Soft Sensing and Optimal Power Control for Cognitive Radio

Sudhir Srinivasa and Syed Ali Jafar
Electrical Engineering and Computer Science
University of California Irvine, Irvine, CA 92697-2625
Email: *sudhirs@uci.edu, syed@uci.edu*

Abstract

We consider a cognitive radio system where the secondary transmitter varies its transmit power based on all the information available from the spectrum sensor. The operation of the secondary user is governed by its peak transmit power constraint and an average interference constraint at the primary receiver. Without restricting the sensing scheme (total received energy, or correlation etc), we characterize the power adaptation strategies that maximize the secondary user's SNR and capacity. We show that, in general, the capacity optimal power adaptation requires decreasing the secondary transmit power from the peak power to zero in a continuous fashion as the probability of the primary user being present increases. We find that that power control that maximizes the SNR is binary, i.e., if there is any transmission, it takes place only at the peak power level. Numerical results for common spectrum sensing schemes show that the SNR and capacity maximizing schemes can be very different. With an average transmit power constraint at the secondary radio, both the SNR and capacity optimal power control schemes are observed to be non-binary. Further, we find that with secondary channel knowledge at the cognitive transmitter, the optimal SNR with an average transmit power constraint is unbounded.

I. Introduction

The widespread acceptance of diverse wireless technologies in recent years has resulted in a huge demand for more bandwidth. The traditional '*divide and set aside*' approach to spectrum regulation has ensured that the licensed (primary) users cause minimal interference to each other. However, it has also created a crowded spectrum with most frequency bands already assigned to different licensees [1]–[3]. The term 'cognitive radio' can be thought of as encompassing several techniques [4]–[11] that seek to overcome the spectral shortage problem by allowing secondary (unlicensed) wireless devices to communicate without interfering with the primary users. This paper will exclusively focus on the 'interweave' (interference avoidance) approach [7]–[11] to cognitive radio, wherein the secondary radio periodically monitors and intelligently detects occupancy in the different frequency bands and then opportunistically communicates over the spectrum holes with minimal interference to the active primary users.

The main challenge to cognitive communication lies in striking a balance between the conflicting goals of minimizing the interference to the primary users and maximizing the performance of the secondary users. This issue has been investigated from a number of perspectives [9], [12]–[27]. In [12], the tradeoff between secondary user performance and primary user interference is optimized by jointly designing the spectrum sensor, the sensing strategy (how the channels to be monitored for primary users are chosen) and the access strategy (whether or not to access a channel given the sensing sensing outcome). [12] discovers that the spectrum sensing strategy can be decoupled from the spectrum access strategy and the spectrum sensor without any loss in performance. Considering queues at the primary and secondary users, [14] investigates the maximum stable throughput of the cognitive link given the primary user's throughput under both perfect and imperfect sensing. [15] explores the capacities achievable by the secondary user with interference constraints at the primary receiver.

The interplay between protection to the primary users and the performance of the secondary users can be handled by adapting the secondary user's transmit power to ensure a certain quality of service (QoS) to the primary user [16]–[26]. Many papers [17]–[21] consider cognitive communication in an interference channel setting, i.e., one where multiple users (some designated 'primary' and the rest 'secondary') communicate simultaneously in the presence of mutual interference. Since all the users transmit concurrently, there is no sensing involved. The power control optimization is formulated as a general multiuser communication problem with different quality of service (QoS) constraints at the different ('primary' and 'secondary') users. [17] proposes an algorithm for capacity optimum power control in the network under interference constraints at the primary receivers. [18], [19] consider minimum SINR (signal to interference noise ratio) constraints at the primary and secondary users and studies the secondary sum rate optimal power adaptation. [19] also considers the extended problem when the different secondary users have different priorities. In a similar setting with the same kind of constraints [20], [21] investigate joint power and admission control in cognitive radio networks. While [15] considers AWGN channels, [16] considers Rayleigh and Nakagami fading channels with power control at the secondary transmitter. It is shown that fading channels allow higher secondary user capacities for the same average primary user interference constraints. [22], [23] consider power spectrum shaping to manage interference in orthogonal frequency division multiplexing (OFDM) and Direct Sequence Spread Spectrum (DSSS) based cognitive radio networks. Defining the problem in terms of spectrum sharing games, [24]–[26] investigate power control exploiting game theory concepts.

Some recent works consider power control for the interweave flavor of cognitive radio, wherein the transmit power is adapted based on information gathered from sensing. The primary user sensing is implemented as a *binary* hypothesis test, i.e., the spectrum sensor (at the secondary user) outputs a *binary decision* (0 or 1) that indicates whether or not the primary user has been detected. The secondary transmit power depends on the sensed signals only through this binary decision. This kind of power adaptation is based on *hard decisions*. In the absence of secondary channel knowledge at the transmitter, it involves transmitting at two power levels - zero when the primary user is detected and at the peak power when no primary radio is deemed present - thereby simplifying implementation at the cognitive transmitter. With binary detection and binary power control, protecting the primary users reduces to satisfying a missed detection probability constraint while maximizing the secondary performance reduces to satisfying a false alarm probability constraint. This idea is used in [9], [27] to calculate the peak secondary transmit power needed to satisfy constraints on the missed detection and false alarm probabilities.

We emphasize that there is a loss of information in translating the (analog) sensed signals to a binary decision. The motivation behind our work stems from the possibility that the soft information from sensing can be used through sophisticated (continuous) power control to improve the system performance. For example, instead of the simple two level power switching (zero or peak power), one can have a power adaptation scheme where the transmit power increases continuously from 0 to the peak power $\mathsf{P}_{\max}$ as a function of the sensed information. With soft sensing based continuous power adaptation, the notions of missed detection and false alarm probabilities are irrelevant. This generalized setting brings us back to the ultimate goals of protecting the primary users and maximizing the performance (SNR or capacity) of the secondary users. With soft power control, these reduce to the more fundamental constraints of minimizing some definition of interference at the primary receiver (for ex. average interference power at the primary receiver) and maximizing some definition of secondary user performance (for example, SNR or capacity of the secondary user). While binary detection and power control are interesting for their simplicity, we explore soft sensing and continuous power adaptation in order to identify optimal cognitive radio design principles. The differences between hard decision and soft decision based power control are summarized in summarized in Table I.

Cognitive Radio Goals	**Hard Decision (Conventional) Based Power Adaptation**	**Soft Decision Based Power Adaptation**
Protection for primary users	Maximizing the probability of detection	Minimizing some definition of 'interference' caused to primary users
Performance of secondary users	Minimizing the probability of false alarm	Maximizing the SNR (or capacity) of the secondary user

TABLE I: Hard decision vs soft decision based power control

We consider a cognitive radio system where the secondary transmitter varies its transmit power based on the value of the sensing metric. The operation of the secondary radio is governed by an average interference constraint at the primary receiver. Without limiting the kind of sensing scheme at the cognitive transmitter, we derive SNR and capacity optimal secondary transmit power adaptation schemes with a peak secondary transmit power constraint. Other considerations such as an average secondary transmit power constraint and availability of secondary channel knowledge at the cognitive transmitter are also explored. The following is a summary of our main results:

- For a peak power constraint at the secondary transmitter, we characterize the power adaptation strategies that maximize the SNR at the secondary receiver and the capacity of the secondary user. We find that *binary* (hard) power adaptation is optimal for SNR regardless of the type of sensing metric, i.e., the SNR optimal power adaptation policy mandates that transmissions take place only at the peak power. We show that this is true regardless of whether or not the secondary transmitter has knowledge of the secondary channel.
- On the other hand, we find that the general capacity optimal power adaptation for a peak power constraint is *not binary* and involves transmissions at non-boundary power levels between zero and the peak power. With numerical results, we show that even for the commom energy sensing scheme, the SNR optimal and capacity optimal power adaptation schemes are very different.
- With an average power constraint at the secondary transmitter, we find that the SNR optimal power adaptation is not binary. Further, when the secondary transmitter has knowledge of the secondary channel, the resulting SNR is shown to be unbounded.

We begin with assumptions about the system model in Section II.

