Sum Power Iterative Water-filling for Multi-Antenna Gaussian Broadcast Channels

Nihar Jindal, Wonjong Rhee, Sriram Vishwanath, Syed Jafar, and Andrea Goldsmith

Dept. of Electrical Engineering

Stanford University, CA, 94305

Email: njindal@wsl.stanford.edu, wonjong@dsl.stanford.edu, sriram@ece.utexas.edu, syed@ece.uci.edu, andrea@wsl.stanford.edu

Abstract

In this paper we consider the problem of maximizing sum rate of a multiple-antenna Gaussian broadcast channel. It was recently found that dirty paper coding achieves the sum capacity of this channel. However, obtaining the optimal transmission policy when employing dirty paper coding is a computationally complex non-convex problem. We use duality to transform this problem into a well-structured convex multiple access channel problem. We exploit the structure of this problem and derive a simple and fast iterative algorithm that provides the optimum transmission policies for the multiple-access channel, which can easily be mapped to the optimal broadcast channel policies.

I. Introduction

There has been a great interest in characterizing and computing the capacity region of downlink channels in recent years. An achievable region for the multiple-antenna downlink channel was found in [3], and this achievable region was shown to be sum rate optimal in [3], [8], [9], [11]. Unfortunately, the sum capacity is not known in closed form in general and is only known as the solution to a computationally

Manuscript submitted to *IEEE Transactions on Information Theory* on July 21, 2004. The material in this paper was presented in part at the *International Symposium on Information Theory*, July, 2003 in Yokohama, Japan and at the *Asilomar Conference on Signals, Systems, and Computers*, Nov. 2002 in Asilomar, CA.

This work was initiated while the authors were at Stanford University. Nihar Jindal is now at Univ. of Minnesota, Wonjong Rhee is at Assia Inc, Sriram Vishwanath is at Univ. of Texas, Austin, Syed Jafar is at Univ. of California, Irvine, and Andrea Goldsmith is at Stanford University.

complex non-convex optimization problem. Therefore, obtaining the optimal rates and transmission policy is difficult. In the single transmit antenna channel, although the problem is still non-convex, it is easily seen that it is optimal to transmit to only the user with the strongest channel. Such a policy is, however, not the optimal policy when the transmitter has multiple antennas.

A duality technique presented in [6], [8] transforms the non-convex downlink problem into a convex sum power *uplink* (MAC) problem, which is much easier to solve. In this sum power uplink or sum power MAC problem, the users in the system have a joint power constraint instead of the individual constraints in the conventional MAC. As in the case of the conventional MAC, there exist standard interior point convex optimization algorithms [2] that solve the sum power MAC problem. A new interior point based method has also been found in [10]. However, employing an interior point convex optimization algorithm to tackle a well structured problem such as the sum capacity problem is often inefficient. In this paper, we exploit the structure of the sum capacity problem to obtain a simple iterative algorithm for calculating sum capacity. This algorithm is inspired by and is very similar to an iterative algorithm for the conventional individual power constraint MAC problem by Yu, Rhee, Boyd and Cioffi [12].

This paper is structured as follows. In the next section, the system model is presented. In Section III, expressions for the sum capacity of the downlink and dual uplink channels are stated. In Section IV, the iterative water-filling algorithm for the multiple-access channel is analyzed, and then in Section V a new iterative water-filling algorithm for the multiple-access channel with a sum power constraint is presented. Finally, a proof of convergence is provided in Section VI and some concluding remarks are furnished in Section VII.

II. System Model

We consider a K user MIMO Gaussian broadcast channel (abbreviated as MIMO BC) where the transmitter has M antennas and the j-th receiver has r_j antennas. This downlink channel is shown in Figure 1 along with the *dual* uplink channel. The dual uplink channel is a K user multiple antenna uplink channel (abbreviated as MIMO MAC) where each of the dual uplink channels is the conjugate transpose of the corresponding downlink channel. The downlink and uplink channel are mathematically described as:

$$\mathbf{y}_i = \mathbf{H}_i\mathbf{x} + \mathbf{n}_i, \quad i = 1, \ldots, K \qquad \text{Downlink channel} \tag{1}$$

$$\mathbf{y}_{MAC} = \sum_{i=1}^{K} \mathbf{H}_i^\dagger \mathbf{x}_i + \mathbf{n} \qquad \text{Dual uplink channel} \tag{2}$$

where $\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_K$ are the channel matrices (with $\mathbf{H}_i \in \mathbb{C}^{r_i \times M}$) of users 1 through K respectively on

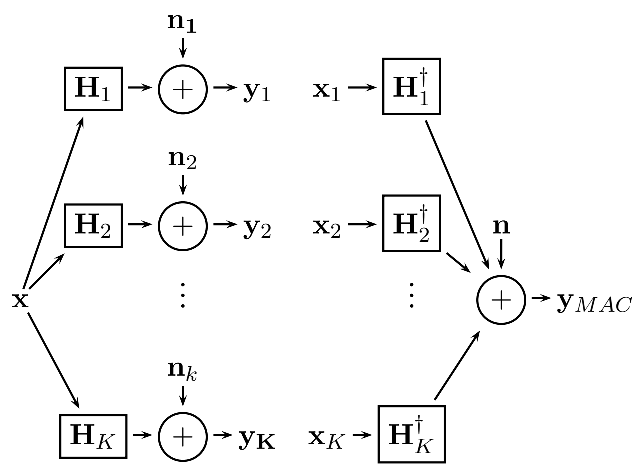

Fig. 1. System models of the MIMO BC(left) and the MIMO MAC (right) channels

the downlink, the vector $\mathbf{x} \in \mathbb{C}^{M\times 1}$ is the downlink transmitted signal, and $\mathbf{x}_1, \ldots, \mathbf{x}_K$ (with $\mathbf{x}_i \in \mathbb{C}^{r_i \times 1}$) are the transmitted signals in the uplink channel. The vectors $\mathbf{n}_1, \ldots, \mathbf{n}_K$ and $\mathbf{n}$ refer to additive Gaussian noise with unit variance on each vector component. We assume there is a sum power constraint of P in the MIMO BC and in the MIMO MAC. Though the computation of the sum capacity of the MIMO BC is of interest, we work with the dual MAC, which is computationally much easier to solve, instead.

