The Optimality of Transmit Beamforming: A Unified View

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***Abstract*— The optimality of transmit beamforming for a multiple antenna system with partial/limited feedback is investigated and a single general necessary and sufficient condition for beamforming to achieve ergodic capacity is derived. The condition obtained is universal - applicable to all partial/limited feedback scenarios in all ergodic fading channel distributions regardless of the number of transmit/receive antennas or transmit power. Using the universal condition we explore the optimality of beamforming for the quantized mean feedback scheme, which generalizes previous results for the separate cases of mean feedback and quantized feedback. Numerical results are provided to complement the analysis.**

***Index Terms*— Beamforming, Ergodic Capacity, Partial Feedback, MIMO channels, Quantized Mean Feedback, Quantized Feedback.**

I. Introduction

Multiple antenna systems with partial/limited feedback strategies have generated a lot of research interest in recent years (mean and covariance feedback [1]–[9], magnitude feedback [10], [11] and quantized feedback [12]–[24]). Characterization of the ergodic capacity of such partial/limited feedback systems is the subject of several publications [1], [3], [4], [10], [18], [25]. In many cases the transmitter optimization problem yields a simple unit rank (beamforming) solution, which is often desired due to the simplicity of the beamforming scheme. In a partial/limited feedback scenario, the optimality of transmit beamforming therefore determines whether or not it is possible to use the beamforming transmit scheme to simultaneously achieve both the maximum throughput and also the low system complexity.

Previous work on partial/limited feedback systems has investigated the optimality of beamforming for different feedback strategies in some well known fading channel models (perfect feedback [25], mean/covariance feedback [2], [4], [5], quantized direction feedback [18], [19], magnitude feedback [20]). While such results are very useful for the specific systems considered, their scope is limited by the underlying assumptions about the feedback scheme and the channel fade distribution. Feedback schemes (or channels) encountered in practice very rarely conform to any of the strategies (or channel distributions) discussed in previous work. In reality, channel and system non-idealities often force these feedback schemes to be combinations of the well known strategies. As an example, consider the typical mean and quantized feedback models. Mean feedback captures the effects of feedback delay and channel estimation error. However it assumes the availability of an unquantized channel mean (infinite capacity feedback channel). Similarly quantized feedback captures the effects of quantization in the feedback channel but assumes that the quantized version of the current channel vector is available instantaneously at the transmitter (zero delay feedback channel). In practice, feedback channels are associated with both a finite feedback rate and a non-zero feedback delay. Consequently the resulting feedback schemes do not correspond to the definitions of mean feedback or quantized feedback. Previously known results on mean or quantized feedback are also not directly pertinent to such practical feedback schemes because the results do not accommodate changes to the underlying feedback strategies and/or channel models. Thus arises the need for a unified view - one that is applicable to any general partial/limited feedback system regardless of the nature or specificities of the feedback strategy or channel model.

The main goal of this work is to provide a unified treatment of the optimality of beamforming for *general* partial/limited feedback scenarios. We are motivated by the observation that the conditions for the optimality of beamforming (w.r.t maximizing ergodic capacity) for mean [2], covariance [4] and quantized direction feedback [19] are obtained by a similar Lagrangian optimization of the corresponding capacity expressions. By unifying all possible partial/limited feedback strategies under a single framework, we find that the same technique can be used to obtain insights into the optimality of beamforming for general feedback schemes. In this work, we first identify a common form for the optimality of beamforming condition that is applicable to all feedback schemes - partial or limited - in all kinds of ergodic fading channel models. In particular, results obtained in previous work [2], [4], [5], [18]–[20] can be easily derived as special cases of the general condition. As the second contribution of this work, we apply the universal optimality condition to obtain results for practically motivated feedback models combining elements of both mean and quantized feedback. While we focus primarily on partial/limited feedback system with multiple transmit antennas and a single receive antenna

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(MISO case), corresponding results for MIMO systems are also provided. We begin with the system model in Section II.

II. System Model

The system model is a point-to-point multiple antenna downlink with a base station consisting of M transmit antennas communicating with a single antenna receiver as in Figure 1. For simplicity of exposition we focus primarily on the MISO scenario in this correspondence. Extensions to the MIMO scenario are dealt with separately in Section III-D.

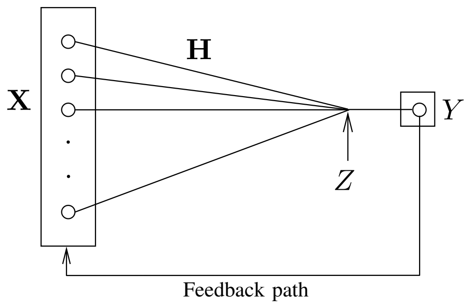

Fig. 1: System Model

$\mathbf{X} = [X_1\ X_2\ \cdots\ X_M]^T \in \mathcal{C}^{(M \times 1)}$ is the signal transmitted from the base station. $\mathbf{H} = [H_1\ H_2\ \cdots\ H_M]^T \in \mathcal{C}^{(M \times 1)}$ is the time varying channel vector, where H_m denotes the channel gain from the m^{th} transmit antenna to the receiver. $Z \sim \mathbb{CN}(0,\ 1)$ represents the circularly symmetric complex Gaussian noise at the receiver. The received signal is denoted by $Y \in \mathcal{C}$. We have suppressed the time dependence of the variables to keep the notation simple. The mathematical relationship between the input and the output is described by the following equation

$$Y = \langle \mathbf{H},\ \mathbf{X} \rangle + Z = \mathbf{H}^\dagger \mathbf{X} + Z. \tag{1}$$

We consider the perfect CSIR scenario, where the receiver is assumed to track the channel perfectly and has complete knowledge of the instantaneous channel vector. The transmitter, on the other hand, obtains partial/limited channel information from the receiver through an error free feedback path.

