

Duality of MIMO Multiple Access channel and Broadcast channel with Amplify-and-Forward Relays

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Abstract

In this work, we consider a two-hop multiuser amplify-and-forward relay network with MIMO nodes. The results are three-fold. *First*, for any relay amplification matrix \mathbf{D} in the multiple-access channel (MAC), we show that duality holds when $\kappa\mathbf{D}^\dagger$ is employed in the broadcast channel (BC), and vice versa, where κ is obtained from switching the total source and relay power constraints. *Second*, under a total network power constraint, we show that MAC-BC duality holds when \mathbf{D} and \mathbf{D}^\dagger are the relaying matrices in the MAC and BC respectively. *Third*, for any \mathbf{D} in the MAC and $c\mathbf{D}^\dagger$ in the BC where c is any positive real scalar, MAC-BC duality under total network power constraint holds *only for the above two cases*.

I. INTRODUCTION

The use of relaying techniques to transfer information via multiple hops and routes offers significant benefits for wireless networks that include throughput enhancement, range extension and power reduction. These benefits make cooperation and relaying a promising candidate for the next generation wireless systems. Various forwarding strategies at the relays have been studied. Among them, memoryless schemes such as amplify and forward (AF) are attractive for their simplicity and the ability to provide soft information [1]. In this work we consider multiuser MIMO AF relay networks. The most basic multiuser scenarios are the multiple access channel and the broadcast channel. For the single-hop communication without relays, the MAC-BC duality [2], [3], which states that the capacity region of a MAC is equal to the capacity region of the reciprocal BC for the same total transmit power, is well known. For AF relay networks with single antenna nodes, the following MAC-BC duality is recently shown in [4]. Consider a MAC with total user power P_S and total relay power P_R as shown in Fig 1 (a). This MAC is

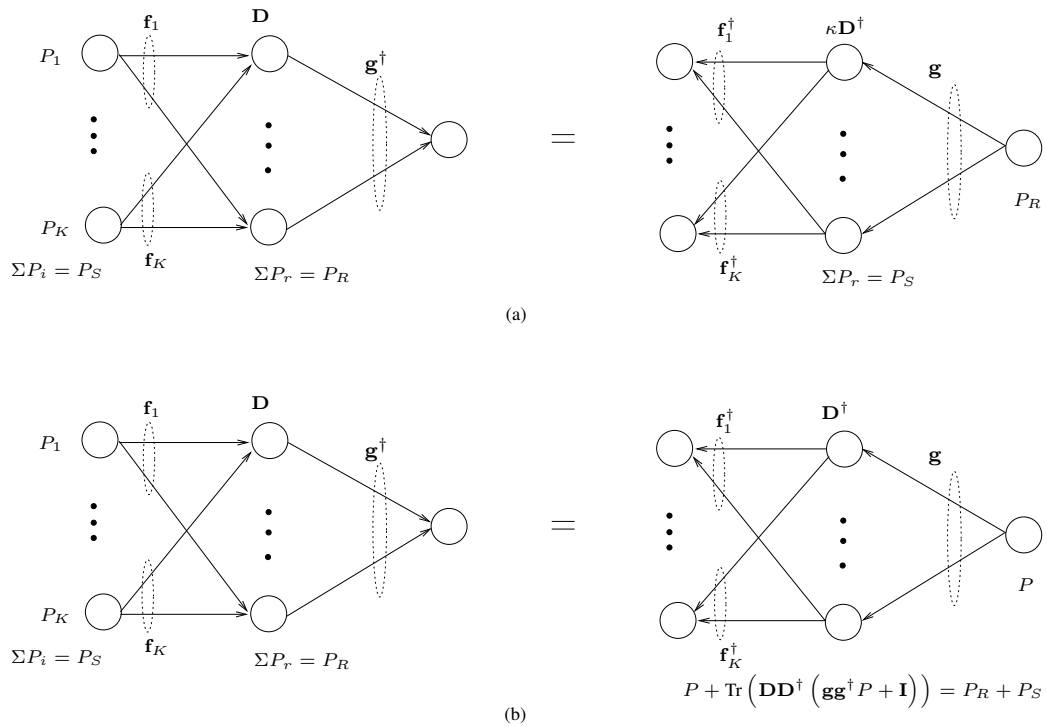


Fig. 1. Duality results for single antenna relay networks

dual to the BC obtained by reversing the direction of communication and swapping the sum relay power and total source power. For single antenna relays, the relay amplification matrix is a diagonal matrix containing the relay scaling factors on its main diagonal. Notice that the BC relay amplification matrix $\kappa\mathbf{D}^\dagger$ is proportional to the MAC relay amplification matrix \mathbf{D} . For multihop networks with more than two hops, MAC-BC duality holds when the powers are shifted such that each hop gets the same transmit power in both the original and the dual channels. Such a duality relationship may be helpful in solving network optimization problems. For example, in [5], a three-hop AF relay network is optimized using the duality relationship in [4].