II. System Model

Consider a communication system with a primary transmitter (PT) and primary receiver (PR) licensed to operate over a certain frequency band as shown in Figure 1. The primary user (primary transmitter - receiver pair, PU) activity follows a block static model with a coherence time $\mathsf{T_c}$ and an ON probability of α, i.e., the primary user switches to an independent ON (or OFF) state (with a probability α of switching to the ON state) every $\mathsf{T_c}$ channel uses. We assume that the primary transmitter uses a Gaussian codebook with an average power $\mathsf{P_t}$ for the primary transmissions.

To allow for higher spectral efficiencies, the channel is also open to be used by a cognitive user (secondary transmitter (ST) - secondary receiver (SR) pair, SU) as Figure 1 shows.

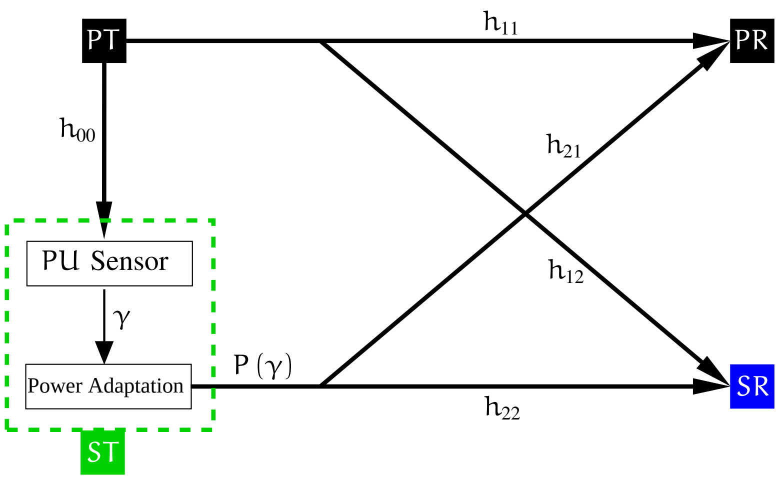


Fig. 1: System Model.

The channel coefficients between each of the primary and secondary nodes are considered to be independent Rayleigh distributed variables with variances that depend on the distances between the nodes, i.e.,

$$h_{ij} = \mathcal{CN}\left(0, \frac{1}{d_{ij}^2}\right), \tag{1}$$

where d_{ij} is the corresponding distance between the associated pair of nodes as shown in Figure 1. We assume no channel state information (CSI) at the transmitting nodes and perfect CSI at the receivers.

Every block, the primary user detector at the secondary transmitter monitors the frequency band for primary transmissions (Figure 1). Based on the signals received, the detector calculates a *sufficient sensing metric* γ as Figure 1 shows. To be as general as possible, we do not restrict the *type* of primary user detector, i.e., γ can represent any sensing metric (for example, γ can denote the total signal power observed, or the correlation between the observed signal and a known signal pattern, etc). We assume that the statistics of γ conditioned on the primary user being $\mathsf{ON/OFF}$ are known *a priori* at the secondary transmitter. We denote the distribution of γ *given that the primary user is* OFF by $f_0(\gamma)$. Similarly, given that the primary user is ON, $\gamma \sim f_1(\gamma)$.

The secondary transmitter adapts its transmit power depending on the value of γ, i.e., if the value of the sensing metric in a certain block is γ, a power $P(\gamma)$ is used to transmit the secondary signals for that block. We assume a peak power constraint at the secondary transmitter, i.e.,

Peak Power Constraint:

$$P(\gamma) \leqslant P_{\max} \quad \forall\, \gamma. \tag{2}$$

The secondary user is allowed to operate within the same frequency band as long as the *average* power received at the primary receiver (when the primary user is ON) does not exceed a certain threshold $\mathbb{I}_0$, i.e.,

Average Interference Constraint:

$$E_\gamma E_{h_{21}}\left[P(\gamma)\,|h_{21}|^2 \middle| \mathsf{PU\ ON}\right] = E_{f_1}\left[P(\gamma)\right]\frac{1}{d_{21}^2} \leqslant \mathbb{I}_0, \tag{3}$$

where $E_{f_1}[\cdot]$ denotes an expectation over the distribution $f_1(\gamma)$.

A. Problem Statement

The performance metrics of interest to us are the average SNR at the secondary receiver and the ergodic capacity of the secondary user. For the system model presented above, we seek answers to the following:

- Does soft sensing help improve the secondary user's SNR (or capacity)?
- What is the optimal power control strategy $P^*(\gamma)$ that maximizes the secondary user's average SNR (or capacity)?

III. Optimal Power Adaptation With A Peak Power Constraint

In this section, we consider the problem of secondary radio SNR and capacity optimization under the average interference (equation (2)) and peak power (equation (3)) constraints.

A. SNR Maximization

The average SNR at the secondary receiver ξ_s can be written as in equation (4). θ is a binary random variable that denotes whether the PU is ON ($\theta = 1$, $\text{Prob}[\theta = 1] = \alpha$) or OFF ($\theta = 0$, $\text{Prob}[\theta = 0] = 1 - \alpha = \bar{\alpha}$). Conditioning on θ, we can write equation (4) as equation (5). Further simplification follows from the fact that $\mathsf{E}_{h_{12}}\left[\frac{1}{1+P_t|h_{12}|^2}\right] = \frac{d_{12}^2}{P_t} e^{\frac{d_{12}^2}{P_t}} \Gamma\left(0, \frac{d_{12}^2}{P_t}\right)$, where $\Gamma(\cdot,\cdot)$ is the incomplete Gamma function. Collecting the constants in $a_0 = \frac{\bar{\alpha}}{d_{22}^2}$, $a_1 = \frac{\alpha}{\bar{\alpha}}$ and $\nu = \mathsf{E}_{h_{12}}\left[\frac{1}{1+P_t|h_{12}|^2}\right] = \left(\frac{d_{12}^2}{P_t}\Gamma\left(0, \frac{d_{12}^2}{P_t}\right) e^{\frac{d_{12}^2}{P_t}}\right)$, the average SNR can be expressed as in equation (7).

$$
\begin{aligned}
\xi_s &= \mathsf{E}_{h_{12},h_{22},\gamma,\theta}\left[\frac{P(\gamma)|h_{22}|^2}{1+\theta P_t|h_{12}|^2}\right] = \mathsf{E}_\theta \mathsf{E}_{h_{12}} \mathsf{E}_{\gamma|\theta}\left[\frac{P(\gamma)\,\mathsf{E}_{h_{22}}\left[|h_{22}|^2\right]}{1+\theta P_t|h_{12}|^2}\right] && (4)\\
&= \mathsf{E}_{h_{22}}\left[|h_{22}|^2\right]\left(\text{Prob}[\theta=0]\,\mathsf{E}_{\gamma|\theta=0}[P(\gamma)] + \text{Prob}[\theta=1]\,\mathsf{E}_{\gamma|\theta=1}\mathsf{E}_{h_{12}}\left[\frac{P(\gamma)}{1+P_t|h_{12}|^2}\right]\right) && (5)\\
&= \mathsf{E}_{h_{22}}\left[|h_{22}|^2\right]\left(\bar{\alpha}\mathsf{E}_{f_0}[P(\gamma)] + \alpha\mathsf{E}_{f_1}\left[P(\gamma)\,\mathsf{E}_{h_{12}}\left[\frac{1}{1+P_t|h_{12}|^2}\right]\right]\right) && (6)\\
&= \frac{\bar{\alpha}}{d_{22}^2}\left[\mathsf{E}_{f_0}[P(\gamma)] + \left(\frac{\alpha}{\bar{\alpha}}\frac{d_{12}^2}{P_t}\Gamma\left(0, \frac{d_{12}^2}{P_t}\right) e^{\frac{d_{12}^2}{P_t}}\right)\mathsf{E}_{f_1}[P(\gamma)]\right]\\
&= a_0\left(\mathsf{E}_{f_0}[P(\gamma)] + a_1\nu\mathsf{E}_{f_1}[P(\gamma)]\right) && (7)
\end{aligned}
$$

The SNR maximization problem can be written as

$$
\max_{\mathsf{E}_{f_1}[P(\gamma)]\leqslant \mathbb{I}_0',\ 0\leqslant P(\gamma)\leqslant P_{\max}} \mathsf{E}_{f_0}[P(\gamma)] + a_1\nu\mathsf{E}_{f_1}[P(\gamma)], \tag{8}
$$

where $\mathbb{I}_0' = \mathbb{I}_0 d_{21}^2$. For the optimization problem of equation (8), we identify the power adaptation strategy $P(\gamma)$ that maximizes the average SNR in the following theorem:

Theorem 1 (SNR Optimal Power Control): For a secondary user operating under the peak transmit power (equation (2)) and average interference (equation (3)) constraints, the power adaptation strategy that maximizes the secondary user's average SNR is binary valued, i.e.,

$$
P^*(\gamma) = \begin{cases} P_{\max} & \text{if } f_0(\gamma) \geqslant (\lambda_1 - a_1\nu) f_1(\gamma) \\ 0 & \text{if } f_0(\gamma) < (\lambda_1 - a_1\nu) f_1(\gamma) \end{cases}, \tag{9}
$$

where γ is the soft information available from sensing and λ_1 is chosen such that equation (9) satisfies the average interference constraint (equation (3)).