We represent the feedback information mathematically by the random variable $\mathbf{F}$. The channel feedback $\mathbf{F}$ is a deterministic function of the time varying channel state $\mathbf{H}$ and/or the time varying statistics of the channel. The set of all possible realizations of the feedback information $\mathbf{F}$ will be denoted by $\mathcal{F}$. This formulation directly captures the nature of the feedback in the variable $\mathbf{F}$ and serves as a common framework for systems with partial/limited feedback [20].

Conditioned on the feedback $\mathbf{F}$, the complex correlation matrix of the input can be written as $K(\mathbf{F}) = \mathsf{E}_{\mathbf{X}|\mathbf{F}}\left[\mathbf{X}\mathbf{X}^\dagger\right]$. The power constraint at the transmitter is given by

$$\mathrm{Tr}\left(K(\mathbf{F})\right) = \mathsf{E}_{\mathbf{X}|\mathbf{F}}\left[\mathbf{X}^\dagger \mathbf{X}\right] \leq P \quad \forall \ \ \mathbf{F} \in \mathcal{F}. \tag{2}$$

We have chosen a 'short-term' power constraint - the transmit power is the same (equal to P) regardless of the value of the feedback $\mathbf{F}$ to the transmitter. While such a constraint simplifies the analysis, our results are applicable even when the transmitter chooses a different transmit power $P(\mathbf{F})$ (with $\mathsf{E}_{\mathbf{F}}\left[P(\mathbf{F})\right] = P$) depending on the feedback $\mathbf{F}$. Such a feedback dependent power adaptation will only introduce an additional optimization in order to calculate the best transmit powers $P^*(\mathbf{F})$.

A. Problem Statement

We are interested in determining whether the simple beamforming strategy is optimal in terms of maximizing the ergodic capacity of the multiple antenna system with partial/limited feedback. Specifically, we seek answers to the following:

1) **Optimality of beamforming**: For a *general* partial/limited CSIT model, given the feedback $\mathbf{F}$, is beamforming the optimal input strategy, i.e., is the optimal input covariance matrix of the form $K^\star(\mathbf{F}) = P\mathbf{b}^\star(\mathbf{F})\,\mathbf{b}^{\star\dagger}(\mathbf{F})$, where $\mathbf{b}^\star(\mathbf{F})$ is an $(M \times 1)$ unit vector?
2) **Quantized Mean Feedback Example**: Using results derived for arbitrary feedback scenarios, what are the insights that can be obtained on the optimality of beamforming for quantized mean feedback systems?

III. Optimality of beamforming

In this section, we introduce the general conditions for the optimality of beamforming for a partial/limited feedback MISO system with partial/limited feedback. Corresponding conditions for the MIMO case are provided in Section III-D.

A. Capacity Expressions

The capacity C^{fb} of the partial/limited feedback system is given by [26]:

$$C^{\text{fb}} = \mathsf{E}_{\mathbf{F}}\left[\max_{K(\mathbf{F}):\mathrm{Tr}(K(\mathbf{F}))\leq P} \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\log\left(1 + \mathbf{H}^\dagger K(\mathbf{F})\,\mathbf{H}\right)\right]\right]. \tag{3}$$

The capacity is achieved by transmitting independent complex circular Gaussian symbols along the eigenvectors of $K^\star(\mathbf{F})$, the optimal input covariance matrix. The eigenvalues of $K^\star(\mathbf{F})$ decide the powers to be allocated to the corresponding complex Gaussian symbols. We notice from equation (3) the capacity optimization problem can be solved as separate optimizations over all $\mathbf{F} \in \mathcal{F}$, i.e.,

$$C^{\text{fb}}(\mathbf{F}) = \max_{K(\mathbf{F}):\mathrm{Tr}(K(\mathbf{F}))\leq P} \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\log\left(1 + \mathbf{H}^\dagger K(\mathbf{F})\,\mathbf{H}\right)\right]. \tag{4}$$

Suppose we constrain $\mathbf{K}(\mathbf{F})$ to be *unit rank*, i.e, fix the transmit strategy to be beamforming, the corresponding *beamforming capacity* can be written as

$$C^{\text{bf}}(\mathbf{F}) = \max_{\mathbf{b}:\|\mathbf{b}\|=1} \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\log\left(1 + \mathbf{H}^\dagger\left(P\mathbf{b}\mathbf{b}^\dagger\right)\mathbf{H}\right)\right]. \tag{5}$$

B. Universal Condition for the Optimality of Beamforming

For the problem definition of Section II-A, we begin with a mathematical statement of the *necessary and sufficient condition* for the optimality of beamforming described in the following theorem:

Theorem 1.A: *Beamforming along* $\mathbf{b}(\mathbf{F})$ *is the optimal transmit strategy if and only if*

$$\lambda_{max}\left(\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{\mathbf{H}\mathbf{H}^\dagger}{1+P|\mathbf{H}_{\|}|^2}\right]\right)=\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{|\mathbf{H}_{\|}|^2}{1+P|\mathbf{H}_{\|}|^2}\right], \quad (6)$$

where $\mathbf{H}_{\|}$ *is the projection of the channel* $\mathbf{H}$ *along the beamforming vector* $\mathbf{b}\left(\mathbf{F}\right)$ *(i.e,* $\mathbf{H}_{\|}=\mathbf{H}^\dagger\mathbf{b}(\mathbf{F})$*) and* $\lambda_{max}\left(\cdot\right)$ *denotes the maximum eigenvalue.*

Proof: The proof is similar to the corresponding result in [19]. While [19] uses matrix differentiation, for the sake of completeness, we provide an alternate proof in Appendix A that only involves taking scalar derivatives.

The necessary and sufficient condition of equation (6) is *universal* - applicable to a wide variety of partial/limited feedback strategies regardless of the distribution of the channel, the amount/quality of feedback, the number of transmit antennas or the amount of transmit power. The condition can also be written in terms of two simplified conditions that are intuitive and straightforward to work with. We now introduce these conditions.