III. Sum Rate Capacity

In [3], [8], [9], [11], the sum rate capacity of the MIMO BC (denoted as $\mathcal{C}_{\mathrm{BC}}(\mathbf{H}_1, \ldots, \mathbf{H}_K, P)$) was shown to be achievable by dirty-paper coding [4]. From these results, the sum rate capacity can be written in terms of the following maximization:

$$\begin{aligned}\mathcal{C}_{\mathrm{BC}}(\mathbf{H}_1, \ldots, \mathbf{H}_K, P) &= \max_{\boldsymbol{\Sigma}_i \geq 0,\ \sum_{i=1}^K \mathrm{Tr}(\boldsymbol{\Sigma}_i) \leq P} \log \left|\mathbf{I} + \mathbf{H}_1 \boldsymbol{\Sigma}_1 \mathbf{H}_1^\dagger\right| + \\ &\quad \log \frac{\left|\mathbf{I} + \mathbf{H}_2(\boldsymbol{\Sigma}_1 + \boldsymbol{\Sigma}_2)\mathbf{H}_2^\dagger\right|}{\left|\mathbf{I} + \mathbf{H}_2 \boldsymbol{\Sigma}_1 \mathbf{H}_2^\dagger\right|} + \cdots + \log \frac{\left|\mathbf{I} + \mathbf{H}_K(\boldsymbol{\Sigma}_1 + \cdots + \boldsymbol{\Sigma}_K)\mathbf{H}_K^\dagger\right|}{\left|\mathbf{I} + \mathbf{H}_K(\boldsymbol{\Sigma}_1 + \cdots + \boldsymbol{\Sigma}_{K-1})\mathbf{H}_K^\dagger\right|}.\end{aligned} \tag{3}$$

The maximization is performed over downlink covariance matrices $\boldsymbol{\Sigma}_1, \ldots, \boldsymbol{\Sigma}_K$, each of which is a $M \times M$ positive semi-definite matrix. In this paper we are interested in finding the covariance matrices that achieve this maximum. It is easily seen that the objective (3) is not a concave function of the covariance matrices. Thus, numerically finding the maximum is a non-trivial problem. However, in [8], a *duality* is established between the uplink and downlink where it is shown that the dirty paper region for the MIMO BC is equal to the capacity region of the dual MIMO MAC (described in (2)). This implies that the sum capacity of the MIMO BC is equal to the sum capacity of the dual MIMO MAC (denoted as $\mathcal{C}_{\mathrm{MAC}}(\mathbf{H}_1, \ldots, \mathbf{H}_K, P)$), i.e.

$$\mathcal{C}_{\mathrm{BC}}(\mathbf{H}_1, \ldots, \mathbf{H}_K, P) = \mathcal{C}_{\mathrm{MAC}}(\mathbf{H}_1^\dagger, \ldots, \mathbf{H}_K^\dagger, P). \tag{4}$$

The sum rate capacity of the MIMO MAC is given by the following expression [8]:

$$\mathcal{C}_{\mathrm{MAC}}(\mathbf{H}_1^\dagger, \ldots, \mathbf{H}_K^\dagger, P) = \max_{\mathbf{Q}_i \geq 0,\ \sum_{i=1}^K \mathrm{Tr}(\mathbf{Q}_i) \leq P} \log \left| \mathbf{I} + \sum_{i=1}^K \mathbf{H}_i^\dagger \mathbf{Q}_i \mathbf{H}_i \right|, \tag{5}$$

where the maximization is performed over uplink covariance matrices $\mathbf{Q}_1, \ldots, \mathbf{Q}_K$ ($\mathbf{Q}_i$ is an $r_i \times r_i$ positive semi-definite matrix), subject to the same power constraint P. The objective in (5) is a concave function of the covariance matrices. Furthermore, in [8, Equations 8-10], a transformation is provided (this mapping is reproduced in Appendix I for convenience) that maps from uplink covariance matrices to the downlink covariance matrices (i.e. from $\mathbf{Q}_1, \ldots, \mathbf{Q}_K$ to $\mathbf{\Sigma}_1, \ldots, \mathbf{\Sigma}_K$) that achieve the same rates and use the same sum power. Therefore, finding the optimal uplink covariance matrices leads directly to the optimal downlink covariance matrices.

In this paper, we develop a specialized algorithm that efficiently computes (5). This algorithm converges, and utilizes the water-filling structure of the optimal solution, first identified for the individual power constraint MAC in [12].