Proof: See Section VI-A. ■

Theorem 1 shows that a binary power control scheme is optimal, i.e., the secondary transmitter simply transmits at either of the boundary points (0 or the peak power $P_{\max}$) based on the roots of the equation $f_0(\gamma) - (\lambda_1 - a_1\nu) f_1(\gamma) = 0$. Transmission does not take place at any intermediate power values. This result is somewhat surprising since it establishes that there is no SNR advantage to the soft information available from primary user sensing *regardless of the sensing scheme or the form of the a priori probabilities*. The soft sensing metric output from the sensing block can be replaced with a binary output without any loss in the average SNR while maintaining the interference level at the primary receiver.

B. Capacity Maximization

The ergodic capacity of the secondary user can be written as in equation (10) by conditioning on the value of θ.

$$
C_s = \mathsf{E}_{h_{12},h_{22},\gamma,\theta}\log\left[1+\frac{P(\gamma)|h_{22}|^2}{1+\theta P_t|h_{12}|^2}\right] = \mathsf{E}_{h_{22},\gamma|\theta=0}\log\left[1+P(\gamma)|h_{22}|^2\right]\bar{\alpha} + \mathsf{E}_{h_{12},h_{22},\gamma|\theta=1}\log\left[1+\frac{P(\gamma)|h_{22}|^2}{1+P_t|h_{12}|^2}\right]\alpha \tag{10}
$$

The capacity optimization problem is: $\max_{\mathsf{E}_{f_1}[P(\gamma)]\leqslant \mathbb{I}_0',\ 0\leqslant P(\gamma)\leqslant P_{\max}} C_s$. The power adaptation scheme that maximizes the capacity is characterized in the following theorem:

Theorem 2 (Capacity Optimal Power Control): For a secondary user operating under the peak transmit power (equation (2)) and average interference (equation (3)) constraints, the power adaptation strategy that maximizes the ergodic capacity of the secondary receiver is given by equation (11), where γ is the sensing metric. λ_1 is chosen to satisfy the average interference constraint.

$$P^*(\gamma)=\begin{cases}0 & \text{if } \frac{\bar{\alpha}}{d_{22}^2}f_0(\gamma)+\alpha f_1(\gamma)\frac{1}{d_{22}^2}\mathsf{E}_{h_{12},h_{22}}\left[\frac{1}{1+P_t|h_{12}|^2}\right]-\lambda_1 f_1(\gamma)\leqslant 0\\ P_{\max} & \text{if } \mathsf{E}_{h_{22}}\left[\bar{\alpha}\frac{f_0(\gamma)|h_{22}|^2}{1+P_{\max}|h_{22}|^2}\right]+\mathsf{E}_{h_{12},h_{22}}\left[\alpha\frac{f_1(\gamma)|h_{22}|^2}{1+P_t|h_{12}|^2+P_{\max}|h_{22}|^2}\right]-\lambda_1 f_1(\gamma)\geqslant 0\\ P(\gamma) & \text{elsewhere. } P(\gamma) \text{ is the solution to } \mathsf{E}_{h_{22}}\left[\bar{\alpha}\frac{f_0(\gamma)|h_{22}|^2}{1+P(\gamma)|h_{22}|^2}\right]+\mathsf{E}_{h_{12},h_{22}}\left[\alpha\frac{f_1(\gamma)|h_{22}|^2}{1+P_t|h_{12}|^2+P(\gamma)|h_{22}|^2}\right]-\lambda_1 f_1(\gamma)=0\end{cases} \tag{11}$$

Proof: See Section VI-B. ∎

Notice that unlike the SNR optimal power adaptation policy, the power adaptation that maximizes the capacity is, in general, *not* a binary one, i.e., it can involve transmission at non-boundary power levels between 0 and $P_{\max}$.

IV. Power Based Sensing

In this section, we consider a power based sensing scheme and characterize the SNR maximizing power control strategy. The sensing metric is the total primary signal power in a number of independent signal samples, i.e.,

$$\gamma(N)=\sum_{n=0}^{N-1}|y(n)|^2, \tag{12}$$

where N is the observation time. We assume that N is small compared to the primary user coherence time T_c. We consider the case of fast fading, i.e., where the channel coefficients change every sample. The received signal at the detector $y(n)$ is of the form

$$y(n)=\begin{cases}h_{00}(n)\,x_p(n)+z(n) & \text{PU is ON}\\ z(n) & \text{PU is OFF}\end{cases} \tag{13}$$

where $x_p(n)$ is the primary signal, $h_{00}(n)$ the coefficient of the channel between the primary and secondary transmitters, $z(n)$ the unit variance white Gaussian noise at the primary detector and n is the sample index.

Notice that conditioned on the presence/absence of the primary user, $\gamma(N)$ is a sum of independent and identically distributed random variables. When a primary signal is present, the sensing metric of equation (12) can be approximated by a Gaussian random variable (Central Limit Theorem) for large N with a distribution

$$f_1(\gamma)=\frac{1}{\sigma_1\sqrt{2\pi}}\exp\left(-\frac{(\gamma-\mu_1)^2}{2\sigma_1^2}\right) \tag{14}$$

where μ_1 and σ_1 are given by

$$\mu_1 = N\mathsf{E}\left[|y(0)|^2\right]=N\left(\frac{P_t}{d_{00}^2}+1\right) \tag{15}$$

$$\begin{aligned}\sigma_1^2 &= N\left(\mathsf{E}\left[|y(0)|^4\right]-\left(\mathsf{E}\left[|y(0)|^2\right]\right)^2\right)\\ &= 2N\left(\frac{P_t}{d_{00}^2}+1\right)^2\end{aligned} \tag{16}$$

Similarly when there is no primary signal, the distribution $f_0(\gamma)$ can be written as

$$f_0(\gamma)=\frac{1}{\sigma_0\sqrt{2\pi}}\exp\left(-\frac{(\gamma-\mu_0)^2}{2\sigma_0^2}\right), \tag{17}$$

where $\mu_0=N$ and $\sigma_0^2=2N$.