C. Simplified Universal Conditions

Let $\mathcal{U}=\{\mathbf{u}_1=\mathbf{b}\left(\mathbf{F}\right),\ \mathbf{u}_2,\ \cdots,\ \mathbf{u}_{M-1},\ \mathbf{u}_M\}$ be any set of orthonormal vectors. Let $H_{\perp k}$ be the projection of the channel $\mathbf{H}$ along $\mathbf{u}_k$, i.e, $H_{\perp k}=\mathbf{H}^\dagger\mathbf{u}_k$. As defined earlier, let $H_{\|}$ be the projection of the channel $\mathbf{H}$ along the beamforming vector $\mathbf{b}\left(\mathbf{F}\right)$, i.e, $H_{\|}=\mathbf{H}^\dagger\mathbf{b}(\mathbf{F})$. Note that $H_{\perp 1}=H_{\|}$. We state the following theorem from [19]:

Theorem 1.B: *Beamforming along* $\mathbf{b}(\mathbf{F})$ *is the optimal transmit strategy if and only if (necessary and sufficient) both the following conditions are simultaneously satisfied:*

1) **Condition 1:**

$$\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{H_{\|}H_{\perp j}^*}{1+P|H_{\|}|^2}\right] \quad = \quad 0 \qquad \forall\ 2\leq j\leq M. \quad (7)$$

2) **Condition 2:**

$$\lambda_{\max}\left(\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{\left(\mathbf{I}-\mathbf{b}\mathbf{b}^\dagger\right)^\dagger\mathbf{H}\mathbf{H}^\dagger\left(\mathbf{I}-\mathbf{b}\mathbf{b}^\dagger\right)}{1+P|H_{\|}|^2}\right]\right)\leq \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{|H_{\|}|^2}{1+P|H_{\|}|^2}\right]. \quad (8)$$

Proof: Extension of Theorem 2 of [19].

It can be shown (similar to Theorem 1, [19]) that Condition 1 is *by itself a necessary condition* for the optimality of beamforming because it will be satisfied by the *best beamforming vector*, i.e., the vector that maximizes the beamforming capacity of equation (5).

The correspondence between Theorems 1.A and 1.B is captured in the following corollary:

Corollary 1: The general condition of Theorem 1.A is equivalent to the simplified conditions of Theorem 1.B.

Proof: Appendix B.

In many partial/limited feedback scenarios, due to the symmetry of the channel distribution given the feedback, the possible optimal beamforming direction can be easily guessed. In such cases, Condition 1 is an invaluable tool to analytically ascertain whether or not a particular direction is indeed the optimal beamforming vector. In general feedback scenarios when no such guesses for the possible optimal beamforming vector exist, analytically checking for the conditions of equations (7) and (8) is hard because all feasible beamforming vectors will have to be tested. We point out that numerically testing for Condition 1 involves validating an *equality* (equation (7)) and is subject to the precision employed in the generation of the beamforming vectors and the channel realizations. On the other hand, numerically checking Condition 2 is easier because the inequality is strict in most cases.

D. The MIMO Case

Previously, we have focused on situations where the receiver is equipped with a single antenna, i.e, the MISO scenario. In this section, we consider the MIMO scenario, where the receiver is equipped with L receive antennas. The input-output equation can be written as $\mathbf{Y}=\mathbf{H}^\dagger\mathbf{X}+\mathbf{Z}$, where $\mathbf{X}\in\mathcal{C}^{(M\times 1)}$ is the signal from the base station, $\mathbf{H}\in\mathcal{C}^{(M\times L)}$ is the channel matrix, $\mathbf{Z}\sim\mathbb{CN}\left(\mathbf{0},\ \mathbf{I}_L\right)\in\mathcal{C}^{(L\times 1)}$ denotes the circularly symmetric complex Gaussian noise at the receiver and $\mathbf{Y}\in\mathcal{C}^{(L\times 1)}$ is the received signal vector. The feedback model follows that described for the MISO case in Section II with $\mathbf{F}$ denoting the feedback and $\mathcal{F}$ representing the set of possible feedback realizations. Similar to the MISO case, one can derive a necessary and sufficient condition for the optimality of beamforming with partial/limited feedback for the MIMO scenario:

Theorem 1.C: *Beamforming along* $\mathbf{b}\left(\mathbf{F}\right)$ *is the optimal transmit strategy if and only if*

$$\lambda_{max}\left[\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\mathbf{H}\left(\mathbf{I}_L+P\mathbf{H}^\dagger\mathbf{b}\left(\mathbf{F}\right)\mathbf{b}^\dagger\left(\mathbf{F}\right)\mathbf{H}\right)^{-1}\mathbf{H}^\dagger\right]\right]=$$
$$\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\mathbf{b}^\dagger\left(\mathbf{F}\right)\mathbf{H}\left(\mathbf{I}_L+P\mathbf{H}^\dagger\mathbf{b}\left(\mathbf{F}\right)\mathbf{b}^\dagger\left(\mathbf{F}\right)\mathbf{H}\right)^{-1}\mathbf{H}^\dagger\mathbf{b}\left(\mathbf{F}\right)\right]. \quad (9)$$

Proof: The proof for the MIMO case is similar to Theorem 7 of [19].

IV. Quantized Mean Feedback

Conditions for the optimality of beamforming in some well known fading scenarios have been provided for a few partial/limited feedback strategies in earlier work [2], [4], [5], [18]–[20]. Using the simplified optimality conditions introduced in Section III, [20] considers a variety of existing feedback strategies and derives optimality results for some specific channel fading scenarios. We emphasize here that the scope of the universal optimality conditions is not limited to previously known feedback strategies. They can also be used to identify if beamforming is optimal for feedback schemes (and channel distributions) where such results are not known. As an example we introduce the 'quantized mean' feedback scheme and derive conditions to be satisfied for beamforming to be optimal.