Another version of duality in AF relay networks can be found in [6]. Here the authors consider a multiuser AF relay network with memory where the relays can transmit a linear combination of previously received signals. For this model, the authors show an interesting duality relationship under a total network power constraint. It is shown that duality holds when the same relaying matrix is used in both the MAC and BC without any scaling factor as illustrated in Fig 1 (b). Duality under total network power constraint holds for the result in [4] as well. Thus MAC-BC duality for single antenna nodes, subject to the network power constraint, holds for two cases. Two important questions arise:

- Does duality hold for any scaled version of \mathbf{D}^\dagger in the dual channel (with \mathbf{D} in the original channel)?
- Do these duality relationships hold when the nodes have multiple antennas?

These are the questions that we answer in this paper. In the following, we consider a SIMO relay network and its

dual channel. Here we highlight the differences with respect to a single antenna relay network that make it hard to establish a duality relationship.

Consider a SIMO relay network as shown in Fig. 2 where a single antenna source communicates to an M -antenna destination through a bunch of parallel relays. The relays may have multiple antennas and let N be the total number of relaying antennas. In the first slot, the source transmits x while the relays receive $\mathbf{r} = \mathbf{g}x + \mathbf{n}_R$ where \mathbf{g} represents the $N \times 1$ source-relay channel. The components of \mathbf{n}_R are $\mathcal{CN}(0, 1)$ additive white Gaussian noise (AWGN). In the second slot, the relays collectively transmit $\mathbf{D}\mathbf{r}$ where \mathbf{D} is the relay amplification matrix. Note that \mathbf{D} in general is a block diagonal matrix as the relays can have multiple antennas. The received signal vector at the destination is

$$\mathbf{y} = \mathbf{H}^\dagger \mathbf{D} \mathbf{g} x + \mathbf{H}^\dagger \mathbf{D} \mathbf{n}_R + \mathbf{n} \quad (1)$$

where \mathbf{H}^\dagger is the $M \times N$ relay-destination channel and the components of \mathbf{n} are $\mathcal{CN}(0, 1)$ AWGN. In normalized-form, the input-output relationship of the relay network is

$$\mathbf{y}' = \left(\mathbf{H}^\dagger \mathbf{D} \mathbf{D}^\dagger \mathbf{H} + \mathbf{I} \right)^{-\frac{1}{2}} \mathbf{H}^\dagger \mathbf{D} \mathbf{g} x + \mathbf{n}' \quad (2)$$

where $\mathbf{n}' \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$.

Now consider a MISO relay network obtained by reversing the communication direction. Here the multiple antenna source communicates to the single antenna destination through the relays that use $\kappa \mathbf{D}^\dagger$ as their relaying matrix. Here κ may denote the scaling factor used in the dual channel similar to those in [4], [6]. The received signal at the destination can be expressed as

$$y = \kappa \mathbf{g}^\dagger \mathbf{D}^\dagger \mathbf{H} \mathbf{x} + \kappa \mathbf{g}^\dagger \mathbf{D}^\dagger \mathbf{n}_R + n \quad (3)$$

where n is unit variance AWGN. In normalized form, the input-output relationship is

$$y' = \frac{\kappa \mathbf{g}^\dagger \mathbf{D}^\dagger \mathbf{H}}{\sqrt{1 + \kappa^2 \|\mathbf{g}^\dagger \mathbf{D}^\dagger\|^2}} \mathbf{x} + n'. \quad (4)$$

Notice that (2) and (4) have different effective channel directions. Compare these channels with the point to point SIMO and MISO channels,

$$y = \mathbf{h}^\dagger \mathbf{x} + n \quad \mathbf{y} = \mathbf{h} x + \mathbf{n}$$

that have essentially the same spatial direction. To establish duality, it is not clear whether we need a different relaying matrix for the dual channel instead of the scaled (and conjugate-transposed) versions of the original relaying matrix.

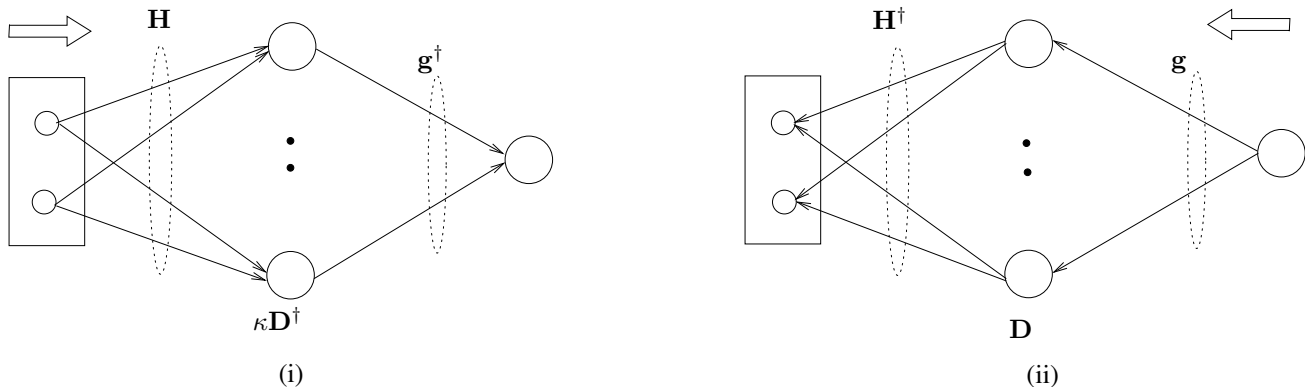


Fig. 2. Two-hop SIMO and MISO relay networks

II. DUALITY IN SIMO AND MISO RELAY NETWORKS

Theorem 1: The capacity of the single user SIMO relay network modeled by (1) is unchanged when the roles of the transmitter and the receiver are reversed while the power constraint at the source and the relays are switched in the dual MISO network.