IV. Iterative Water-Filling with Individual Power Constraints

The iterative water-filling algorithm for the conventional MIMO MAC problem was obtained by Yu, Rhee, Boyd, and Cioffi in [12]. This algorithm finds the sum capacity of a MIMO MAC with *individual* power constraints $P_1, \ldots, P_K$ on each user, which is equal to:

$$\mathcal{C}_{\mathrm{MAC}}(\mathbf{H}_1^\dagger, \ldots, \mathbf{H}_K^\dagger, P_1, \ldots, P_K) \quad = \quad \max_{\{\mathbf{Q}_i \geq 0,\ \mathrm{Tr}(\mathbf{Q}_i) \leq P_i\}} \log \left| \mathbf{I} + \sum_{i=1}^K \mathbf{H}_i^\dagger \mathbf{Q}_i \mathbf{H}_i \right|. \tag{6}$$

This differs from (5) only in the power constraint structure. Notice that the objective is a concave function of the covariance matrices, and that the constraints in (6) are separable because there is an individual trace constraint on *each* covariance matrix. In such situations, it is generally sufficient to optimize with respect to the first variable while holding all other variables constant, then optimize with respect to the second variable, etc., in order to reach a globally optimum point. This is referred to as the block-coordinate ascent algorithm and convergence can be shown under relatively general conditions [1, Section 2.7]. If we define the function $f(\cdot)$ as

$$f(\mathbf{Q}_1,\ldots,\mathbf{Q}_K) \triangleq \log\left|\mathbf{I}+\sum_{i=1}^{K}\mathbf{H}_i^\dagger\mathbf{Q}_i\mathbf{H}_i\right|, \tag{7}$$

then in the $(l+1)$-th iteration of the block-coordinate ascent algorithm,

$$\mathbf{Q}_j^{(l+1)} \triangleq \arg\max_{\mathbf{Q}_j\geq 0,\ \mathrm{Tr}(\mathbf{Q}_j)\leq P_j} f(\mathbf{Q}_1^{(l)},\ldots,\mathbf{Q}_{j-1}^{(l)},\mathbf{Q}_j,\mathbf{Q}_{j+1}^{(l)},\ldots,\mathbf{Q}_K^{(l)}) \tag{8}$$

for $j=(l \bmod K)+1$ and $\mathbf{Q}_j^{(l+1)}=\mathbf{Q}_j^{(l)}$ for all other j. Notice that only one of the covariances is updated in each iteration.

The key to the iterative water-filling algorithm is noticing that $f(\mathbf{Q}_1,\ldots,\mathbf{Q}_K)$ can be rewritten as:

$$\begin{aligned} f(\mathbf{Q}_1,\ldots,\mathbf{Q}_K) &= \log\left|\mathbf{I}+\sum_{i\neq j}\mathbf{H}_i^\dagger\mathbf{Q}_i\mathbf{H}_i+\mathbf{H}_j^\dagger\mathbf{Q}_j\mathbf{H}_j\right| \\ &= \log\left|\mathbf{I}+\sum_{i\neq j}\mathbf{H}_i^\dagger\mathbf{Q}_i\mathbf{H}_i\right| + \\ &\quad \log\left|\mathbf{I}+\left(\mathbf{I}+\sum_{i\neq j}\mathbf{H}_i^\dagger\mathbf{Q}_i\mathbf{H}_i\right)^{-1/2}\mathbf{H}_j^\dagger\mathbf{Q}_j\mathbf{H}_j\left(I+\sum_{i\neq j}\mathbf{H}_i^\dagger\mathbf{Q}_i\mathbf{H}_i\right)^{-1/2}\right| \end{aligned}$$

for any j. Thus, the iteration in (8) can be rewritten as:

$$\mathbf{Q}_j^{(l+1)} = \arg\max_{\mathbf{Q}_j\geq 0,\ \mathrm{Tr}(\mathbf{Q}_j)\leq P_j} \log\left|\mathbf{I}+\mathbf{G}_j^\dagger\mathbf{Q}_j\mathbf{G}_j\right| \tag{9}$$

where $\mathbf{G}_j=\mathbf{H}_j\left(\mathbf{I}+\sum_{i\neq j}\mathbf{H}_i^\dagger\mathbf{Q}_i^{(l)}\mathbf{H}_i\right)^{-1/2}$. This maximization is clearly equal to the expression for the capacity of a point-to-point MIMO channel with channel matrix $\mathbf{G}_j$ and power constraint P_j. It is well known that the capacity of such a MIMO channel is achieved by choosing the input covariance along the eigenvectors of the channel matrix and by water-filling on the eigenvalues of the channel matrix [7]. Thus, $\mathbf{Q}_j^{(l+1)}$ should be chosen as a *water-fill* of the channel $\mathbf{G}_j$, i.e. the eigenvectors of $\mathbf{Q}_j^{(l+1)}$ should equal the left eigenvectors of $\mathbf{G}_j$, with the eigenvectors chosen by the water-filling procedure.

At each step of the algorithm, exactly one user optimizes his covariance matrix while treating the signals from all other users as noise. In the next step, the next user (in numerical order) optimizes his covariance while treating all other signals, including the updated covariance of the previous user, as noise. This intuitively appealing algorithm can easily be shown to satisfy the conditions of [1, Section 2.7] and thus provably converges. Furthermore, the optimization in each step of the algorithm simplifies to water-filling over an effective channel, which is computationally efficiently.

If we let $\mathbf{Q}_1^*, \ldots, \mathbf{Q}_K^*$ denote the optimal covariances, notice that

$$f(\mathbf{Q}_1^*, \ldots, \mathbf{Q}_K^*) = \max_{\mathbf{Q}_j \geq 0, \mathrm{Tr}(\mathbf{Q}_j) \leq P_j} f(\mathbf{Q}_1^*, \ldots, \mathbf{Q}_{j-1}^*, \mathbf{Q}_j, \mathbf{Q}_{j+1}^*, \ldots, \mathbf{Q}_K^*). \tag{10}$$

Thus, $\mathbf{Q}_1^*$ is a water-fill of the noise and the signals from all other users, and simultaneously $\mathbf{Q}_2^*$ is a water-fill of the noise and the signals from all other users, and so on. Thus, the sum capacity achieving covariance matrices simultaneously water-fill each of their respective effective channels (which for User j depends on the covariance matrices of all other users) [12], with the water-filling levels (i.e. the eigenvectors) of each user determined by the power constraints P_j. In the next section, we will see that similar intuition describes the sum capacity achieving covariance matrices in the MIMO MAC when there is a sum power constraint instead of individual power constraints.