Consider a typical mean feedback scenario [1] where the channel is distributed as $\mathbf{H}\sim\mathbb{CN}\left(\boldsymbol{\mu},\ \frac{\triangle}{M}\mathbf{I}_M\right)$, with a mean vector $\boldsymbol{\mu}$. The channel vector can therefore be regarded as a superposition of a mean vector $\boldsymbol{\mu}$ and an independent isotropic 'estimation error' $\mathbf{g}$, and written as $\mathbf{H} = \boldsymbol{\mu} + \mathbf{g}$, where $\mathbf{g}$ is a circularly symmetric complex Gaussian vector distributed as $\mathbf{g} \sim \mathbb{CN}\left(\mathbf{0},\ \frac{\Delta}{M}\mathbf{I}_M\right)$. In a practical feedback channel, finite bandwidth constraints force a quantization of the channel mean information at the receiver. Suppose the receiver quantizes the channel mean $\boldsymbol{\mu}$ via a given[1] set of $N = 2^B$ quantization vectors $\mathcal{Q} = \{\mathbf{q}_1,\ \mathbf{q}_2,\ \cdots,\ \mathbf{q}_N\}$ known to both the transmitter and receiver[2]. If the index that is fed back to the transmitter is n, the channel mean $\boldsymbol{\mu} \in D_n$, where D_n is the quantization region of $\mathbf{q}_n$. Since the feedback involves sending a quantized version of the channel mean, we refer to this hybrid scheme as the '*quantized mean*' feedback strategy.

A. Optimality of Beamforming for Quantized Mean Feedback

The simplified universal conditions of equation (7) and (8) will be used to derive optimality of beamforming results for the quantized mean feedback scheme. We are particularly interested in determining whether or not *beamforming along the quantization vector fed back* is optimal. For simplicity we assume (without loss of generality) that the index fed back is $n = 1$ (the quantization vector closest to the channel mean $\boldsymbol{\mu}$ is $\mathbf{q}_1$) and that the corresponding quantization vector[3] is $\mathbf{q}_1 = [1\ 0\ \cdots\ 0]$. We have $H_{\|} = \mathbf{H}^\dagger\mathbf{b} = (\boldsymbol{\mu} + \mathbf{g})^\dagger\,\mathbf{b} = (\mu_1 + g_1)^*$ and $H_{\perp j} = (\mu_j + g_j)^*$, where μ_k and g_k are the k^{th} components of the corresponding vectors. Since $\boldsymbol{\mu}$ and $\mathbf{g}$ are independent, the two simplified conditions for the quantized mean strategy can be (from equations (7) and (8)) written as

1) **Condition 1** (for quantized mean feedback):

$$\mathsf{E}_{\boldsymbol{\mu}\in D_1}\mathsf{E}_{\mathbf{g}}\left[\frac{(\mu_1+g_1)\,(\mu_l+g_l)^*}{1+P|\,(\mu_1+g_1)\,|^2}\right] = 0 \quad \forall\ 2 \le l \le M. \tag{10}$$

2) **Condition 2** (for quantized mean feedback):

$$\mathsf{E}_{\boldsymbol{\mu}\in D_1}\mathsf{E}_{\mathbf{g}}\left[\frac{1+P\,|(\mu_j+g_j)|^2}{1+P\,|(\mu_1+g_1)|^2}\right] \le 1 \quad \forall\ 1 \le j \le M. \tag{11}$$

Beamforming along the quantization vector fed back will be the optimal transmit strategy when both equations (10) and (11) are satisfied. An inspection of the form of equations (10) and (11) reveals that form of Condition 1 is similar to the definition of a centroid, i.e., Condition 1 relates to the *symmetry* of the channel distribution around the beamforming vector - it tests the direction of the beamforming vector. On the other hand, it can be seen that Condition 2 tests the *strength* of the channel along the beamforming vector relative to the strength of the channel in the null space of the beamforming vector.

B. Optimality Results

Consider the quantized mean feedback scheme discussed in Section IV. Let the quantization vectors be chosen according to the Grassmannian criterion [13]. Since condition 1 is always satisfied by the *best beamforming vector* (necessary condition), the optimality of beamforming is effectively decided only by Condition 2, which depends on the angular spread of the quantization region and the distribution of the channel given the mean vector. It is not mathematically tractable to determine the exact geometry of the quantization regions for general M and N. To understand the impact of the angular spread of the quantization region on the optimality of beamforming, we approximate the decision regions to be conical volumes symmetric around the associated quantization vector. The conical volume (Figure 2) is defined by the half-angle[4] θ, i.e.,

$$D_n = \left\{\boldsymbol{\mu} : \left|\left\langle \frac{\boldsymbol{\mu}}{\|\boldsymbol{\mu}\|},\ \mathbf{q}_n\right\rangle\right| \ge \cos(\theta)\right\}. \tag{12}$$

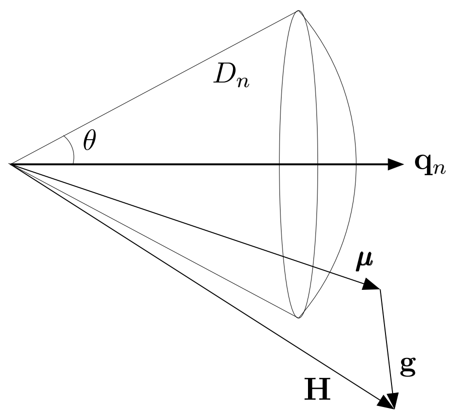

Fig. 2: Illustration of the conical quantization region in the three dimensional real domain $\mathbb{R}^3$.

The half-angle θ can be thought of as a measure of the *quantization error* - as θ decreases, the amount of feedback (number of quantization vectors N) increases. When θ is small (large N), quantized mean feedback resembles mean feedback. Similarly Δ, related to the feedback delay, is a measure of the error in the estimation of the channel mean - the *estimation error*. The higher the Δ, the lower the conditional probability that the channel vector $\mathbf{H} \in D_n$ given $\boldsymbol{\mu} \in D_n$. When Δ is small, quantized mean feedback is similar to quantized feedback.

When the estimation error (governed by Δ) and the quantization error (governed by θ) are small, one would expect

[1]We assume that the quantization set is predetermined - we are only concerned with the problem of optimality of beamforming and do not deal with the problem of finding the optimal quantization vectors.

[2]While the quantization is performed just as in quantized feedback [12]–[22], [27], notice that the channel mean $\boldsymbol{\mu}$ is quantized (instead of the instantaneous channel $\mathbf{H}$).