Proof: To prove it, we need to show that the dual MISO channel is at least as capable as the original channel and vice versa.

A. Part 1 (MISO \geq SIMO)

Let P and P_R denote the source and the relay power constraint respectively in the SIMO relay network. The input signal at the source is a scalar x that satisfies the power constraint P while the relay amplification factor \mathbf{D} satisfies the relay power constraint. For this model (given by (1)), the capacity optimal strategy is to use Gaussian input at the transmitter and employ maximal ratio combining at the receiver. Let \mathbf{b} be any unit norm vector used for receive combining, then the achievable SNR is given by

$$\text{SNR}_1 = \frac{|\mathbf{b}^\dagger \mathbf{H}^\dagger \mathbf{D} \mathbf{g}|^2 P}{1 + \text{Tr}(\mathbf{D}^\dagger \mathbf{H} \mathbf{b} \mathbf{b}^\dagger \mathbf{H}^\dagger \mathbf{D})} \quad (5)$$

and \mathbf{D} should conform to the power constraint which is given by $\text{Tr}(\mathbf{D}(\mathbf{g} \mathbf{g}^\dagger P + \mathbf{I}) \mathbf{D}^\dagger) = P_R$. Absorbing the power constraint into the SNR expression we have

$$\text{SNR}_1 = \frac{|\mathbf{b}^\dagger \mathbf{H}^\dagger \mathbf{D} \mathbf{g}|^2 P P_R}{\text{Tr}(\mathbf{D}(\mathbf{g} \mathbf{g}^\dagger P + \mathbf{I}) \mathbf{D}^\dagger) + P_R \text{Tr}(\mathbf{D}^\dagger \mathbf{H} \mathbf{b} \mathbf{b}^\dagger \mathbf{H}^\dagger \mathbf{D})}.$$

The capacity of this system is

$$C_1 = \max_{\mathbf{b}: \|\mathbf{b}\|=1} \log(1 + \text{SNR}_1). \quad (6)$$

For the dual MISO network, we construct an achievable scheme that achieves the capacity in (6). Let the

transmitted symbol be $\mathbf{x} = \mathbf{b}x'$ where $\mathcal{E}[|x'|^2] = P_R$. Let $\kappa \mathbf{D}^\dagger$ be the relay amplification factor where

$$\kappa^2 = \frac{P}{\text{Tr}(\mathbf{D}^\dagger (\mathbf{H}\mathbf{b}\mathbf{b}^\dagger \mathbf{H}^\dagger P_R + \mathbf{I}) \mathbf{D})}.$$

Here κ ensures that the total relay power is P . The achievable SNR for this scheme is

$$\text{SNR}_2 = \frac{\kappa^2 |\mathbf{g}^\dagger \mathbf{D}^\dagger \mathbf{H}\mathbf{b}|^2 P_R}{1 + \kappa^2 \text{Tr}(\mathbf{D}\mathbf{g}\mathbf{g}^\dagger \mathbf{D}^\dagger)}$$

The achievable SNR after substituting κ is given by

$$\begin{aligned} \text{SNR}_2 &= \frac{|\mathbf{g}^\dagger \mathbf{D}^\dagger \mathbf{H}\mathbf{b}|^2 P P_R}{\text{Tr}(\mathbf{D}^\dagger (\mathbf{H}\mathbf{b}\mathbf{b}^\dagger \mathbf{H}^\dagger P_R + \mathbf{I}) \mathbf{D}) + P \text{Tr}(\mathbf{D}\mathbf{g}\mathbf{g}^\dagger \mathbf{D}^\dagger)} \\ &= \text{SNR}_1. \end{aligned} \tag{7}$$

Thus the MISO relay network can achieve the capacity of the SIMO relay network with a beamforming vector equal to the receive combining vector in the SIMO case.

B. Part 2 (MISO \leq SIMO)

Consider the MISO case with source transmit power P_R and total relay power P . Let \mathbf{D} be the relaying matrix. The capacity of this system is the result of the following optimization problem.

$$\begin{aligned} C &= \max_{\mathbf{Q}_x, \mathbf{D}} I(\mathbf{x}; y) \\ \text{s.t.} \quad &\text{Tr}(\mathbf{Q}_x) = P_R \quad \text{and} \quad \text{Tr}(\mathbf{D}^\dagger (\mathbf{H}\mathbf{Q}_x \mathbf{H}^\dagger + \mathbf{I}) \mathbf{D}) = P. \end{aligned} \tag{8}$$

For a given \mathbf{D} , Gaussian inputs are optimal and the capacity achieving strategy is beamforming. However there is a power constraint on \mathbf{D} that depends on the input covariance matrix \mathbf{Q}_x . Therefore it is not clear whether beamforming is optimal or not. Furthermore it is not clear whether Gaussian inputs are optimal.