V. Sum Power Iterative Water-Filling

In the previous section we described an iterative water-filling algorithm that computes the sum capacity of a MIMO MAC subject to individual power constraints. We are instead concerned with computing the sum capacity, along with the corresponding optimal covariance matrices, of a MIMO BC. As stated earlier, this is equivalent to computing the sum capacity (and the corresponding optimal covariance matrices) of a MIMO MAC subject to a sum power constraint, i.e. computing:

$$\mathcal{C}_{\mathrm{MAC}}(\mathbf{H}_1^\dagger, \ldots, \mathbf{H}_K^\dagger, P) = \max_{\{\mathbf{Q}_i \geq 0,\ \sum_{i=1}^K \mathrm{Tr}(\mathbf{Q}_i) \leq P\}} \log \left| \mathbf{I} + \sum_{i=1}^K \mathbf{H}_i^\dagger \mathbf{Q}_i \mathbf{H}_i \right|. \tag{11}$$

If we let $\mathbf{Q}_1^*, \ldots, \mathbf{Q}_K^*$ denote a set of covariance matrices that achieve the above maximum, it is easy to see that similar to the individual power constraint problem, each covariance must be a water-fill of the noise and signals from all other users. More precisely, this means that for every j, the eigenvectors of $\mathbf{Q}_j^*$ are aligned with the left eigenvectors of $\mathbf{H}_j \left(\mathbf{I} + \sum_{i \neq j} \mathbf{H}_i^\dagger \mathbf{Q}_i^* \mathbf{H}_i \right)^{-1/2}$. However, since there is a *sum* power constraint on the covariances, the water level of all users must be equal. This is akin to saying that no advantage will be gained by transferring power from one user with a higher water-filling level to another user with a lower water-filling level. In the individual power constraint channel, since each user's water-filling level was determined by his own power constraint, the covariances of each user could be updated one at a time. With a sum power constraint, however, we must update all covariances *simultaneously* to maintain a constant water-level.

Motivated by the individual power algorithm, we propose the following algorithm in which all K covariances are simultaneously updated during each step, based on the covariance matrices from the previous step. This is a natural extension of the per-user sequential update described in Section IV. At each iteration step we generate an effective channel for *each* user based on the covariances of all other users. In order to maintain a common water-level, we simultaneously water-fill across all K effective channels, i.e. we maximize the sum of rates on the K effective channels. The $(l+1)$-th iteration of the algorithm is described by the following:

1) Generate effective channels $\mathbf{G}_j = \mathbf{H}_j(\mathbf{I} + \sum_{i \neq j} \mathbf{H}_i^\dagger \mathbf{Q}_i^{(l)} \mathbf{H}_i)^{-1/2}$ for $j = 1, \dots, K$.
2) Treating these effective channels as parallel, non-interfering channels, obtain the new covariance matrices $\{\mathbf{Q}_i^{(l+1)}\}_{i=1}^K$ by water-filling with total power P:
$$\{\mathbf{Q}_i^{(l+1)}\}_{i=1}^K = \arg \max_{\mathbf{Q}_i \geq 0,\ \sum_{i=1}^K \mathrm{Tr}(\mathbf{Q}_i) \leq P} \sum_{i=1}^K \log \left| \mathbf{I} + \mathbf{G}_i^\dagger \mathbf{Q}_i \mathbf{G}_i \right| .$$
This maximization is equivalent to water-filling the block diagonal channel with diagonals equal to $\mathbf{G}_1, \dots, \mathbf{G}_K$. If the SVD of $\mathbf{G}_j \mathbf{G}_j^\dagger$ is written as $\mathbf{G}_j \mathbf{G}_j^\dagger = \mathbf{U}_j \mathbf{D}_j \mathbf{U}_j^\dagger$ with $\mathbf{U}_j$ unitary and $\mathbf{D}_j$ square and diagonal, then the new covariance matrices are given by:
$$\mathbf{Q}_j^{(l+1)} = \mathbf{U}_j \mathbf{\Lambda}_j \mathbf{U}_j^\dagger \tag{12}$$
where $\mathbf{\Lambda}_j = \left[\mu \mathbf{I} - (\mathbf{D}_j)^{-1}\right]^+$ and the operation $[\mathbf{A}]^+$ denotes a component-wise minimum with zero. Here the water-filling level μ is chosen such that $\sum_{i=1}^K \mathrm{Tr}(\mathbf{\Lambda}_i) = P$.

Perhaps surprisingly, this algorithm does not always lead to an increase in the objective function and does not always converge to the optimum when $K > 2$. Even though the algorithm converges to the maximum sum rate for a two-user channel, the algorithm needs to be modified to guarantee convergence when there are more than two users. In the following section we discuss the modification and the proof of convergence.

VI. Convergence Proof

In this section we show that the sum power iterative water-filling algorithm converges when $K = 2$, but does not always converge when $K > 2$. For $K > 2$, we describe a modified version of the algorithm that provably converges to the optimum.