[3]Given any arbitrary set of quantization vectors in a specific coordinate system, the coordinate system can be rotated (regardless of the kind of channel distribution) so that $\mathbf{q}_1$ (the first quantization vector) is along the vector $[1\ 0\ \cdots\ 0]^T$. We emphasize here that the channel distribution and the orientation of the quantization vectors (with the corresponding quantization regions) will also correspondingly change.

[4]Given the number of Grassmannian quantization vectors N, the half-angle θ is half the smallest angle between any two vectors in the Grassmannian quantization set, i.e., $\theta = \frac{1}{2}\arccos\left[\max_{i,\ j}|\mathbf{q}_i\mathbf{q}_j^\dagger|\right]$. We note that θ monotonically decreases with N.

beamforming along the channel mean to be optimal. However, it is not obvious whether or not beamforming is capacity achieving at higher estimation and/or quantization errors. The optimality conditions derived in Section IV-A can be used to numerically determine if beamforming is optimal for a particular $(\Delta,\ \theta)$ pair.

We consider a $M = 4$ antenna quantized mean feedback system. For different values of the transmit power P, the expectations in equations (10) and (11) are calculated by averaging over realizations of $\boldsymbol{\mu}$ and $\mathbf{g}$. While the $\boldsymbol{\mu}$ are generated so that they are uniformly distributed within the conical region D_1, the realizations of $\mathbf{g}$ follow $\mathbf{g} \sim \mathbb{CN}\left(\mathbf{0},\ \frac{\Delta}{M}\mathbf{I}_M\right)$. We fix $\mathsf{E}\left[\|\boldsymbol{\mu}\|^2\right] = \left(1 - \frac{\Delta}{M}\right)$ so that the resulting channel vectors have $\mathsf{E}\left[\|\mathbf{H}\|^2\right] = 1$. Figure 3 plots the boundary between the beamforming optimal and suboptimal regions for different values of P.

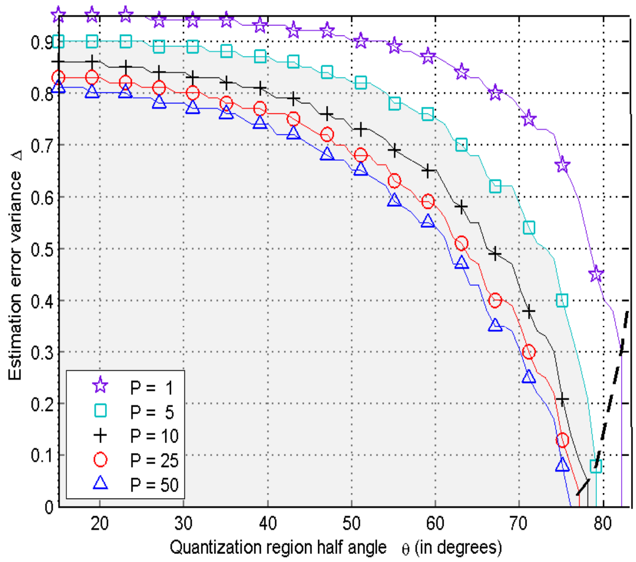

Fig. 3: Beamforming optimal regions for a quantized mean feedback system with $M = 4$ transmit antennas. For any P, the region enclosed by the corresponding plot and the axes defines the $(\Delta,\ \theta)$ values for which beamforming along the quantized mean achieves capacity. The shaded area represents the region where beamforming is optimal for $P = 5$.

For any given P, the region enclosed by the corresponding curve in Figure 3 and the axes represents all the $(\Delta,\ \theta)$ pairs for which beamforming is optimal. When the angular spread of the quantization region θ is small and the estimation error Δ is low, it can be seen from Figure 3 that beamforming is optimal. For fixed values of P and Δ, beamforming should become suboptimal once the half angle θ is increased beyond a certain threshold. This is because as θ increases, the quantization region becomes bigger and higher throughputs may be achieved by distributing power along two or more directions. This threshold boundary is reflected by the dotted line in Figure 3. The abscissa of the point of intersection of the dotted line and the curve for a given P is the value of θ beyond which beamforming is suboptimal for all Δ. As an illustrative example, the shaded area in figure 3 represents all the points in the $(\Delta,\ \theta)$ plane where beamforming is optimal for $P = 5$. It can be seen from the figure that beamforming is suboptimal for $P = 5$ when $\theta > 79°$ regardless of the value of Δ.

From the figure, it can also be seen that as the transmit power increases, the optimality region becomes smaller. This is consistent with the waterfilling analogy - beamforming becomes suboptimal when the amount of water increases (larger transmit power) or when the modes become very deep (higher estimation error - larger Δ).

V. Numerical Results: Less feedback more often, or more feedback less often?

Consider a typical limited feedback system with perfect CSIR. Let the forward channel be described by an autoregressive model [1], so that the channel realization at time t is given by

$$\mathbf{H}\left(t\right) = a\mathbf{H}\left(t-1\right) + \sqrt{1-a^2}\mathbf{w}, \tag{13}$$

where $\mathbf{w} \sim \mathcal{N}\left(\mathbf{0},\ \frac{1}{M}\mathbf{I}\right)$, a is the forgetting factor ($0 \leq a \leq 1$). Let the feedback channel allow a finite rate of B bits every D feedback channel uses. In efficiently utilizing the available feedback bandwidth, the receiver can use one of the following strategies:

- **Strategy 1**: Once every D channel uses, quantize the instantaneous channel using B bits (feedback more channel information less often).
- **Strategy 2**: Every channel use, quantize the instantaneous channel using $\frac{B}{D}$ bits (feedback less channel information more often).