Notice that the relay power constraint depends only on the input covariance matrix and not on the input distribution. For any $(\mathbf{D}, \mathbf{Q}_x)$ pair that satisfies the power constraints, the best input distribution is Gaussian as it maximizes the mutual information. Let $(\mathbf{D}^*, \mathbf{Q}_x^*)$ be the optimal pair. Now consider the eigenvalue decomposition of \mathbf{Q}_x^*

$$\mathbf{Q}_x^* = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\dagger \Rightarrow \mathbf{x} = \sum_{i=1}^M \sqrt{\lambda_i} \mathbf{u}_i x_i$$

where $\sum_{i=1}^M \lambda_i = P_R$ and \mathbf{u}_i is the i th column of \mathbf{U} .

We can view the multi-antenna transmitter as M independent sources each with power λ_i as shown in Fig. 3. The resultant MAC achieves the same capacity as in (8). In the MAC, the channel from the i th source to the relays

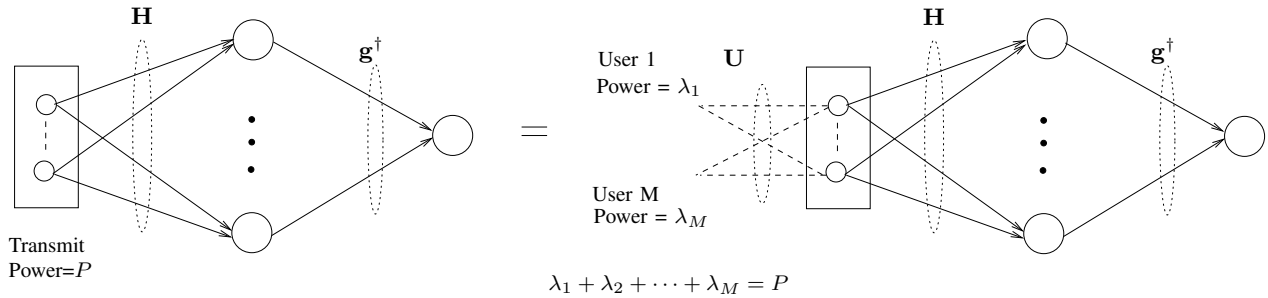


Fig. 3. Transformation of MISO relay channel to a SISO relay MAC.

is $\mathbf{H}\mathbf{u}_j$. This MAC relay channel subscribes to the framework in [4]. This channel is dual to the broadcast channel which in turn is inferior to the SIMO relay channel. Thus the SIMO relay network is at least as capable as the MISO relay network. ■

Discussion: Let us understand why (2) and (4) take different form making it hard to spot a duality connection. For the SIMO channel, let \mathbf{D} be any relay amplification vector that satisfies the relay power constraint. Then the optimal receive combining vector (MRC) is given by

$$\mathbf{b}^* = \frac{(\mathbf{H}^\dagger \mathbf{D} \mathbf{D}^\dagger \mathbf{H} + \mathbf{I})^{-\frac{1}{2}} \mathbf{H}^\dagger \mathbf{D} \mathbf{g}}{\| (\mathbf{H}^\dagger \mathbf{D} \mathbf{D}^\dagger \mathbf{H} + \mathbf{I})^{-\frac{1}{2}} \mathbf{H}^\dagger \mathbf{D} \mathbf{g} \|}.$$

From Theorem 1, this vector is also optimal in the MISO relay channel. It is important to note that the effective channel in the MISO channel is not in the same direction as that of the optimal precoding vector. This is because any vector that is parallel to the effective channel may affect the relay power constraint (more power expenditure at the relays). Therefore the optimum precoding vector need not be along the direction of the effective channel. This explains the difference in the direction of the channel in (4) and (2). This also suggests that the duality is not mainly due to use of the scaled versions of \mathbf{D} but it is rather due to the relay power constraint that couples the two channels.

III. MISO MAC AND DUAL BC

Consider a MIMO broadcast channel with AF relays as shown in Fig. 4. The channel from the transmitter to the relays is denoted by \mathbf{H} while the row vector \mathbf{g}_i^\dagger represents the channel from the set of relays to the single antenna user i . Let the source transmit power be P^B . The relay amplification matrix employed in this broadcast channel is \mathbf{D}^B . The received signal at the k th user is given by

$$y_k = \mathbf{g}_k^\dagger \mathbf{D}^B \mathbf{H} \mathbf{x} + \mathbf{g}_k^\dagger \mathbf{D}^B \mathbf{n}_R + n_k, \quad (9)$$

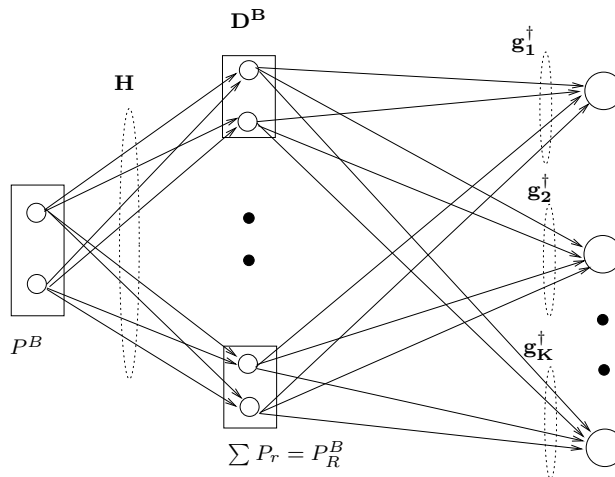


Fig. 4. MIMO broadcast AF relay network

where n_k and the components of \mathbf{n}_R are i.i.d. $\mathcal{CN}(0, 1)$. The total power consumed by the relays can be calculated as

$$P_R^B = \text{Tr} \left(\mathbf{D}^B \left(\mathbf{H} \mathbf{Q}_x \mathbf{H}^\dagger + \mathbf{I} \right) \mathbf{D}^{B\dagger} \right). \quad (10)$$