To obtain the SNR optimal power adaptation policy, we consider the roots of the LHS of equation (37). Substituting equations (14) and (17) into equation (37), we have

$$\frac{(\gamma-\mu_0)^2}{2\sigma_0^2}-\frac{(\gamma-\mu_1)^2}{2\sigma_1^2}+\ln\left(\frac{\sigma_0(\lambda_1-a_1\nu)}{\sigma_1}\right)\leqslant 0. \tag{18}$$

Based on the discussion in Section III-A, the power adaptation can be calculated as follows:

$$P(\gamma)=\begin{cases}P_{\max}^* & \gamma\in[\rho_1(\lambda_1),\ \rho_2(\lambda_1)]\\ 0 & \text{elsewhere}\end{cases}, \tag{19}$$

where $\rho_1(\lambda_1)$ and $\rho_2(\lambda_1)\geqslant\rho_1(\lambda_1)$ are given by equation (20).

$$\rho_1(\lambda_1),\rho_2(\lambda_1)=\frac{\left(\frac{\mu_0}{\sigma_0^2}-\frac{\mu_1}{\sigma_1^2}\right)\pm\sqrt{\left(\frac{\mu_0}{\sigma_0^2}-\frac{\mu_1}{\sigma_1^2}\right)^2-2\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\left(\frac{\mu_0^2}{2\sigma_0^2}-\frac{\mu_1^2}{2\sigma_1^2}+\ln\left(\frac{\sigma_0(\lambda_1-a_1\nu)}{\sigma_1}\right)\right)}}{\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)} \tag{20}$$

The value of λ_1 is calculated based on the interference constraint at the primary receiver (equation (3)), i.e.,

$$\mathbb{I}_0 d_{21}^2 = P_{max} \left(\int_{\rho_1(\lambda_1)}^{\rho_2(\lambda_1)} f_1(\gamma)\, d\gamma \right) = P_{max} \left(Q\left(\frac{\rho_1(\lambda_1) - \mu_1}{\sigma_1} \right) - Q\left(\frac{\rho_2(\lambda_1) - \mu_1}{\sigma_1} \right) \right)$$

The resulting SNR at the secondary receiver can be written as

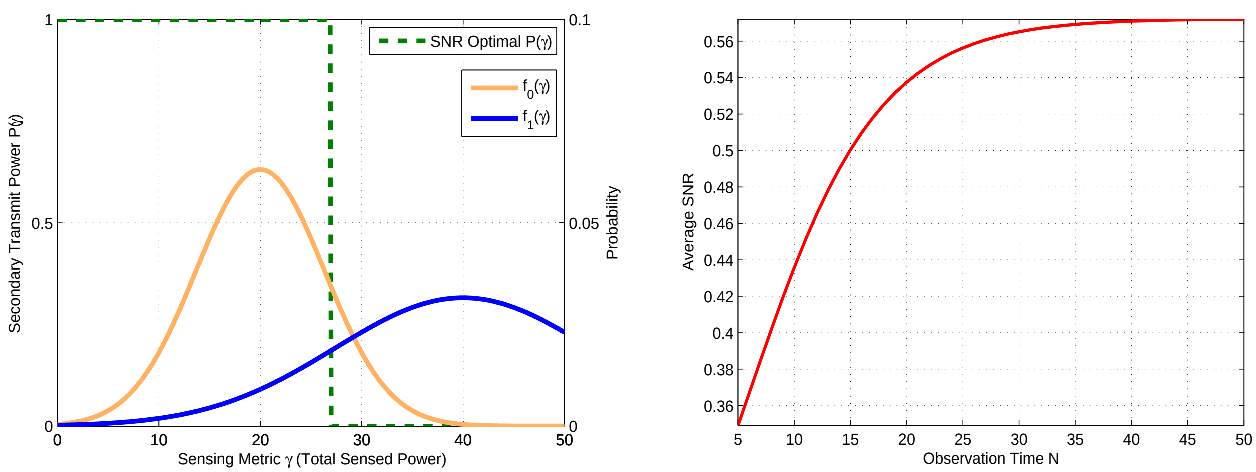
$$\xi_s = a_0 P_{max} \left(a_1 \nu \mathbb{I}_0 d_{21}^2 + \left(Q\left(\frac{\rho_1(\lambda_1) - \mu_0}{\sigma_0} \right) - Q\left(\frac{\rho_2(\lambda_1) - \mu_0}{\sigma_0} \right) \right) \right)$$

It is difficult to analytically determine the capacity optimal power adaptation from equation (11). We instead provide numerical results comparing the optimal power adaptation strategies for SNR and capacity.

A. Numerical Results

We consider a scenario where the primary user is ON for half the time, i.e., the average ON time is $\alpha = 0.5$. The power based sensing scheme at the secondary user calculates the total power in $N = 20$ samples of the primary signal. We assume that the primary transmit power $P_t = 1$ and that the peak secondary transmit power constraint $P_{max} = 1$.

We first examine the case where the primary and secondary nodes are located such that $d_{11} = d_{22} = d_{00} = 1$, $d_{12} = d_{21} = \sqrt{2}$ and the tolerable interference at the primary user is $I_0 = 0.075$ (15% of $\frac{\bar{\alpha} P_{max}}{d_{21}^2}$). The SNR optimal power adaptation is plotted in Figure 2(a). Notice that the optimal adaptation is a step function, with $\gamma_1 = 0$ and $\gamma_2 = 26.93$. The dependence of the SNR on the observation time N is explored in Figure 2(b). It can be seen that as N increases, the secondary transmitter has more accurate knowledge of whether or not the primary user is active. Consequently, the SNR increases while the interference to the primary user is maintained at $\mathbb{I}_0$.



(a) Power allocation for optimal SNR

(b) Variation of SNR with N

Fig. 2: Figure 2(a) shows the SNR optimal power adaptation with 15% (w.r.t $\frac{P_{max}}{d_{22}^2}$) interference tolerance at the primary receiver.

We next consider a case with $d_{00} = 4$, $d_{12} = \sqrt{17}$, $d_{21} = 1$ and $d_{22} = 1$ and $I_0 = 0.05$ (10% of $\frac{P_{max}}{d_{21}^2}$). The SNR optimal and capacity optimal power adaptation policies in Figure 3. The first interesting observation from Figure 3 is that the SNR optimal power adaptation, unlike the previous case, is a step function. Second, the SNR and capacity optimal power adaptation policies are very different. While the SNR optimal power adaptation policy is a binary strategy, i.e. mandates transmission either at zero power or at the peak power P_{max}, the capacity optimal strategy involves transmission at intermediate power values.

We now return to the first scenario ($d_{11} = d_{22} = d_{00} = 1$, $d_{12} = d_{21} = \sqrt{2}$ and $I_0 = 0.075$ (15% of $\frac{\bar{\alpha} P_{max}}{d_{21}^2}$)). Figure 4(a) shows the SNR and capacity optimal power adaptation policies for different values of P_{max} while fixing the interfernce constraint. Notice that the width of the SNR optimal power adaptation policy decreases with P_{max} to maintain the same interference I_0. Therefore the optimal SNR increases with P_{max}. On the other hand, we observe that the secondary user's capacity does not increase beyond $P_{max} = 3$. Figure 4(b) compares the capacities of the SNR optimal and capacity optimal power adaptation policies. It is interesting to note that the capacity of the SNR optimal policy *decreases* with P_{max}. This is due to the fact that the SNR optimal policy dictates transmission only at zero power of the peak power P_{max}.

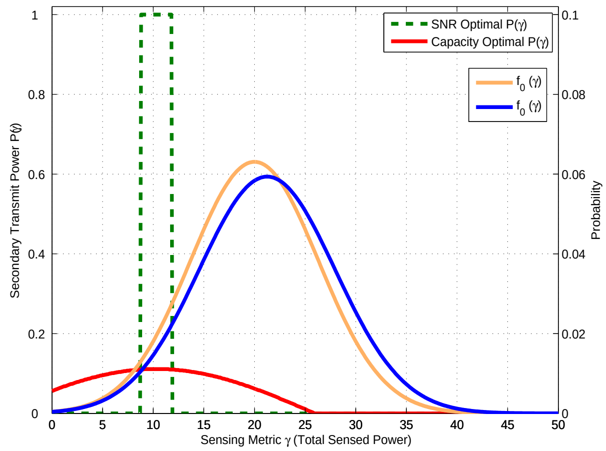


Fig. 3: Figure 3 compares the SNR and capacity optimal power adaptation with 16% (w.r.t $\frac{P_{\max}}{d_{22}^2}$) interference tolerance at the primary receiver.

V. Extensions

In this section we discuss extensions of the SNR optimal power adaptation result of Section III-A to cognitive radio systems with an average power constraint. We also explore SNR optimal power control policies to more complex models with secondary channel knowledge at both the secondary transmitter and receiver with both peak and average power constraints. While we exclusively focus on the SNR optimal policies in this section, capacity optimal policies similar to those in Section III-B can also be derived.