A. Two User Analysis

In order to prove convergence of the algorithm for $K = 2$, let us consider the following optimization problem:
$$\begin{aligned} \max_{\mathrm{Tr}(\mathbf{A}_1+\mathbf{A}_2) \leq P,\ \mathrm{Tr}(\mathbf{B}_1+\mathbf{B}_2) \leq P} \quad & \frac{1}{2} \log \left| \mathbf{I} + \mathbf{H}_1^\dagger \mathbf{A}_1 \mathbf{H}_1 + \mathbf{H}_2^\dagger \mathbf{B}_2 \mathbf{H}_2 \right| \\ & + \frac{1}{2} \log \left| \mathbf{I} + \mathbf{H}_1^\dagger \mathbf{B}_1 \mathbf{H}_1 + \mathbf{H}_2^\dagger \mathbf{A}_2 \mathbf{H}_2 \right| . \end{aligned} \tag{13}$$

We first show that the solutions to the original sum rate maximization problem in (11) and (13) are the same. If we let $\mathbf{A}_1 = \mathbf{B}_1 = \mathbf{Q}_1$ and $\mathbf{A}_2 = \mathbf{B}_2 = \mathbf{Q}_2$, we see that any sum rate achievable in (11) is also achievable in the modified sum rate in (13). Also, since the $\log(\det(\cdot))$ function is concave we have

$$\log\left|\mathbf{I} + \mathbf{H}_1^\dagger \mathbf{Q}_1 \mathbf{H}_1 + \mathbf{H}_2^\dagger \mathbf{Q}_2 \mathbf{H}_2\right| \geq \frac{1}{2}\log\left|\mathbf{I} + \mathbf{H}_1^\dagger \mathbf{A}_1 \mathbf{H}_1 + \mathbf{H}_2^\dagger \mathbf{B}_2 \mathbf{H}_2\right| + \frac{1}{2}\log\left|\mathbf{I} + \mathbf{H}_1^\dagger \mathbf{B}_1 \mathbf{H}_1 + \mathbf{H}_2^\dagger \mathbf{A}_2 \mathbf{H}_2\right|$$

if we let $\mathbf{Q}_1 = \frac{1}{2}(\mathbf{A}_1 + \mathbf{B}_1)$ and $\mathbf{Q}_2 = \frac{1}{2}(\mathbf{A}_2 + \mathbf{B}_2)$. Since $\text{Tr}(\mathbf{Q}_1) + \text{Tr}(\mathbf{Q}_2) = \frac{1}{2}\text{Tr}(\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{B}_1 + \mathbf{B}_2) \leq P$, any sum rate achievable in (13) is also achievable in the original (11). Thus, every set of maximizing covariances $(\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2)$ map directly to a set of maximizing $(\mathbf{Q}_1, \mathbf{Q}_2)$. Therefore, we can equivalently solve (13) to find the uplink covariances that maximize the sum-rate expression in (11).

Now notice that the maximization in (13) has separable constraints on $(\mathbf{A}_1, \mathbf{A}_2)$ and $(\mathbf{B}_1, \mathbf{B}_2)$. Thus, we can use the block coordinate ascent method in which we maximize with respect to $(\mathbf{A}_1, \mathbf{A}_2)$, then with respect to $(\mathbf{B}_1, \mathbf{B}_2)$, and so on. The maximization of (13) with respect to $(\mathbf{A}_1, \mathbf{A}_2)$ can be written as:

$$\max_{\text{Tr}(\mathbf{A}_1 + \mathbf{A}_2) \leq P} \log\left|\mathbf{I} + \mathbf{G}_1^\dagger \mathbf{A}_1 \mathbf{G}_1\right| + \log\left|\mathbf{I} + \mathbf{G}_2^\dagger \mathbf{A}_2 \mathbf{G}_2\right| \tag{14}$$

where $\mathbf{G}_1 = \mathbf{H}_1(\mathbf{I} + \mathbf{H}_2^\dagger \mathbf{B}_2 \mathbf{H}_2)^{-1/2}$ and $\mathbf{G}_2 = \mathbf{H}_2(\mathbf{I} + \mathbf{H}_1^\dagger \mathbf{B}_1 \mathbf{H}_1)^{-1/2}$. Clearly, this is equivalent to the iterative water-filling step described in the previous section where $\mathbf{B}_1, \mathbf{B}_2$ play the role of the covariance matrices from the previous step. Similarly, when maximizing with respect to $\mathbf{B}_1, \mathbf{B}_2$, the covariances $\mathbf{A}_1, \mathbf{A}_2$ are the covariance matrices from the previous step. Therefore, performing the cyclic coordinate ascent algorithm on (13) is exactly equivalent to the sum power iterative water-filling algorithm described in Section V.

Furthermore, notice that each iteration is equal to the calculation of the capacity of a point-to-point MIMO channel. Water-Filling is known to be optimal in this setting, and in Appendix II we show that the water-filling solution is also the unique solution. Therefore, by [13, pg. 228] [1, Chapter 2.7], the block coordinate ascent algorithm converges because at each step of the algorithm there is a unique maximizing solution. Thus, the iterative water-filling algorithm given in Section V converges to the maximum sum rate when $K = 2$.