The total power expended across the network is $P_T^B = P^B + P_R^B$. Now consider the multiple access channel as shown in Fig. 5 obtained from reversing the direction of communication in the broadcast channel. Let \mathbf{D}^M be the relay amplification matrix employed in this channel. The received signal at the base station can be expressed as

$$\mathbf{y} = \mathbf{H}^\dagger \mathbf{D}^M \sum_{k=1}^K \mathbf{g}_k x_k + \mathbf{H}^\dagger \mathbf{D}^M \mathbf{n}_R + \mathbf{n} \quad (11)$$

where x_k is the signal transmitted by user k with power $\mathcal{E}[|x_k|^2] = P_k$. The total power utilized by the relays is

$$P_R^M = \text{Tr} \left(\mathbf{D}^M \left(\sum_{k=1}^K \mathbf{g}_k \mathbf{g}_k^\dagger P_k + \mathbf{I} \right) \mathbf{D}^{M\dagger} \right) \quad (12)$$

and the total power expended by the network is $P_T^M = P_R^M + P^M$ where $P^M = \sum_{k=1}^K P_k$.

Remark: For any fixed relaying matrix \mathbf{D}^M in the MAC that satisfies the power constraint, the relay network reduces to a single hop MAC for which the capacity region can be calculated. Successive decoding of users' messages at the receiver is capacity optimal. By optimizing \mathbf{D}^M for every point on the boundary of the capacity region, the capacity region of the original relay network can be obtained. With regard to the broadcast relay network, the capacity region is obtained through joint optimization of the transmit covariance matrix and the relaying matrix \mathbf{D}^B . Note that Gaussian inputs are optimal. However it is not immediately clear whether dirty paper coding is optimal. This is because the choice of \mathbf{Q}_x not only affects the transmit power constraint but also the relay power constraint.

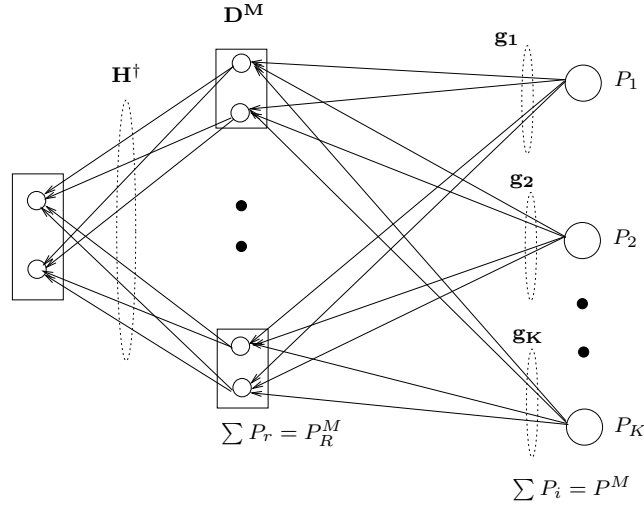


Fig. 5. MIMO multiple access AF relay network

For any $(\mathbf{Q}_x, \mathbf{D}^B)$ pair that satisfies the power constraints, the relay network can be reduced to a single hop MIMO broadcast channel with a covariance matrix constraint \mathbf{Q}_x . It is shown in [7] that dirty paper coding region is indeed the capacity region of MIMO broadcast channel with a covariance matrix constraint. This result allows us to calculate the capacity region of the MIMO BC with AF relays. The union of the dirty paper coding regions of all feasible $(\mathbf{Q}_x, \mathbf{D}^B)$ gives the capacity region of the BC relay network.

The following theorem reveals the duality relationship between the relay MAC and its reciprocal BC for fixed relaying matrices.

Theorem 2: Consider the BC and MAC AF relay networks described by (9) and (11). Let \mathbf{D} be any relaying matrix in the BC. For any $c \in \mathbb{R}_+$, let $c\mathbf{D}^\dagger$ be the relay amplification matrix in the reciprocal MAC. Then the following statements are true.

- 1) MAC-BC duality holds when the total source and relay power are switched in the dual network, i.e. $P_R^M = P^B$, $P_R^B = P^M$ and

$$c^2 = \frac{P_R^M}{\text{Tr} \left(\mathbf{D}^\dagger \left(\sum_{j=1}^K \mathbf{g}_j \mathbf{g}_j^\dagger P_j + \mathbf{I} \right) \mathbf{D} \right)}. \quad (13)$$

- 2) Under a total network power constraint, MAC-BC duality holds when \mathbf{D} and \mathbf{D}^\dagger are the relaying matrices used in the MAC and BC respectively. In other words, MAC-BC duality holds when

$$c = 1, \quad P_R^M + P^M = P_R^B + P^B. \quad (14)$$

- 3) The values for c given by (13) and (14) are the only cases where MAC-BC duality holds for a total network power constraint.