A. Average Power Constraint

We now consider the case when the power constraint at the transmitter follows:

Average Power Constraint:

$$\mathsf{E}\left[P\left(\gamma\right)\right] = \bar{\alpha}\mathsf{E}_{f_0}\left[P\left(\gamma\right)\right] + \alpha\mathsf{E}_{f_1}\left[P\left(\gamma\right)\right] \leqslant P_{avg} \tag{21}$$

The SNR maximization problem can be written as

$$\max_{\mathsf{E}_{f_1}\left[P(\gamma)\right]\leqslant \mathbb{I}_0',\ 0\leqslant P(\gamma),\ \mathsf{E}\left[P(\gamma)\right]\leqslant P_{avg}} \mathsf{E}_{f_0}\left[P\left(\gamma\right)\right] + a_1\nu\mathsf{E}_{f_1}\left[P\left(\gamma\right)\right], \tag{22}$$

where $\mathbb{I}_0' = \mathbb{I}_0 d_{21}^2$. The optimization problem of equation (22) is solved in Theorem 3:

Theorem 3 (SNR **Optimal Power Control with an Average Power Constraint):** For a secondary user operating under an average transmit power (equation (21)) and average interference (equation (3)) constraints, the optimal SNR is given by

$$\xi_s = \max_{x:x\geqslant 0} \min\left\{\mathbb{I}_0' a_0\left(a_1\nu + x\right),\ \frac{P_{avg}a_0}{\bar{\alpha}}\frac{\left(\frac{\alpha}{\bar{\alpha}}\nu + x\right)}{\left(\frac{\alpha}{\bar{\alpha}} + x\right)}\right\}. \tag{23}$$

The SNR optimal power adaptation strategy is given by

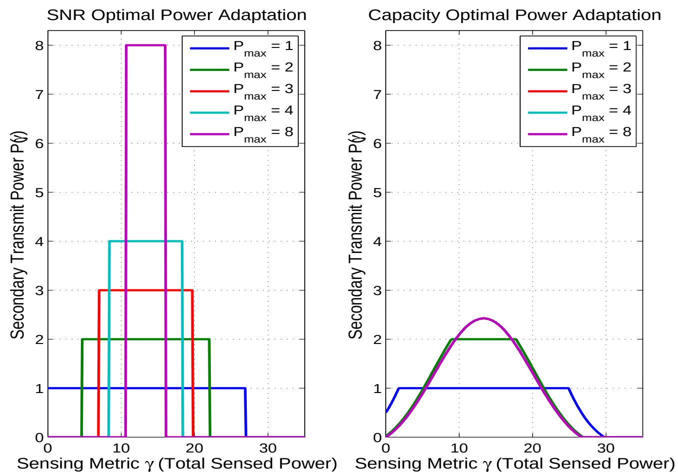
$$P^*\left(\gamma\right) = \sum_{i=0}^{K} P\left(\gamma_i\right)\delta\left(\gamma - \gamma_i\right), \tag{24}$$

where γ_i are the roots of the equation

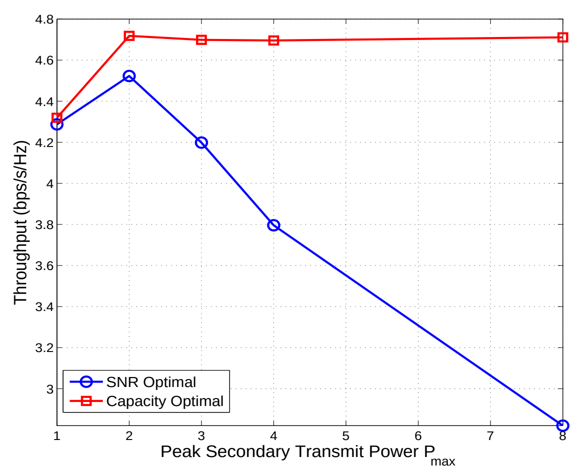
$$\frac{f_0\left(\gamma\right)}{f_1\left(\gamma\right)} = \arg\max_{x:x\geqslant 0} \min\left\{\mathbb{I}_0' a_0\left(a_1\nu + x\right),\ \frac{P_{avg}a_0}{\bar{\alpha}}\frac{\left(\frac{\alpha}{\bar{\alpha}}\nu + x\right)}{\left(\frac{\alpha}{\bar{\alpha}} + x\right)}\right\}. \tag{25}$$

Proof: See Section VI-C. ■

Theorem 3 shows that in a cognitive system with an average power constraint and an average interference constraint at the primary transmitter, the soft information provides an SNR advantage. Therefore, unlike the peak power constraint case, soft information helps the secondary user achieve a higher SNR.



(a) Power allocation for optimal SNR



(b) Variation of SNR with N

Fig. 4: Figure 4(a) shows the SNR and capacity optimal power adaptations with increasing $P_{\max}$. Figure 4(b) shows the capacities of the SNR and capacity optimal power adaptation policies.

B. Secondary Channel Knowledge At The Cognitive Transmitter

We now consider a more involved model where the cognitive transmitter also has secondary channel knowledge, i.e., h_{22} is known to the secondary transmitter. The secondary transmitter therefore adapts its transmit power based on both γ and h_{22}, i.e., by using a power $P(\gamma, h_{22})$ in a time block where the sensing metric is γ and the secondary channel gain is h_{22}. As before, we assume perfect CSI at the receivers. We consider both a peak and an average power constraint and analyze the SNR optimal power control schemes.

1) Peak Power Constraint: We first consider a peak power constraint at the secondary transmitter, i.e.,

$$P(\gamma, h_{22}) \leqslant P_{\max} \qquad \forall\, \gamma, h_{22}. \tag{26}$$

Theorem 4 shows that with a peak power constraint, the optimal power adaptation is binary valued regardless of the availability of channel information at the secondary transmitter:

Theorem 4: For a secondary user with channel knowledge operating under the peak transmit power (equation (2)) and average interference (equation (3)) constraints, the power adaptation strategy that maximizes the secondary user's average SNR is binary valued, i.e.,

$$P^{*}(\gamma, \omega) = \begin{cases} P_{\max} & \text{if } f_0(\gamma) \leqslant f_1(\gamma)\left[\frac{\lambda_1 - a_1 \nu \omega}{\omega}\right] \\ 0 & \text{if } f_0(\gamma) > f_1(\gamma)\left[\frac{\lambda_1 - a_1 \nu \omega}{\omega}\right] \end{cases}, \tag{27}$$

where γ is the soft information available from sensing, $\omega = h_{22}^2$ is the secondary channel gain. λ_1 is chosen such that equation (27) satisfies the average interference constraint (equation (3)).

Proof: See Section VI-D. ■

2) Average Power Constraint: We now consider an average power constraint of the form:

$$\mathsf{E}\left[\mathsf{P}\left(\gamma, h_{22}\right)\right] = \bar{\alpha}\mathsf{E}_{f_0,h_{22}}\left[\mathsf{P}\left(\gamma, h_{22}\right)\right] + \alpha\mathsf{E}_{f_1,h_{22}}\left[\mathsf{P}\left(\gamma, h_{22}\right)\right] \leqslant P_{avg}, \tag{28}$$

and prove that the optimal SNR in this case is infinite in Theorem 5.

Theorem 5: For a secondary user with channel knowledge operating under an average transmit power (equation (21)) and average interference (equation (3)) constraints, the optimal SNR is unbounded.