B. More Than Two Users

If there are more than two users, the original algorithm is easily shown by example not to always converge. Thus, the algorithm needs to be slightly modified in order to guarantee convergence. For simplicity, we consider three users and then generalize. Similar to the previous section, consider the following maximization:

$$
\begin{aligned}
\max \quad & \frac{1}{3}\log\left|\mathbf{I}+\mathbf{H}_1^\dagger\mathbf{A}_1\mathbf{H}_1+\mathbf{H}_2^\dagger\mathbf{B}_2\mathbf{H}_2+\mathbf{H}_3^\dagger\mathbf{C}_3\mathbf{H}_3\right| \\
& +\frac{1}{3}\log\left|\mathbf{I}+\mathbf{H}_1^\dagger\mathbf{C}_1\mathbf{H}_1+\mathbf{H}_2^\dagger\mathbf{A}_2\mathbf{H}_2+\mathbf{H}_3^\dagger\mathbf{B}_3\mathbf{H}_3\right| \\
& +\frac{1}{3}\log\left|\mathbf{I}+\mathbf{H}_1^\dagger\mathbf{B}_1\mathbf{H}_1+\mathbf{H}_2^\dagger\mathbf{C}_2\mathbf{H}_2+\mathbf{H}_3^\dagger\mathbf{A}_3\mathbf{H}_3\right| \qquad (15)
\end{aligned}
$$

subject to the constraints Tr$(\mathbf{A}_1+\mathbf{A}_2+\mathbf{A}_3)\leq P$, Tr$(\mathbf{B}_1+\mathbf{B}_2+\mathbf{B}_3)\leq P$, and Tr$(\mathbf{C}_1+\mathbf{C}_2+\mathbf{C}_3)\leq P$. By the same argument used for the two user case, any set of covariances is a solution to the original optimization problem in (11) if and only if it is a solution to the above problem (with $\mathbf{A}_i=\mathbf{B}_i=\mathbf{C}_i$ for $i=1,2,3$). In order to maximize (15), we can again use the cyclic coordinate ascent algorithm. We first maximize with respect to $(\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3)$, then with respect to $(\mathbf{B}_1,\mathbf{B}_2,\mathbf{B}_3)$, and so on. As before, convergence is guaranteed by [1, Section 2.7]. In the two user case, the cyclic coordinate ascent method applied to the modified optimization problem yields the same iterative water-filling algorithm proposed in Section V where the effective user of each channel was based on the covariance matrices from the previous step. If there are more than two users, however, the effective channel of each user depends on covariances which are up to $K-1$ steps old, instead of just one step old. It is easily seen that the effective channel of User j in the n-th step is:

$$
\mathbf{G}_j^{(n)}=\mathbf{H}_j\left(\mathbf{I}+\sum_{i=1}^{K-1}\mathbf{H}_{[j+i]_K}^\dagger\mathbf{Q}_{[j+i]_K}^{(n-K+i)}\mathbf{H}_{[j+i]_K}\right)^{-1/2} \qquad (16)
$$

where $[x]_K=x+lK$ where l is an integer such that $1\leq x+lK\leq K$. For the three user case, the update of $\mathbf{Q}_1^{(n)}$ depends on $\mathbf{Q}_2^{(n-2)}$ and $\mathbf{Q}_3^{(n-1)}$, $\mathbf{Q}_2^{(n)}$ depends on $\mathbf{Q}_3^{(n-2)}$ and $\mathbf{Q}_1^{(n-1)}$, and $\mathbf{Q}_3^{(n)}$ depends on $\mathbf{Q}_1^{(n-2)}$ and $\mathbf{Q}_2^{(n-1)}$. Thus, the previous $K-1$ states of the algorithm must be stored. If (16) is used to generate each effective channel in step 1 of the sum power iterative water-filling algorithm in Section V, then the algorithm provably converges to the optimum due to the convergence of the block coordinate ascent method.

C. Numerical Results

In Figures 2 and 3, plots of sum rate vs. iteration number are provided for a randomly chosen 10 user channel with 4 transmit and receive antennas. In Fig. 2 the original algorithm converges to the optimum, and is seen to converge considerably faster than the modified, provably convergent algorithm. Given that the original algorithm converges in this scenario, it is not surprising that its convergence rate is much