Proof: Refer to the Appendix for the proof. ■

Discussion: Due to the existence of two duals for the same network, for any relay MAC with total network power $P + P_R$ and relaying matrix \mathbf{D} there exists an equivalent relay MAC with total network power $P + P_R$ and relaying matrix $\lambda\mathbf{D}$ where $\lambda \neq 1$. Similarly for any BC, there is an equivalent BC that expends the same total network power and uses a scaled version of the original relaying matrix.

As a result of Theorem 2, we have the following duality relationships for the capacity region of MAC and BC relay networks.

- The capacity region of the MISO relay network with total source power P and total relay power P_R is equal to the capacity region of the dual BC with source power P_R and total relay power P .
- The capacity region of the MISO relay network under total network power P_N is equal to the capacity region of the dual BC with total network power P_N .

A. Extensions

1) *Multi-antenna users:* Theorem 2 also holds when the users have multiple antennas. Using eigenvalue decomposition of the user covariance matrix, the multiple antennas can be reduced to independent single antenna nodes. Consequently the result in Theorem 2 can be utilized to establish the duality relationship.

2) *Linear processing:* Theorem 2 holds even when the user signals are encoded and decoded independently at the transmitter and the receiver respectively, i.e., without successive encoding (dirty paper coding) or successive interference cancelation. Refer to the Appendix for the proofs.

IV. CONCLUSION

In this work, we showed that the duality relationships in AF relay networks, that were known for single antenna nodes, hold even when the nodes have multiple antennas. Similar to [4], the dual channel uses a scaled and conjugate-transposed version of the relaying matrix used in the original channel. In addition the source and relay powers are switched so that each hop has the same transmit power in both the channels. In another version of duality, the same relaying matrix is used in both the channels. Here the total network power expenditure is the same in the MAC and BC. It is shown in this work that such a duality also holds for MIMO networks. Since both the duality results use a scaled form of the original relaying matrix in the dual channel, an interesting question arises: How many possible duality relationships can be found by using different scaled versions of the relaying matrix in the dual channel? We showed that duality under total network power constraint holds only for the above two cases.

APPENDIX

A. Proof of Theorem 2

For any power allocation α_i and precoding vector \mathbf{u}_i for user i and for a given encoding order in BC, consider a MAC that uses a decoding order that is the reverse of the encoding order and employs \mathbf{u}_i^\dagger as the receive combining vector for decoding user i 's signal. The goal is to find a user power allocation strategy in the MAC that would result in achieving the same rate tuple as in the BC. The source in the broadcast channel transmits K independent messages. Let $\mathbf{D}^B = \mathbf{D}$ be the relay amplification matrix. For a fixed \mathbf{D} , the network reduces to a single hop MIMO broadcast channel whose capacity region is equal to the dirty paper coding region. Without loss of generality, let user 1 be encoded first, user 2 next and so on. Let α_i be the power allocated for the i th user such that $\sum \alpha_i = P^B$. Let \mathbf{u}_i denote an arbitrary unit norm precoding vector at the source for user i . Due to successive encoding, the received signal at user i will be

$$y_i = \mathbf{g}_i^\dagger \mathbf{D} \mathbf{H} \mathbf{u}_i x_i + \mathbf{g}_i^\dagger \mathbf{D} \mathbf{H} \sum_{j=i+1}^K \mathbf{u}_j x_j + \mathbf{g}_i^\dagger \mathbf{D} \mathbf{n}_R + n_i$$

The achievable SNR for user i is

$$\text{SNRB}_i = \frac{|\mathbf{g}_i^\dagger \mathbf{D} \mathbf{H} \mathbf{u}_i|^2 \alpha_i}{\sum_{j=i+1}^K |\mathbf{g}_i^\dagger \mathbf{D} \mathbf{H} \mathbf{u}_j|^2 \alpha_j + \text{Tr}(\mathbf{D}^\dagger \mathbf{g}_i \mathbf{g}_i^\dagger \mathbf{D}) + 1} \quad (15)$$

The total power utilized by the relays is

$$P_R^B = \text{Tr} \left(\mathbf{D} \mathbf{H} \left(\sum_{i=1}^K \mathbf{u}_i \mathbf{u}_i^\dagger \alpha_i \right) \mathbf{H}^\dagger \mathbf{D}^\dagger \right).$$

Now consider a MAC that is the flipped version of the BC. Let $\mathbf{D}^M = c \mathbf{D}^\dagger$ be the relay amplification matrix where c is any real scalar. We choose a decoding order that is reverse to the encoding order in BC, i.e., user 1 is decoded last. Let \mathbf{u}_i^\dagger be the receive combining vector for user i . The received signal at the destination after canceling the interference of sources K to $i+1$ is

$$\mathbf{y} = c \sum_{j=1}^i \mathbf{H}^\dagger \mathbf{D}^\dagger \mathbf{g}_j x_j + c \mathbf{H}^\dagger \mathbf{D}^\dagger \mathbf{n}_R + \mathbf{n}.$$