Proof: See Section VI-E. ■

C. Multiple primary users

In previous sections, we have derived SNR optimal power control schemes assuming a single primary user in the frequency band. We now consider a scenario with *multiple* primary users in the same frequency band and show that the previous results are applicable to this case. For the sake of simplicity, consider two primary users (user 1 and user 2) with different transmit powers $P_t^{[1]}$ and $P_t^{[2]}$; and different average interference constraints $\mathbb{I}_1$ and $\mathbb{I}_2$. The spectrum sensor at the secondary user (user 3) calculates the sensing metric γ based on the received signals. We assume that the statistics of γ conditioned on the activity of the two primary users is known apriori to the secondary user. The probability distributions are denoted by $f_{00}\left(\gamma\right)$, $f_{10}\left(\gamma\right)$, $f_{01}\left(\gamma\right)$ and $f_{11}\left(\gamma\right)$ depending on whether the two primary users are ON or OFF. The interference constraint of equation (3) will be replaced by the following two interference constraints:

$$\begin{aligned} q_1\mathsf{E}_{f_{10}}\left[\mathsf{P}\left(\gamma\right)\right] + q_2\mathsf{E}_{f_{11}}\left[\mathsf{P}\left(\gamma\right)\right] &\leqslant \mathbb{I}_1 &(29)\\ r_1\mathsf{E}_{f_{01}}\left[\mathsf{P}\left(\gamma\right)\right] + r_2\mathsf{E}_{f_{11}}\left[\mathsf{P}\left(\gamma\right)\right] &\leqslant \mathbb{I}_2, &(30)\end{aligned}$$

where q_i and r_i are known constants. It can further be shown that the average SNR expression is of the form $s_1\mathsf{E}_{f_{00}}\left[\mathsf{P}\left(\gamma\right)\right] + s_2\mathsf{E}_{f_{01}}\left[\mathsf{P}\left(\gamma\right)\right] + s_3\mathsf{E}_{f_{10}}\left[\mathsf{P}\left(\gamma\right)\right] + s_4\mathsf{E}_{f_{11}}\left[\mathsf{P}\left(\gamma\right)\right]$, where the s_i are constants that depend on the channel distributions. We observe that the fundamental form of the SNR optimization will remain the same, and therefore results similar to Theorems 1, 3, 4 and 5 can be derived for the two user case. This also extends to the case with more than two primary users.

VI. Proofs

A. Proof of Theorem 1

The Lagrangian $L_S\left[\mathsf{P}\left(\gamma\right), \lambda_1, \{\lambda_2\left(\gamma\right)\}, \{\lambda_3\left(\gamma\right)\}\right]$ for the objective function in the SNR maximization of equation (8) can be written as in equation (31), where λ_1, $\lambda_2\left(\gamma\right)$ and $\lambda_3\left(\gamma\right)$ are the Lagrangian variables.

$$L_S\left[\mathsf{P}\left(\gamma\right), \lambda_1, \{\lambda_2\left(\gamma\right)\}, \{\lambda_3\left(\gamma\right)\}\right] = \left[\mathsf{E}_{f_0}\left[\mathsf{P}\left(\gamma\right)\right] + a_1\nu\mathsf{E}_{f_1}\left[\mathsf{P}\left(\gamma\right)\right] - \lambda_1\left(\mathsf{E}_{f_1}\left[\mathsf{P}\left(\gamma\right)\right] - \mathbb{I}_0'\right) + \int_0^\infty \lambda_2\left(\gamma\right)\mathsf{P}\left(\gamma\right)d\gamma - \int_0^\infty \lambda_3\left(\gamma\right)\left(\mathsf{P}\left(\gamma\right) - P_{\max}\right)d\gamma\right] \tag{31}$$

It is easy to show that the objective function is concave in $\mathsf{P}\left(\gamma\right)$ and that the constraint set (equation (3)) is convex. Taking the derivative of $L_S\left[\mathsf{P}\left(\gamma\right), \lambda_1, \{\lambda_2\left(\gamma\right)\}, \{\lambda_3\left(\gamma\right)\}\right]$ with respect to $\mathsf{P}\left(\gamma\right)$ and setting it to zero, the necessary and sufficient KKT conditions are:

$$\begin{aligned} f_0\left(\gamma\right) + a_1\nu f_1\left(\gamma\right) - \lambda_1 f_1\left(\gamma\right) + \lambda_2\left(\gamma\right) - \lambda_3\left(\gamma\right) &= 0 &(32)\\ \lambda_1\left(\mathsf{E}_{f_1}\left[\mathsf{P}\left(\gamma\right)\right] - \mathbb{I}_0'\right) &= 0 &(33)\\ \lambda_2\left(\gamma\right)\mathsf{P}\left(\gamma\right) &= 0 \quad \forall\,\gamma &(34)\\ \lambda_3\left(\gamma\right)\left(\mathsf{P}\left(\gamma\right) - P_{\max}\right) &= 0 \quad \forall\,\gamma &(35)\end{aligned}$$

For each value of γ, the optimal power adaptation $\mathsf{P}^*\left(\gamma\right)$ can be 0, $P_{\max}$ or take a value in the open interval $\left(0, P_{\max}\right)$. This directly gives rise to the following three cases:

- *Case 1*: Suppose $\mathsf{P}^*\left(\gamma\right) = 0$ for some γ, equation (35) requires that $\lambda_3\left(\gamma\right) = 0$. Substituting this into equation (32), this is possible when (since $\lambda_2\left(\gamma\right) \geqslant 0$),

$$f_0\left(\gamma\right) + \left(a_1\nu - \lambda_1\right)f_1\left(\gamma\right) \leqslant 0. \tag{36}$$

- *Case 2*: Suppose $\mathsf{P}^*\left(\gamma\right) = P_{\max}$ for some γ, equation (34) requires that $\lambda_2\left(\gamma\right) = 0$. Substituting this into equation (32) and noting that $\lambda_3\left(\gamma\right) \geqslant 0$, we have

$$f_0\left(\gamma\right) + \left(a_1\nu - \lambda_1\right)f_1\left(\gamma\right) \geqslant 0. \tag{37}$$

  Therefore $\mathsf{P}^*\left(\gamma\right) = P_{\max}$ for all γ satisfying equation (37).
- *Case 3*: Suppose $0 < \mathsf{P}^*\left(\gamma\right) < P_{\max}$ for some γ. From equations (34) and (35), we have $\lambda_2\left(\gamma\right) = \lambda_3\left(\gamma\right) = 0$. From equation (32), we require

$$f_0(\gamma) = (\lambda_1 - a_1\nu) f_1(\gamma) \tag{38}$$

In general, the solution set to equation (38) (for a given value of λ) will have a measure of zero. The power allocation at the roots of equation (38) will have to be expressed as impulse functions (i.e., of the form $P(\gamma_0)\delta(\gamma-\gamma_0)$), that are excluded by definition because they do not satisfy the peak power constraint.

The optimal power allocation policy can therefore be written as in equation (9), where the value of λ_1 is dictated by the average interference constraint (equation (3)).

B. *Proof of Theorem 2*

The Lagrangian for maximizing the capacity function of equation (10) can be written as in equation (39).

$$L_C\left[P(\gamma), \lambda_1, \{\lambda_2(\gamma)\}, \{\lambda_3(\gamma)\}\right] = \bar{\alpha}\mathsf{E}_{h_{22},\gamma|\theta=0} \log\left[1 + P(\gamma)|h_{22}|^2\right] + \alpha\mathsf{E}_{h_{12},h_{22},\gamma|\theta=1}\log\left[1 + \frac{P(\gamma)|h_{22}|^2}{1+P_t|h_{12}|^2}\right] - \lambda_1\left(\mathsf{E}_{f_1}[P(\gamma)] - \mathbb{I}_0'\right) + \int_0^\infty \lambda_2(\gamma)P(\gamma)\,d\gamma - \int_0^\infty \lambda_3(\gamma)\left(P(\gamma) - P_{\max}\right)d\gamma \tag{39}$$

The derivative of the Lagrangian with respect to $P(\gamma)$ in equation (40) and the complementary slackness conditions of equations (42)-(44) form the KKT conditions for this optimization.

$$\begin{aligned}
\mathsf{E}_{h_{22}}\left[\bar{\alpha}\frac{f_0(\gamma)|h_{22}|^2}{1+P(\gamma)|h_{22}|^2}\right] + \mathsf{E}_{h_{12},h_{22}}\left[\alpha\frac{f_1(\gamma)|h_{22}|^2}{1+P_t|h_{12}|^2+P(\gamma)|h_{22}|^2}\right] - \lambda_1 f_1(\gamma) + \lambda_2(\gamma) - \lambda_3(\gamma) &= 0 && (40)\\
b_0(\gamma) f_0(\gamma) + b_1(\gamma) f_1(\gamma) - \lambda_1 f_1(\gamma) + \lambda_2(\gamma) - \lambda_3(\gamma) &= 0 && (41)\\
\lambda_1\left(\mathsf{E}_{f_1}[P(\gamma)] - \mathbb{I}_0'\right) &= 0 && (42)\\
\lambda_2(\gamma)P(\gamma) &= 0 \quad \forall\gamma && (43)\\
\lambda_3(\gamma)\left(P(\gamma) - P_{\max}\right) &= 0 \quad \forall\gamma && (44)
\end{aligned}$$