Notice that in both cases the average amount of feedback is the same, equal to $\frac{B}{D}$ bits per feedback channel use. A natural question that arises out of the above discussion is: which strategy offers a higher throughput? To answer this, we compare the throughputs of the two strategies assuming vector quantization at the receiver based on the Grassmannian criterion[5] [13] - in Strategy 1, the receiver uses a set of $N_1 = 2^B$ quantization vectors while Strategy 2 employs $N_2 = 2^{\frac{B}{D}}$ vectors. In Strategy 1, the channel model implies that the B bit feedback at the beginning of the block (D channel uses) is the quantized channel mean for the entire block. Therefore Strategy 1 is essentially quantized mean feedback. Similarly Strategy 2 involves feeding back a quantized version of the instantaneous channel every block - it is quantized feedback.

Figure 4 shows the average throughputs of the two strategies with increasing delay in feedback for forgetting factors $a = 0,\ 0.5,\ 0.9$ and 1. The input covariance matrix in Strategy 1 is calculated as in [1] while beamforming is considered as the transmit scheme in Strategy 2. The number of transmit antennas considered is $M = 3$, $B = 6$ bits and the average receive SNR is assumed to be 5dB. We use the best known line packings provided by [28] for the quantization vector sets. The throughput plots for perfect feedback and no-feedback are also shown for comparison.

When $D = 1$, for all a, both quantized mean (Strategy 1) and quantized (Strategy 2) feedback strategies receive $B =$

[5]Note that the channel has memory and one may be able to achieve a better quantization by utilizing this memory. However we consider only the static quantization set model for simplicity.

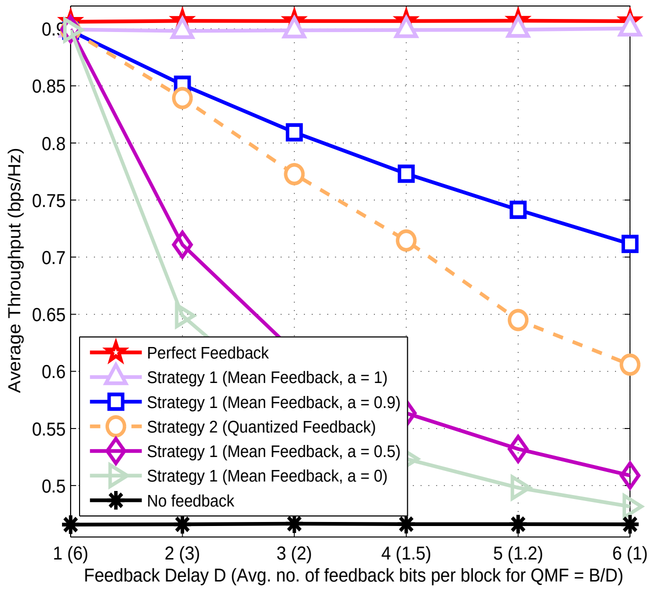

Fig. 4: Throughput plots of the two strategies for different values of the feedback delay D. The throughput of quantized direction feedback is independent of a. Throughputs for the perfect and no feedback cases are shown for the sake of comparison.

6 bits every channel use and consequently have the same throughput. As D increases, quantized mean feedback conveys less precise information about the instantaneous state of the channel ($a \neq 0$). The performance of Strategy 1 therefore deteriorates with an increase in D. For large a ($a \approx 1$, high memory) the channel does not change much and quantized mean feedback receives 6 bits of channel information while quantized feedback (Strategy 2) receives $\frac{6}{D}$ bits of channel information. Quantized mean feedback therefore performs better than quantized feedback for large a. As a decreases, the quality of the feedback decreases and consequently the average throughput offered by quantized mean feedback decreases.

An intuitive result emerges from the the behavior of the throughput plots - when the channel changes rapidly (low a, fast fading), it is better to have less feedback more frequently. On the other hand, when the channel changes slowly (high a, slow fading), it is more advantageous to have more feedback information with a higher delay between updates.

VI. Conclusions

We explore the optimality of beamforming in a general multiple antenna channel with perfect channel knowledge at the receiver and partial/limited channel information at the transmitter. Without restricting the kind of partial/limited feedback or the type of channel distribution, we derive a universal necessary and sufficient condition for beamforming to achieve capacity. Using the optimality condition, we obtain results on the optimality of beamforming for the quantized mean feedback strategy. With numerical results, we show that beamforming is optimal for quantized mean feedback when the angular spread of the quantization regions and the mean estimation error are small. Exploring the tradeoff between the frequency and amount of feedback, we find that frequency of feedback is more important in fast fading channels while the amount of feedback is more important for slow fading channels.

Appendix

A. *Proof of Theorem 1.A*

Let $\mathbf{b}$ be any given unit vector (the beamforming vector). Let K be any arbitrary feasible input covariance matrix (K is positive semi-definite and $\mathrm{Tr}\,[K] \leq P$). We consider the corresponding set of input covariance matrices defined by

$$\left\{(1-\rho)P\mathbf{b}\mathbf{b}^\dagger + \rho K, \ \forall \rho \in [0, \ 1]\right\}. \tag{14}$$

For each feasible K, we have a corresponding set of matrices. The union of the sets associated with all feasible K yields the entire domain of optimization - all possible positive semi-definite matrices satisfying the power constraint of equation (2).

Consider the objective function of the optimization problem of equation (4). Given $\mathbf{b}$, we can write the objective function as (for some $0 \leq \rho \leq 1$ and some K)

$$f(\rho) \triangleq \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\log\left(1 + \mathbf{H}^\dagger\left(P\mathbf{b}\mathbf{b}^\dagger + \rho\left(K - P\mathbf{b}\mathbf{b}^\dagger\right)\right)\mathbf{H}\right)\right]. \tag{15}$$

We note here that the point $\rho = 0$ corresponds to the beamforming capacity, i.e., beamforming along $\mathbf{b}$. Further, it is easy to verify that the function $f(\rho)$ is a concave function in ρ.

Necessary Condition: The condition to be satisfied for beamforming along $\mathbf{b}$ to be optimal is $\frac{d}{d\rho}f(\rho)\,|_{\rho=0} \leq 0$ for all K, i.e.,

$$\max_{K:\mathrm{Tr}[K]\leq P}\left(\frac{d}{d\rho}f(\rho)\,|_{\rho=0}\right) \leq 0. \tag{16}$$

If this condition does not hold, then there exists an input covariance matrix $\left((1-\rho)P\mathbf{b}\mathbf{b}^\dagger + \rho K\right)$ for some $\rho \neq 0$ and some $K \neq P\mathbf{b}\mathbf{b}^\dagger$ that yields a higher capacity.