In this channel, the achievable SNR for user i is

$$\text{SNRM}_i = \frac{c^2 |\mathbf{u}_i^\dagger \mathbf{H}^\dagger \mathbf{D}^\dagger \mathbf{g}_i|^2 P_i}{c^2 \sum_{j=1}^{i-1} |\mathbf{u}_i^\dagger \mathbf{H}^\dagger \mathbf{D}^\dagger \mathbf{g}_j|^2 P_j + c^2 \text{Tr}(\mathbf{D} \mathbf{H} \mathbf{u}_i \mathbf{u}_i^\dagger \mathbf{H}^\dagger \mathbf{D}^\dagger) + 1}. \quad (16)$$

The total power utilized by the relays in the MAC is

$$P_R^M = c^2 \text{Tr} \left(\mathbf{D}^\dagger \left(\sum_{j=1}^K \mathbf{g}_j \mathbf{g}_j^\dagger P_j + \mathbf{I} \right) \mathbf{D} \right).$$

Now we are interested in finding the power allocation to users in the MAC such that the rate tuple in the BC is achieved in the MAC. The total network power expended in the MAC reduces to

$$\begin{aligned} P_T^M &= \sum_{j=1}^K P_j \left(1 + c^2 \text{Tr} \left(\mathbf{D}^\dagger \mathbf{g}_j \mathbf{g}_j^\dagger \mathbf{D} \right) \right) + c^2 \text{Tr}(\mathbf{D} \mathbf{D}^\dagger) \\ &= c^2 \sum_{j=1}^K P_j \left(1 + \text{Tr} \left(\mathbf{D}^\dagger \mathbf{g}_j \mathbf{g}_j^\dagger \mathbf{D} \right) \right) + c^2 \text{Tr}(\mathbf{D} \mathbf{D}^\dagger) + (1 - c^2) \sum_{j=1}^K P_j. \end{aligned} \quad (17)$$

Equating (15) and (16), the power allocation for user i that will achieve the same rate in the BC will be

$$P_i = \alpha_i \frac{\sum_{j=1}^{i-1} \left| \mathbf{u}_i^\dagger \mathbf{H}^\dagger \mathbf{D}^\dagger \mathbf{g}_j \right|^2 P_j + \text{Tr} \left(\mathbf{D} \mathbf{H} \mathbf{u}_i \mathbf{u}_i^\dagger \mathbf{H}^\dagger \mathbf{D}^\dagger \right) + \frac{1}{c^2}}{\sum_{j=i+1}^K \left| \mathbf{g}_i^\dagger \mathbf{D} \mathbf{H} \mathbf{u}_j \right|^2 \alpha_j + \text{Tr} \left(\mathbf{D}^\dagger \mathbf{g}_i \mathbf{g}_i^\dagger \mathbf{D} \right) + 1}. \quad (18)$$

Using (18) and employing the following relation

$$\sum_{i=1}^K \sum_{j=1}^{i-1} \left| \mathbf{u}_i^\dagger \mathbf{H}^\dagger \mathbf{D}^\dagger \mathbf{g}_j \right|^2 P_j \alpha_i = \sum_{i=1}^K \sum_{j=i+1}^K \left| \mathbf{g}_i^\dagger \mathbf{D} \mathbf{H} \mathbf{u}_i \right|^2 \alpha_j P_i$$

the total network power in the MAC can be obtained as

$$P_T^M = \sum_{i=1}^K \alpha_i \left(1 + c^2 \text{Tr} \left(\mathbf{D} \mathbf{H} \mathbf{u}_i \mathbf{u}_i^\dagger \mathbf{H}^\dagger \mathbf{D}^\dagger \right) \right) + (1 - c^2) \sum_{j=1}^K P_j + c^2 \text{Tr} \left(\mathbf{D}^\dagger \mathbf{D} \right). \quad (19)$$

For the broadcast channel, the total power spent across the network is

$$P_T^B = \sum_{i=1}^K \alpha_i + \text{Tr} \left(\mathbf{D} \left(\mathbf{I} + \sum_{i=1}^K \mathbf{H} \mathbf{u}_i \mathbf{u}_i^\dagger \mathbf{H}^\dagger \alpha_i \right) \mathbf{D}^\dagger \right). \quad (20)$$

To achieve the same rate tuple in both the BC and MAC, the difference in the total network power when \mathbf{D} and $c\mathbf{D}^\dagger$ are the relaying matrices for the BC and MAC respectively is

$$\begin{aligned} \Delta P &= P_T^M - P_T^B \\ &= (c^2 - 1) \left(\text{Tr} \left(\mathbf{D} \left(\mathbf{I} + \sum_{i=1}^K \mathbf{H} \mathbf{u}_i \mathbf{u}_i^\dagger \mathbf{H}^\dagger \alpha_i \right) \mathbf{D}^\dagger \right) - \sum_{j=1}^K P_j \right) \\ &= (c^2 - 1) (P_R^B - P^M). \end{aligned} \quad (21)$$

Similarly, if we start with a MAC that uses \mathbf{D} and find the power allocation for a dual BC that employs $c\mathbf{D}^\dagger$, it can be shown that $\Delta P = P_T^B - P_T^M = (c^2 - 1) (P_R^M - P^B)$. For MAC-BC duality to hold under a total network

power constraint, we need to find c such that $\Delta P = 0$. From (21), it is clear that the only solutions are