As in the case of SNR, we consider the the three cases ($P^*(\gamma) = 0$, $P^*(\gamma) = P_{\max}$ and $0 < P^*(\gamma) < P_{\max}$):

- *Case 1*: Suppose $P^*(\gamma) = 0$ for some γ, equation (44) requires that $\lambda_3(\gamma) = 0$. Substituting this into equation (40), this is possible when (since $\lambda_2(\gamma) \geqslant 0$),

$$\bar{\alpha} f_0(\gamma)\frac{1}{d_{22}^2} + \alpha f_1(\gamma)\frac{1}{d_{22}^2}\mathsf{E}_{h_{12},h_{22}}\left[\frac{1}{1+P_t|h_{12}|^2}\right] - \lambda_1 f_1(\gamma) \leqslant 0. \tag{45}$$

- *Case 2*: Suppose $P^*(\gamma) = P_{\max}$ for some γ. From equation (43) we know that $\lambda_2(\gamma) = 0$. Further since $\lambda_3(\gamma) \geqslant 0$, this is possible if

$$\mathsf{E}_{h_{22}}\left[\bar{\alpha}\frac{f_0(\gamma)|h_{22}|^2}{1+P_{\max}|h_{22}|^2}\right] + \mathsf{E}_{h_{12},h_{22}}\left[\alpha\frac{f_1(\gamma)|h_{22}|^2}{1+P_t|h_{12}|^2+P_{\max}|h_{22}|^2}\right] - \lambda_1 f_1(\gamma) \geqslant 0. \tag{46}$$

  Therefore $P^*(\gamma) = P_{\max}$ for all γ satisfying equation (37).

- *Case 3*: Suppose $0 < P^*(\gamma) < P_{\max}$ for some γ. From equations (43) and (44), we have $\lambda_2(\gamma) = \lambda_3(\gamma) = 0$. From equation (32), we require

$$\mathsf{E}_{h_{22}}\left[\bar{\alpha}\frac{f_0(\gamma)|h_{22}|^2}{1+P(\gamma)|h_{22}|^2}\right] + \mathsf{E}_{h_{12},h_{22}}\left[\alpha\frac{f_1(\gamma)|h_{22}|^2}{1+P_t|h_{12}|^2+P(\gamma)|h_{22}|^2}\right] - \lambda_1 f_1(\gamma) = 0. \tag{47}$$

Equations (45) - (47) directly yield Theorem 2.

C. *Proof of Theorem 3*

The Lagrangian $L_S^{avg}\left[P(\gamma), \lambda_1, \{\lambda_2(\gamma)\}, \lambda_3\right]$ for the objective function in the SNR maximization of equation (22) can be written as:

$$L_S\left[P(\gamma), \lambda_1, \{\lambda_2(\gamma)\}, \lambda_3\right] = \left[\mathsf{E}_{f_0}[P(\gamma)] + a_1\nu\mathsf{E}_{f_1}[P(\gamma)] - \lambda_1\left(\mathsf{E}_{f_1}[P(\gamma)] - \mathbb{I}_0'\right) + \int_0^\infty \lambda_2(\gamma)P(\gamma)\,d\gamma - \lambda_3\left(\bar{\alpha}\mathsf{E}_{f_0}[P(\gamma)] + \alpha\mathsf{E}_{f_1}[P(\gamma)] \leqslant P_{avg}\right)\right] \tag{48}$$

The derivative of $L_S^{avg}\left[P(\gamma),\lambda_1,\{\lambda_2(\gamma)\},\lambda_3\right]$ and the associated KKT constraints can be written as

$$\begin{aligned} f_0(\gamma)+a_1\nu f_1(\gamma)-\lambda_1 f_1(\gamma)+\lambda_2(\gamma)-\lambda_3\left(\bar{\alpha}f_0(\gamma)+\alpha f_1(\gamma)\right) &= 0 && (49)\\ \lambda_1\left(\mathsf{E}_{f_1}\left[P(\gamma)\right]-\mathbb{I}_0^{'}\right) &= 0 && (50)\\ \lambda_2(\gamma)P(\gamma) &= 0 \quad \forall\, \gamma && (51)\\ \lambda_3\left(\bar{\alpha}\mathsf{E}_{f_0}\left[P(\gamma)\right]+\alpha\mathsf{E}_{f_1}\left[P(\gamma)\right]\leqslant P_{avg}\right) &= 0 && (52)\end{aligned}$$

If $P(\gamma)>0$ for some γ, equation (51) implies that $\lambda_2(\gamma)=0$. From equation (49), we have

$$f_0(\gamma)=\frac{\left[\lambda_1+\lambda_3\alpha-a_1\nu\right]}{\left[1-\lambda_3\bar{\alpha}\right]}f_1(\gamma)=g(\lambda_1,\lambda_3)\,f_1(\gamma). \tag{53}$$

Since λ_1 and λ_3 are independent of γ, the roots of equation (53) determine the points at which the secondary transmit power is non-zero. The transmit power can therefore be expressed as

$$P(\gamma)=\sum_{j=0}^{K}P(\gamma_j)\,\delta(\gamma-\gamma_j), \tag{54}$$

where $\{\gamma_j,\ 1\leqslant j\leqslant K\}$ are the roots of equation (53). Notice that impulse functions are *not* excluded since there is no peak power constraint. The average power and interference constraints can be written as

$$\begin{aligned} P_{avg} &\geqslant \sum_{j=0}^{K}P(\gamma_j)\left(\alpha f_1(\gamma_j)+\bar{\alpha}f_0(\gamma_j)\right)=\left[\sum_{j=0}^{K}P(\gamma_j)\,f_1(\gamma_j)\right]\left(\alpha+\bar{\alpha}g(\lambda_1,\lambda_3)\right) && (55)\\ \mathbb{I}_0^{'} &\geqslant \left[\sum_{j=0}^{K}P(\gamma_j)\,f_1(\gamma_j)\right] && (56)\end{aligned}$$

Equations (55) and (56) can be used to obtain an upperbound on the average SNR:

$$\begin{aligned} \xi_s &= \left[\sum_{j=0}^{K}P(\gamma_j)\,f_1(\gamma_j)\right]a_0\left(a_1\nu+g(\lambda_1,\lambda_3)\right) && (57)\\ &\leqslant \min\left\{\mathbb{I}_0^{'}a_0\left(a_1\nu+g(\lambda_1,\lambda_3)\right),\ \frac{P_{avg}a_0\left(a_1\nu+g(\lambda_1,\lambda_3)\right)}{\left(\alpha+\bar{\alpha}g(\lambda_1,\lambda_3)\right)}\right\} && (58)\\ &= \min\left\{\mathbb{I}_0^{'}a_0\left(a_1\nu+g(\lambda_1,\lambda_3)\right),\ \frac{P_{avg}}{d_{22}^2}\frac{\left(\frac{\alpha}{\bar{\alpha}}\nu+g(\lambda_1,\lambda_3)\right)}{\left(\frac{\alpha}{\bar{\alpha}}+g(\lambda_1,\lambda_3)\right)}\right\}. && (59)\end{aligned}$$

Since $\Gamma(0,\ x)\,e^x x\leqslant 1\ \ \forall\ x\geqslant 0$, we have $\nu\leqslant 1$. The first term inside the minimum increases with $g(\lambda_1,\lambda_3)$. On the other hand, the second term decreases with $g(\lambda_1,\lambda_3)$. Consequently,

$$\xi_s=\max_{g(\lambda_1,\lambda_3)}\min\left\{\mathbb{I}_0^{'}a_0\left(a_1\nu+g(\lambda_1,\lambda_3)\right),\ \frac{P_{avg}a_0}{\bar{\alpha}}\frac{\left(\frac{\alpha}{\bar{\alpha}}\nu+g(\lambda_1,\lambda_3)\right)}{\left(\frac{\alpha}{\bar{\alpha}}+g(\lambda_1,\lambda_3)\right)}\right\}, \tag{60}$$

and the SNR is bounded.