Sufficient Condition: If equation (16) is satisfied for all K, then due to the concavity of $f(\rho)$ in ρ, the point $\rho = 0$ is a global maximum, i.e., beamforming along $\mathbf{b}$ yields the highest capacity and is therefore optimal.

Therefore equation (16) is both a *necessary and sufficient* condition for beamforming along $\mathbf{b}$ to be optimal.

Let $K = \sum_{i=0}^{M}\alpha_i\mathbf{s}_i\mathbf{s}_i^\dagger$, where $\{\mathbf{s}_1, \cdots, \mathbf{s}_M\}$ and $\{\alpha_1, \cdots, \alpha_M\}$ are the eigenvectors and corresponding eigenvalues of K. Note that the α_i are all nonnegative and sum to P. The optimality of beamforming condition of equation (16) can be simplified to

$$\begin{aligned} 0 &\geq \max_{\{\alpha_j\},\ \{\mathbf{s}_j\}} \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{\sum_{j=1}^{M}\alpha_j\mathbf{H}^\dagger\mathbf{s}_j\mathbf{s}_j^\dagger\mathbf{H} - P|\mathbf{H}^\dagger\mathbf{b}|^2}{1+P|\mathbf{H}^\dagger\mathbf{b}|^2}\right] \\ &= \max_{\{\alpha_j\},\ \{\mathbf{s}_j\}} \sum_{j=1}^{M}\alpha_j\mathbf{s}_j^\dagger\Lambda\mathbf{s}_j - \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{P|\mathbf{H}^\dagger\mathbf{b}|^2}{1+P|\mathbf{H}^\dagger\mathbf{b}|^2}\right] \\ &= P\left(\lambda_{max}(\Lambda) - \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{|\mathbf{H}^\dagger\mathbf{b}|^2}{1+P|\mathbf{H}^\dagger\mathbf{b}|^2}\right]\right), \end{aligned} \tag{17}$$

where $\Lambda = \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{\mathbf{H}\mathbf{H}^\dagger}{1+P|\mathbf{H}^\dagger\mathbf{b}|^2}\right]$. Equation (17) follows from the fact that a convex combination of terms is less than or equal to the maximum term. Equation (17) can be rewritten as

$$\lambda_{max}(\Lambda) \leq \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{|\mathbf{H}^\dagger\mathbf{b}|^2}{1+P|\mathbf{H}^\dagger\mathbf{b}|^2}\right] \tag{18}$$

Since $\mathbf{b}$ is a unit vector, we also have

$$\lambda_{max}(\Lambda) \geq \mathbf{b}^\dagger\Lambda\mathbf{b} = \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{|\mathbf{H}^\dagger\mathbf{b}|^2}{1+P|\mathbf{H}^\dagger\mathbf{b}|^2}\right],$$

and consequently equation (18) reduces to an equality. Substituting $H_{\|} = \mathbf{H}^\dagger\mathbf{b}$ in equation (18), we have the result of Theorem 1.A. ■

B. Proof of Corollary 1

Given a unit vector $\mathbf{b}$, we have the following identity for any positive semidefinite Hermitian matrix $\mathbf{A}$:

$$\begin{aligned}
&\lambda_{\max}\left(\left(\mathbf{I}-\mathbf{b}\mathbf{b}^\dagger\right)^\dagger \mathbf{A}\left(\mathbf{I}-\mathbf{b}\mathbf{b}^\dagger\right)\right) \\
&= \max_{\mathbf{v}:\|\mathbf{v}\|^2=1}\left[\mathbf{v}^\dagger\left(\mathbf{I}-\mathbf{b}\mathbf{b}^\dagger\right)^\dagger \mathbf{A}\left(\mathbf{I}-\mathbf{b}\mathbf{b}^\dagger\right)\mathbf{v}\right] \\
&= \max_{\alpha,\ \beta,\ \mathbf{b}_\perp:|\alpha|^2+|\beta|^2=1,\ \mathbf{b}_\perp^\dagger\mathbf{b}=0,\ \|\mathbf{b}_\perp\|=1}\Big[(\alpha\mathbf{b}+\beta\mathbf{b}_\perp)^\dagger \\
&\qquad\times\left(\mathbf{I}-\mathbf{b}\mathbf{b}^\dagger\right)^\dagger \mathbf{A}\left(\mathbf{I}-\mathbf{b}\mathbf{b}^\dagger\right)(\alpha\mathbf{b}+\beta\mathbf{b}_\perp)\Big] \\
&= \max_{\beta,\mathbf{b}_\perp:|\beta|\leq 1,\mathbf{b}_\perp^\dagger\mathbf{b}=0,\ \|\mathbf{b}_\perp\|=1}\left[|\beta|^2\,\mathbf{b}_\perp^\dagger\mathbf{A}\mathbf{b}_\perp\right] \\
&= \max_{\mathbf{b}_\perp:\mathbf{b}_\perp^\dagger\mathbf{b}=0,\ \|\mathbf{b}_\perp\|=1}\left[\mathbf{b}_\perp^\dagger\mathbf{A}\mathbf{b}_\perp\right] \qquad (19)
\end{aligned}$$

Let $\mathcal{U} = \{\mathbf{u}_1 = \mathbf{b},\ \mathbf{u}_2,\ \cdots,\ \mathbf{u}_{M-1},\ \mathbf{u}_M\}$ be any set of orthonormal vectors.