- $c = 1$
- $P_R^B = P^M$ which leads to $P_R^M = P^B$ due to equal network power. The value of c can be found from the relay power constraint in the MAC which is given by

$$c^2 = \frac{P_R^M}{\text{Tr} \left(\mathbf{D}^\dagger \left(\sum_{j=1}^K \mathbf{g}_j \mathbf{g}_j^\dagger P_j + \mathbf{I} \right) \mathbf{D} \right)}.$$

This concludes the proof of Theorem 2. ■

B. Proof for extensions

1) *Linear Processing*: For any power allocation α_i and precoding vector \mathbf{u}_i for user i , the achievable SNR in the BC (treating all undesired signals as interference) is

$$\text{SNRB}_i = \frac{|\mathbf{g}_i^\dagger \mathbf{D} \mathbf{H} \mathbf{u}_i|^2 \alpha_i}{\sum_{j \neq i} |\mathbf{g}_i^\dagger \mathbf{D} \mathbf{H} \mathbf{u}_j|^2 \alpha_j + \text{Tr} \left(\mathbf{D}^\dagger \mathbf{g}_i \mathbf{g}_i^\dagger \mathbf{D} \right) + 1}. \quad (22)$$

The total power utilized across the network can be calculated as

$$P_T^B = P^B + P_R^B = \sum_{j=1}^K \alpha_j + \text{Tr} \left(\mathbf{D} \left(\mathbf{I} + \mathbf{H} \left(\sum_{i=1}^K \mathbf{u}_i \mathbf{u}_i^\dagger \alpha_i \right) \mathbf{H}^\dagger \right) \mathbf{D}^\dagger \right).$$

Now consider a MAC that employs $c\mathbf{D}^\dagger$ as its relaying matrix. With \mathbf{u}_i as the receive combining vector, the SNR for user i with power P_i is given by

$$\text{SNRM}_i = \frac{c^2 |\mathbf{u}_i^\dagger \mathbf{H}^\dagger \mathbf{D}^\dagger \mathbf{g}_i|^2 P_i}{c^2 \sum_{j \neq i} |\mathbf{u}_i^\dagger \mathbf{H}^\dagger \mathbf{D}^\dagger \mathbf{g}_j|^2 P_j + c^2 \text{Tr} \left(\mathbf{D} \mathbf{H} \mathbf{u}_i \mathbf{u}_i^\dagger \mathbf{H}^\dagger \mathbf{D}^\dagger \right) + 1}. \quad (23)$$

The total power utilized in the MAC is

$$P_T^M = P^M + P_R^M = \sum_{j=1}^K P_j + c^2 \text{Tr} \left(\mathbf{D}^\dagger \left(\sum_{j=1}^K \mathbf{g}_j \mathbf{g}_j^\dagger P_j + \mathbf{I} \right) \mathbf{D} \right). \quad (24)$$

The user power allocation in the MAC that achieves the rate tuple of the BC can be obtained as

$$P_i = \alpha_i \frac{\sum_{j \neq i} |\mathbf{u}_i^\dagger \mathbf{H}^\dagger \mathbf{D}^\dagger \mathbf{g}_j|^2 P_j + \text{Tr} \left(\mathbf{D} \mathbf{H} \mathbf{u}_i \mathbf{u}_i^\dagger \mathbf{H}^\dagger \mathbf{D}^\dagger \right) + \frac{1}{c^2}}{\sum_{j \neq i} |\mathbf{g}_i^\dagger \mathbf{D} \mathbf{H} \mathbf{u}_j|^2 \alpha_j + \text{Tr} \left(\mathbf{D}^\dagger \mathbf{g}_i \mathbf{g}_i^\dagger \mathbf{D} \right) + 1}. \quad (25)$$

Substituting (25) in (24) and calculating the difference $P_T^M - P_T^B$ we have

$$P_T^M - P_T^B = (c^2 - 1) (P_R^B - P^M).$$

which is the same as in (21). Therefore all the statements of Theorem 2 hold for linear processing.

2) *Multi-antenna users:* Consider multiple antennas at the users with different number of antennas at the users. First let us consider the multiple access channel. Using eigenvalue decomposition of any covariance matrix \mathbf{Q}_j for user j , the multiple antennas at the user can be decomposed into multiple single-antenna users. Consequently through Theorem 2, the dual broadcast channel is at least as capable as the MAC. Now consider the broadcast channel for which successive encoding of the streams of the users is capacity optimal. Now for any precoding vector \mathbf{u}_{ij} for the i th stream of user j and the receive combining vector \mathbf{v}_{ij}^\dagger at user j , and for a given encoding order, there exists a dual MAC strategy that employs \mathbf{v}_{ij} and \mathbf{u}_{ij}^\dagger as the precoding and receive combining vector respectively for the i th stream of user j . The decoding order at the receiver in the MAC is opposite to the transmitter encoding order in the BC. The proof is largely similar to the proof of Theorem 2 and the only difference is the successive encoding and decoding of the streams belonging to the same user. (The proof of Theorem 2 considers only stream per user.) ■

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