "Packing spheres in the grassmann manifold: A geometric approach to the non-coherent multi-antenna channel," *IEEE Trans. Inform. Theory*, vol. 48, Feb 2002.

[13] S. A. Jafar and S. Shamai, "Degrees of freedom region of the MIMO X channel," *IEEE Transactions on Information Theory*, vol. 54, no. 1, pp. 151–170, Jan. 2008.

[14] V. R. Cadambe, S. A. Jafar, and S. S. (Shitz), "Interference alignment on the deterministic channel and application to fully connected AWGN interference networks," *IEEE Transactions on Information Theory*, vol. 55, pp. 731–745, January 2009.

[15] G. Kramer, "Outer bounds on the capacity of Gaussian interference channels," *IEEE Transactions on Information Theory*, vol. 50, pp. 581–586, March 2004.

[16] A. B. Carleial, "Outer bounds on the capacity of interference channels," *IEEE Trans. Information Theory*, vol. 29, pp. 602–606, July 1983.

[17] H. Chong, M. Motani, and H. Garg, "Capacity theorems for the "Z" channel," *IEEE Transactions on Information Theory*, vol. 53, pp. 1348–1365, Apr. 2007.

[18] R. Etkin and E. Ordentlich, "On the degrees-of-freedom of the K-user Gaussian interference channel," *arXiv:0901.1695*, January 2009.

[19] A. Host-Madsen and A. Nosratinia, "The multiplexing gain of wireless networks," *Information Theory, 2005. ISIT 2005. Proceedings. International Symposium on*, pp. 2065–2069, 4-9 Sept. 2005.

[20] C. Sung, K. Lui, K. Shum, and H. So, "Sum capacity of one-sided parallel Gaussian interference channels," *IEEE Transactions on Information Theory*, vol. 54, pp. 468–472, Jan. 2008.

[21] L. Sankar, X. Shang, E. Erkip, and V. Poor, "Ergodic two-user interference channels: Is separability optimal?," *46th Annual Allerton Conference*, Sep 2008.

[22] X. Shang, B. Chen, and G. Kramer, "On sum-rate capacity of parallel Gaussian interference channels," *IEEE GLOBECOM Commun. Theory Symp.*, Dec. 2008.

[23] B. Nazer, M. Gastpar, S. Jafar, and S. Vishwanath, "Ergodic interference alignment," *arxiv/0901.4379*, 2009.

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