D. Proof of Theorem 4

Let $\omega=|h_{22}|^2$. The Lagrangian $L_S\left[P(\gamma,\omega),\lambda_1,\{\lambda_2(\gamma,\omega)\},\{\lambda_3(\gamma,\omega)\}\right]$ is

$$\begin{aligned} L_S\left[P(\gamma,\omega),\lambda_1,\{\lambda_2(\gamma,\omega)\},\{\lambda_3(\gamma,\omega)\}\right] &= \Big[\mathsf{E}_{f_0,\omega}\left[\omega P(\gamma,\omega)\right]+a_1\nu\mathsf{E}_{f_1,\omega}\left[\omega P(\gamma,\omega)\right]-\lambda_1\left(\mathsf{E}_{f_1,\omega}\left[P(\gamma,\omega)\right]-\mathbb{I}_0^{'}\right)+\\ &\quad \mathsf{E}_{\omega}\left[\int_0^{\infty}\lambda_2(\gamma,\omega)P(\gamma,\omega)\,d\gamma\right]-\mathsf{E}_{\omega}\left[\int_0^{\infty}\lambda_3(\gamma,\omega)\left(P(\gamma,\omega)-P_{\max}\right)d\gamma\right]\Big] && (61)\end{aligned}$$

Derivative of Lagrangian yields

$$\begin{aligned} f(\omega)\left[\omega f_0(\gamma)+a_1\nu\omega f_1(\gamma)-\lambda_1 f_1(\gamma)+\lambda_2(\gamma,\omega)-\lambda_3(\gamma,\omega)\right] &= 0 && (62)\\ \lambda_1\left(\mathsf{E}_{f_1,\omega}\left[P(\gamma,\omega)\right]-\mathbb{I}_0^{'}\right) &= 0 && (63)\\ \lambda_2(\gamma,\omega)P(\gamma,\omega) &= 0 \quad \forall\,\gamma,\omega && (64)\\ \lambda_3(\gamma,\omega)\left(P(\gamma,\omega)-P_{\max}\right) &= 0 \quad \forall\,\gamma,\omega && (65)\end{aligned}$$

For each value of γ, the optimal power adaptation $P^*(\gamma)$ can be 0, P_{max} or take a value in the open interval $(0, P_{max})$. This directly gives rise to the following three cases:

- *Case 1*: Suppose $P^*(\gamma,\omega) = 0$ for some γ and ω, equation (65) requires that $\lambda_3(\gamma,\omega) = 0$. Substituting this into equation (62), this is possible when

$$f_0(\gamma) \geqslant f_1(\gamma)\left[\frac{\lambda_1 - a_1\nu\omega}{\omega}\right]. \tag{66}$$

- *Case 2*: Suppose $P^*(\gamma,\omega) = P_{max}$ for some γ and ω, equation (64) requires that $\lambda_2(\gamma,\omega) = 0$. Substituting this into equation (62) and noting that $\lambda_3(\gamma,\omega) \geqslant 0$, we have

$$f_0(\gamma) \leqslant f_1(\gamma)\left[\frac{\lambda_1 - a_1\nu\omega}{\omega}\right]. \tag{67}$$

  Therefore $P^*(\gamma) = P_{max}$ for all γ satisfying equation (67).
- *Case 3*: Suppose $0 < P^*(\gamma,\omega) < P_{max}$ for some γ and ω. From equations (64) and (65), we have $\lambda_2(\gamma,\omega) = \lambda_3(\gamma,\omega) = 0$. From equation (62), we require

$$f_0(\gamma) = f_1(\gamma)\left[\frac{\lambda_1 - a_1\nu\omega}{\omega}\right]. \tag{68}$$

  Since this involves impulse functions, this case will have to be excluded owing to the peak power constraint.

The optimal power allocation policy is therefore binary valued.

E. *Proof of Theorem 5*

Let $\omega = |h_{22}|^2$. Consider a power allocation policy of the form

$$P(\gamma,\omega) = P(\gamma_0,\omega_0)\,\delta(\gamma-\gamma_0)\,\delta(\omega-\omega_0). \tag{69}$$

The average power and interference constraints can be expressed as

$$\mathbb{I}_0' \geqslant f_1(\gamma_0)\,f(\omega_0)\,P(\gamma_0,\omega_0) \tag{70}$$

$$P_{avg} \geqslant \alpha f_1(\gamma_0)\,f(\omega_0)\,P(\gamma_0,\omega_0) + \bar{\alpha} f_0(\gamma_0)\,f(\omega_0)\,P(\gamma_0,\omega_0) = P(\gamma_0,\omega_0)\,f(\omega_0)\left[\alpha f_1(\gamma_0) + \bar{\alpha} f_0(\gamma_0)\right] \tag{71}$$

The constraints of both equations (70) and (71) will be satisfied if we choose

$$P(\gamma_0,\omega_0) = \frac{1}{f(\omega_0)}\min\left\{\frac{\mathbb{I}_0'}{f_1(\gamma_0)}, \frac{P_{avg}}{\alpha f_1(\gamma_0) + \bar{\alpha} f_0(\gamma_0)}\right\}. \tag{72}$$

Further, from equation (72), the average SNR can be expressed as

$$\begin{aligned}
\xi_s &= a_0\omega_0\left[\alpha\nu f_1(\gamma_0)\,f(\omega_0)\,P(\gamma_0,\omega_0) + \bar{\alpha} f_0(\gamma_0)\,f(\omega_0)\,P(\gamma_0,\omega_0)\right] \\
&= a_0\omega_0 f(\omega_0)\,P(\gamma_0,\omega_0)\left[\alpha\nu f_1(\gamma_0) + \bar{\alpha} f_0(\gamma_0)\right] \\
&= a_0\omega_0\min\left\{\frac{\mathbb{I}_0'\left[\alpha f_1(\gamma_0) + \bar{\alpha} f_0(\gamma_0)\right]}{f_1(\gamma_0)},\ \frac{P_{avg}\left[\alpha\nu f_1(\gamma_0) + \bar{\alpha} f_0(\gamma_0)\right]}{\left(\alpha f_1(\gamma_0) + \bar{\alpha} f_0(\gamma_0)\right)}\right\} \\
&= a_0\omega_0\min\left\{\mathbb{I}_0'\alpha\left[1 + \frac{\bar{\alpha}}{\alpha}\frac{f_0(\gamma_0)}{f_1(\gamma_0)}\right],\ \frac{P_{avg}\left[\nu + \frac{\bar{\alpha}}{\alpha}\frac{f_0(\gamma_0)}{f_1(\gamma_0)}\right]}{\left[1 + \frac{\bar{\alpha}}{\alpha}\frac{f_0(\gamma_0)}{f_1(\gamma_0)}\right]}\right\}
\end{aligned}$$

It is easy to see that as $w_0 \to \infty$, the average SNR becomes unbounded.

VII. Discussion and Conclusion

We consider a cognitive radio system where the secondary transmitter adapts its transmit power depending on the soft information obtained from the spectrum sensor. We have a peak power constraint at the secondary transmitter and an average interference constraint at the primary receiver. We characterize the SNR and capacity optimal power adaptation strategies for arbitrary sensing schemes. Binary power control is SNR optimal, which shows that one can simultaneously obtain the dual benefits of optimum SNR performance and low power control complexity. On the other hand, the capacity optimal power adaptation scheme is, in general, not binary and dictates transmission at power levels other than 0 and P_{max}.

We point out here that past work has considered different kinds of interference constraints to protect the primary users [13], [19], [28]. For the average interference constraint considered in equation (3), a natural question that arises is: From the *primary* user's perspective, is it better to have binary power control, based on sensing; or have the secondary transmitter employ continuous power adaptation such that the primary user sees the same average interference? Suppose we are interested in the primary user's rate, notice that the logarithmic form of the capacity expression implies that variable interference power is preferred to constant interference power [29]. While continuous power adaptation ensures that the secondary user's capacity is maximized, binary power adaptation at the secondary transmitter therefore is primary user friendly because it ensures that the interference seen at the primary receiver is varying.

For a power based spectrum sensing scheme, we find that the SNR optimal power control scheme directs transmission at peak power if the sensing metric lies within a certain range, regardless of the availability of secondary channel knowledge at the secondary transmitter. With an average secondary transmit power constraint, we show that the optimal SNR is unbounded with channel state information at the secondary transmitter.

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