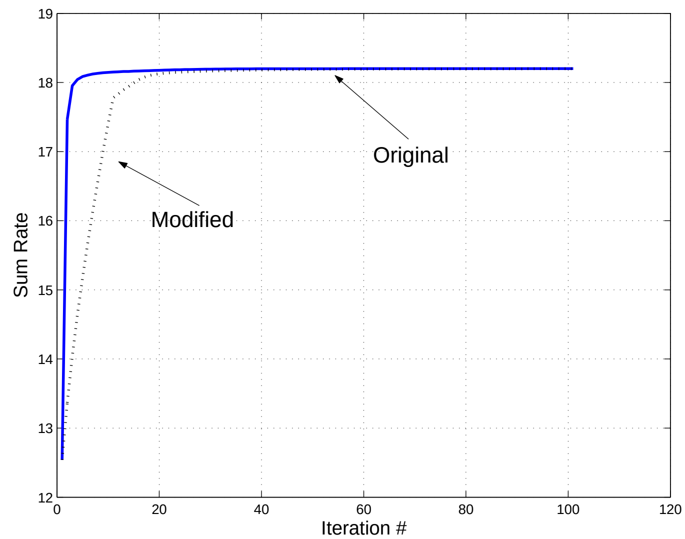

Fig. 2. Algorithm comparison for convergent scenario

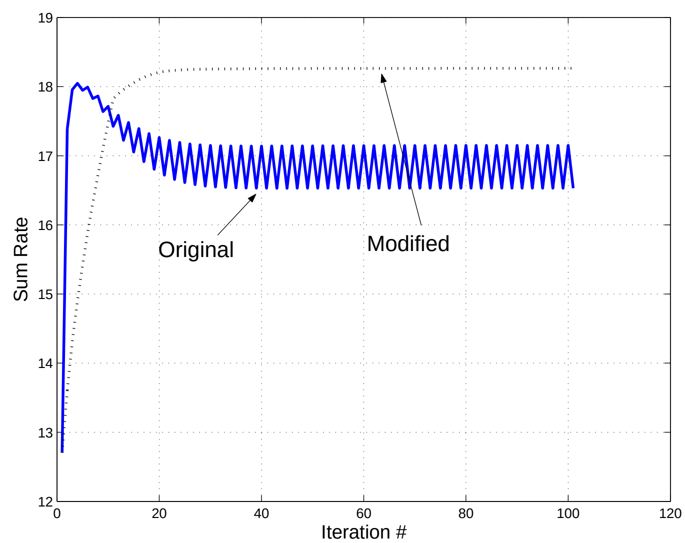

Fig. 3. Algorithm comparison for divergent scenario

faster. The modified algorithm is intuitively slower because updates are based on covariance matrices from up to $K-1$ iterates ago, as opposed to only the previous iterate. In Fig. 3, however, the original algorithm diverges, and oscillates between two sub-optimal points. In general, it is not difficult to find similar examples of divergence for a large number of users. When convergence speed is of concern, it appears to be beneficial to use the original algorithm for the first few iterations (or until sum rate is decreased in the next iteration) and then use the modified algorithm thereafter. The modified algorithm converges from any starting point, and thus convergence is still guaranteed.

VII. Conclusions

In this paper we proposed an algorithm to find the sum capacity achieving transmission strategies for the multiple antenna broadcast channel. We use the fact that the Gaussian broadcast and multiple-access channels are duals in the sense that their capacity regions, and therefore their sum capacities, are equal. Our algorithm computes the sum capacity achieving strategy for the dual multiple-access channel, which can easily be converted to the equivalent optimal strategies for the broadcast channel. The algorithm exploits the inherent structure of the multiple-access channel and employs a simple iterative water-filling procedure that provably converges to the optimum.

Appendix I
MAC-BC Transformation

In this appendix, we restate the mapping from uplink covariance matrices to downlink matrices. Given uplink covariances $\mathbf{Q}_1, \ldots, \mathbf{Q}_K$, the transformation in [8, Equations 8-10] outputs downlink covariance matrices $\mathbf{\Sigma}_1, \ldots, \mathbf{\Sigma}_K$ that achieve the same rates (on a user-by-user basis, and thus also in terms of sum rate) using the same sum power, i.e. with $\sum_{i=1}^K \mathrm{Tr}(\mathbf{Q}_i) = \sum_{i=1}^K \mathrm{Tr}(\mathbf{\Sigma}_i)$. For convenience, we first define the following two quantities:

$$\mathbf{A}_j \triangleq \mathbf{I} + \mathbf{H}_j \left(\sum_{l=1}^{j-1} \mathbf{\Sigma}_l \right) \mathbf{H}_j^\dagger, \qquad \mathbf{B}_j \triangleq \mathbf{I} + \sum_{l=j+1}^{K} \mathbf{H}_l^\dagger \mathbf{Q}_l \mathbf{H}_l \tag{17}$$

for $j = 1, \ldots, K$. Furthermore, assume the matrix $\mathbf{B}_j^{-1/2}\mathbf{H}_j^\dagger \mathbf{A}_j^{-1/2}$ can be decomposed via the SVD as $\mathbf{B}_j^{-1/2}\mathbf{H}_j^\dagger \mathbf{A}_j^{-1/2} = \mathbf{F}_j \mathbf{D}_j \mathbf{G}_j^\dagger$, where $\mathbf{D}_j$ is a square and diagonal matrix[1]. Then, the equivalent downlink covariance matrices can be computed via the following transformation:

$$\mathbf{\Sigma}_j = \mathbf{B}_j^{-1/2}\mathbf{F}_j \mathbf{G}_j^\dagger \mathbf{A}_j^{1/2} \mathbf{Q}_j \mathbf{A}_j^{1/2} \mathbf{G}_j \mathbf{F}_j^\dagger \mathbf{B}_j^{-1/2}, \tag{18}$$

beginning with $j = 1$. See [8] for a derivation and more detail.

[1]Note that the standard SVD command in MATLAB does not return a square and diagonal $\mathbf{D}_j$. This is accomplished by using the “0” option in the SVD command in MATLAB, and is referred to as the “economy size” decomposition.

APPENDIX II
UNIQUENESS OF WATER-FILLING SOLUTION

In this appendix we show that there is a unique solution to the following problem:

$$\max_{\mathbf{Q}\geq 0,\ \mathrm{Tr}(\mathbf{Q})\leq P} \log\left|\mathbf{I}+\mathbf{H}\mathbf{Q}\mathbf{H}^{\dagger}\right| \tag{19}$$

for any non-zero $\mathbf{H} \in \mathbb{C}^{N\times M}$ for arbitrary M, N. This proof is identical to the proof of optimality of water-filling in [7, Section 3.2], with the addition of a simple proof of uniqueness.

Since $\mathbf{H}^{\dagger}\mathbf{H} \in \mathbb{C}^{M\times M}$ is Hermitian and positive semi-definite, we can diagonalize it and write $\mathbf{H}^{\dagger}\mathbf{H} = \mathbf{U}\mathbf{D}\mathbf{U}^{\dagger}$ where $\mathbf{U}$ is unitary and $\mathbf{D}$ is diagonal with non-negative entries. Using the identity $|\mathbf{I}+\mathbf{A}\mathbf{B}| = |\mathbf{I}+\mathbf{B}\mathbf{A}|$, we can rewrite the objective function as

$$\log\left|\mathbf{I}+\mathbf{H}\mathbf{Q}\mathbf{H}^{\dagger}\right| = \log\left|\mathbf{I}+\mathbf{Q}\mathbf{H}^{\dagger}\mathbf{H}\right| = \log\left|\mathbf{I}+\mathbf{Q}\mathbf{U}\mathbf{D}\mathbf{U}^{\dagger}\right| = \log\left|\mathbf{I}+\mathbf{U}^{\dagger}\mathbf{Q}\mathbf{U}\mathbf{D}\right|. \tag{20}$$