1) Forward Part: (Equation (6) ⇒ Equations (7) and (8)): Equation (6) can be equivalently written as

$$\max_{\mathbf{v}:\|\mathbf{v}\|=1}\left(\mathbf{v}^\dagger\Lambda\mathbf{v}\right) = \mathbf{b}^\dagger\Lambda\mathbf{b}, \tag{20}$$

where $\Lambda = \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{\mathbf{H}\mathbf{H}^\dagger}{1+P|\mathbf{H}^\dagger\mathbf{b}|^2}\right]$ and $\mathbf{b}$ is the beamforming vector. Therefore $\mathbf{b}$ is an eigenvector of Λ and the corresponding eigenvalue is $\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{|\mathbf{H}^\dagger\mathbf{b}|^2}{1+P|\mathbf{H}^\dagger\mathbf{b}|^2}\right] = \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{|\mathbf{H}_{\|}|^2}{1+P|\mathbf{H}_{\|}|^2}\right]$. We have

$$\Lambda\mathbf{b} = \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{|\mathbf{H}_{\|}|^2}{1+P|\mathbf{H}_{\|}|^2}\right]\mathbf{b}. \tag{21}$$

From equation (21), we can see that for any unit vector $\mathbf{b}_\perp$ orthogonal to $\mathbf{b}$, we have $\mathbf{b}_\perp^\dagger\Lambda\mathbf{b} = \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{|\mathbf{H}_{\|}|^2}{1+P|\mathbf{H}_{\|}|^2}\right]\mathbf{b}_\perp^\dagger\mathbf{b} = 0$. In particular, considering $\mathbf{b}_\perp = \mathbf{u}_i$ (with $i \geq 2$) directly yields Condition 1 (equation (7)) of Theorem 1.B.

Suppose, in the LHS of equation (20), we consider a smaller constraint set consisting only of all unit vectors orthogonal to $\mathbf{b}$, we can write

$$\max_{\mathbf{b}_\perp:\mathbf{b}_\perp^\dagger\mathbf{b}=0,\ \|\mathbf{b}_\perp\|=1}\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\mathbf{b}_\perp^\dagger\Lambda\mathbf{b}_\perp\right] \leq \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{|\mathbf{H}_{\|}|^2}{1+P|\mathbf{H}_{\|}|^2}\right], \tag{22}$$

which combined with equation (19) directly gives Condition 2 (equation (8)) of Theorem 1.B.

2) Reverse Part: (Equations (7) and (8) ⇒ Equation (6)): Since any unit vector $\mathbf{b}_\perp$ orthogonal to $\mathbf{b}$ can be written as a linear combination of $\{\mathbf{u}_2,\ \cdots,\ u_M\}$, Condition 1 yields

$$\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{\mathbf{b}_\perp^\dagger\mathbf{H}\mathbf{H}^\dagger\mathbf{b}}{1+P|\mathbf{H}_{\|}|^2}\right] = 0 \quad \forall\ \mathbf{b}_\perp \quad \text{s.t} \quad \mathbf{b}_\perp\mathbf{b} = 0,\ \ \|\mathbf{b}_\perp\| = 1. \tag{23}$$

Further, applying equation (19) to (equation (8)) (Condition 2) yields

$$\max_{\mathbf{b}_\perp:\mathbf{b}_\perp^\dagger\mathbf{b}=0,\ \|\mathbf{b}_\perp\|=1}\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\mathbf{b}_\perp^\dagger\Lambda\mathbf{b}_\perp\right] \leq \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{|\mathbf{H}_{\|}|^2}{1+P|\mathbf{H}_{\|}|^2}\right]. \tag{24}$$

Consider the maximization $\max_{\mathbf{v}:\|\mathbf{v}\|=1}\left(\mathbf{v}^\dagger\Lambda\mathbf{v}\right)$, which can be expanded as in equation (25) by substituting $\mathbf{u} = \alpha\mathbf{b} + \beta\mathbf{b}_\perp$. From equation (23), we can see that the last two terms in the maximization in the RHS of equation (25) are zeros, yielding equation (26). Using (24) in (26), we get equation (27). This directly proves the general condition of Theorem 1.A. ■

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$$\max_{\mathbf{u}:\|\mathbf{u}\|=1}\left(\mathbf{v}^\dagger\Lambda\mathbf{v}\right) = \max_{\alpha,\ \beta,\ \mathbf{b}_\perp:|\alpha|^2+|\beta|^2=1,\ \mathbf{b}_\perp^\dagger\mathbf{b}=0,\ \|\mathbf{b}_\perp\|=1}\Bigg(|\alpha|^2\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{\mathbf{b}^\dagger\mathbf{H}\mathbf{H}^\dagger\mathbf{b}}{1+P|\mathbf{H}_{\|}|^2}\right]+|\beta|^2\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{\mathbf{b}_\perp^\dagger\mathbf{H}\mathbf{H}^\dagger\mathbf{b}_\perp}{1+P|\mathbf{H}_{\|}|^2}\right] + \alpha^*\beta\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{\mathbf{b}^\dagger\mathbf{H}\mathbf{H}^\dagger\mathbf{b}_\perp}{1+P|\mathbf{H}_{\|}|^2}\right]+\alpha\beta^*\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{\mathbf{b}_\perp^\dagger\mathbf{H}\mathbf{H}^\dagger\mathbf{b}}{1+P|\mathbf{H}_{\|}|^2}\right]\Bigg) \tag{25}$$

$$= \max_{\alpha,\ \beta,\ \mathbf{b}_\perp:|\alpha|^2+|\beta|^2=1,\ \mathbf{b}_\perp^\dagger\mathbf{b}=0,\ \|\mathbf{b}_\perp\|=1}\Bigg(|\alpha|^2\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{\mathbf{b}^\dagger\mathbf{H}\mathbf{H}^\dagger\mathbf{b}}{1+P|\mathbf{H}_{\|}|^2}\right]+|\beta|^2\mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{\mathbf{b}_\perp^\dagger\mathbf{H}\mathbf{H}^\dagger\mathbf{b}_\perp}{1+P|\mathbf{H}_{\|}|^2}\right]\Bigg) \tag{26}$$

$$= \mathsf{E}_{\mathbf{H}|\mathbf{F}}\left[\frac{\mathbf{b}^\dagger\mathbf{H}\mathbf{H}^\dagger\mathbf{b}}{1+P|\mathbf{H}_{\|}|^2}\right] \tag{27}$$

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