Let $\tilde{\mathbf{Q}} = \mathbf{U}^{\dagger}\mathbf{Q}\mathbf{U}$. Clearly $\mathbf{Q} = \mathbf{U}\tilde{\mathbf{Q}}\mathbf{U}^{\dagger}$. Since $\mathrm{Tr}(\mathbf{A}\mathbf{B}) = \mathrm{Tr}(\mathbf{B}\mathbf{A})$ and $\mathbf{U}$ is unitary, we have $\mathrm{Tr}(\tilde{\mathbf{Q}}) = \mathrm{Tr}(\mathbf{U}^{\dagger}\mathbf{Q}\mathbf{U}) = \mathrm{Tr}(\mathbf{Q}\mathbf{U}\mathbf{U}^{\dagger}) = \mathrm{Tr}(\mathbf{Q})$. Furthermore, $\tilde{\mathbf{Q}} \geq 0$ if and only if $\mathbf{Q} \geq 0$. Therefore, the maximization can equivalently be carried out over $\tilde{\mathbf{Q}}$, i.e.:

$$\max_{\tilde{\mathbf{Q}}\geq 0,\ \mathrm{Tr}(\tilde{\mathbf{Q}})\leq P} \log\left|\mathbf{I}+\tilde{\mathbf{Q}}\mathbf{D}\right|. \tag{21}$$

with $\mathbf{D} \in \mathcal{R}^{M\times M}$ diagonal and non-negative. In addition, any solution to (19) corresponds to a solution of (21) via the invertible mapping $\tilde{\mathbf{Q}} = \mathbf{U}^{\dagger}\mathbf{Q}\mathbf{U}$. Thus, if the maximization in (19) has multiple solutions, the maximization in (21) must also have multiple solutions. Therefore, it is sufficient to show that (21) has a unique solution, which we prove next.

First notice that Haddamard's inequality [5] gives the following upper bound $\left|\mathbf{I}+\tilde{\mathbf{Q}}\mathbf{D}\right| \leq \prod_{i=1}^{K}(1+\mathbf{Q}_{ii}\mathbf{D}_{ii})$, which is achievable if and only if $\mathbf{Q}$ is diagonal. Since $\mathrm{Tr}(\mathbf{Q}) = \sum_{i=1}^{K}\mathbf{Q}_{ii} \leq P$ and $\mathbf{Q}_{ii} \geq 0$ for $i = 1, \ldots, K$ by the positive semi-definite condition, for any feasible non-diagonal $\mathbf{Q}$ there exists a diagonal $\mathbf{Q}$ corresponding to a strictly larger objective value. Therefore, the optimal solution must be diagonal. If $\mathbf{Q}$ is diagonal, the objective function is equal to $\sum_{i=1}^{M}\log(1+\mathbf{Q}_{ii}\mathbf{D}_{ii})$. Since we can ignore entries of $\mathbf{D}$ that are zero and the assumption that $\mathbf{H}$ is not the zeroes matrix insures that at least one diagonal entry of $\mathbf{D}$ is non-zero, we can without loss of generality assume $\mathbf{D}_{ii} > 0$ for $i = 1, \ldots, M$. Therefore, the objective is a strictly concave function of $\mathbf{Q}_{11}, \ldots, \mathbf{Q}_{MM}$, and thus (21) has a unique solution.

ACKNOWLEDGMENT

The authors wish to thank Daniel Palomar and Tom Luo for helpful discussions regarding convergence issues.

REFERENCES

[1] D. Bertsekas. *Nonlinear Programming*. Athena Scientific, 1999.

[2] S. Boyd and L. Vandenberghe. *Introduction to convex optimization with engineering applications*. Course Reader, 2001.

[3] G. Caire and S. Shamai. On the achievable throughput of a multiantenna Gaussian broadcast channel. *IEEE Trans. Inform. Theory*, 49(7):1691–1706, July 2003.

[4] M. Costa. Writing on dirty paper. *IEEE Trans. Inform. Theory*, 29(3):439–441, May 1983.

[5] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. Wiley, 1991.

[6] N. Jindal, S. Vishwanath, and A. Goldsmith. On the duality of Gaussian multiple-access and broadcast channels. *IEEE Trans. Inform. Theory*, 50(5):768–783, May 2004.

[7] E. Telatar. Capacity of multi-antenna Gaussian channels. *European Trans. on Telecomm. ETT*, 10(6):585–596, November 1999.

[8] S. Vishwanath, N. Jindal, and A. Goldsmith. Duality, achievable rates, and sum-rate capacity of MIMO broadcast channels. *IEEE Trans. Inform. Theory*, 49(10):2658–2668, Oct. 2003.

[9] P. Viswanath and D. N. Tse. Sum capacity of the vector Gaussian broadcast channel and uplink-downlink duality. *IEEE Trans. Inform. Theory*, 49(8):1912–1921, Aug. 2003.

[10] H. Viswanathan, S. Venkatesan, and H. C. Huang. Downlink capacity evaluation of cellular networks with known interference cancellation. *IEEE J. Sel. Areas Commun.*, 21:802–811, June 2003.

[11] W. Yu and J. M. Cioffi. Sum capacity of a Gaussian vector broadcast channel. In *Proc. of Int. Symp. Inform. Theory*, page 498, June 2002.

[12] W. Yu, W. Rhee, S. Boyd, and J. Cioffi. Iterative water-filling for Gaussian vector multiple access channels. *IEEE Trans. Inform. Theory*, 50(1):145–152, Jan. 2004.

[13] W. Zangwill. *Nonlinear Programming: A Unified Approach*. Prentice Hall, 1969.