Interference Alignment and Spatial Degrees of Freedom for the $K$ User Interference Channel

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Abstract—We show that the sum capacity of the $K$ user frequency selective (or time-varying) interference channel is $C(\text{SNR}) = (K/2) \log(\text{SNR}) + o(\log(\text{SNR}))$ meaning that the channel has a total of $K/2$ degrees of freedom per orthogonal time and frequency dimension. Linear schemes of interference alignment and zero forcing suffice to achieve all the degrees of freedom and multi-user detection is not required.

I. INTRODUCTION

The capacity of ad-hoc wireless networks is the much sought after “holy-grail” of network information theory [1]. While capacity characterizations have been found for centralized networks (Gaussian MIMO multiple access and broadcast networks), similar capacity characterizations for most distributed communication scenarios (e.g. interference networks) remain long standing open problems. In the absence of precise capacity characterizations, researchers have pursued asymptotic and/or approximate capacity characterizations. A promising approach in this direction is the degrees of freedom characterization of wireless networks. The degrees of freedom (also referred to as multiplexing gain [2] or capacity pre-log) of a network approximates the capacity of a network as

$$C(\text{SNR}) = d \log(\text{SNR}) + o(\log(\text{SNR}))$$

where $d$ is the number of degrees of freedom of the network and $C(\text{SNR})$ represents the capacity of a network as a function of the signal to noise ratio (SNR). At high SNR, the $o(\log(\text{SNR}))$ term becomes negligible in comparison to $\log(\text{SNR})$ and the accuracy of the approximation approaches 100%. By studying wireless networks at high SNR, the degrees of freedom approach de-emphasizes noise and explicitly addresses the effects of interference in a wireless network.

Researchers have characterized the degrees of freedom of various distributed wireless networks. It has been shown in [3]–[5]) that the 2 user interference channel has only 1 degree of freedom even if we allow for noisy co-operation between the 2 transmitters and receivers. The degrees of freedom of the 2 user MIMO X channel without message sharing has been studied in [6]–[9]. [5], [9], [10] study the degrees of freedom of cognitive interference and X networks i.e. networks with partial message sharing between nodes. In this paper, we study the degrees of freedom of the fully connected $K$ user interference channel with global channel knowledge at all nodes. Unlike [4], [5], [10], we do not allow any message sharing or co-operation between transmitters or receivers. While the best known outerbound (shown in [4]) states that the $K$ user interference channel cannot have more than $K/2$ degrees of freedom, it was conjectured$^1$ that the $K$ user interference channel has 1 degree of freedom. While [4] studies interference channels with constant channel coefficients, in this work we show that if the channel coefficients are frequency-selective (or time-varying) then the outerbound of $K/2$ degrees of freedom is tight. Equivalently, we show that the sum capacity of the $K$ user interference channel with frequency-selective (or time-varying) channel coefficients, is

$$C(\text{SNR}) = K/2 \log(\text{SNR}) + o(\log(\text{SNR}))$$ (1)

The achievable scheme is based on the idea of interference alignment.

A. Interference Alignment

Interference alignment is a powerful scheme that has emerged out of recent work in the context of the MIMO $X$ channel [6]–[9]. The key to interference alignment is the realization that the alignment of signal spaces (in time, frequency, space and codes) is relative to the observer (receiver). Since every receiver sees a different picture, signals may be constructed to cast overlapping shadows at the receivers where they constitute interference while they remain distinguishable at the receivers where they are desired.

We illustrate interference alignment through the following toy example (See also [11]). To create the simplest example possible, we deviate a little from our system model to allow propagation delays in the toy example.

$^1$We refer to this conjecture as the Host-Madsen-Nosratinia conjecture in this paper.
Motivating example - Can everyone speak half the time with no interference?: Consider the $K$ user interference channel where there is a propagation delay from each transmitter to each receiver. Let $T_{ij}$ represent the signal propagation delay from transmitter $i$ to receiver $j$. Suppose the locations of the transmitters and receivers can be configured such that the delay $T_{ij}$ from each transmitter to its intended receiver is an even multiple of a basic symbol duration $T_s$, while the signal propagation delays $T_{ij}, (i \neq j)$ from each transmitter to all unintended receivers are odd multiples of the symbol duration. The communication strategy is the following. All transmissions occur simultaneously at even symbol durations. Note that with this policy, each receiver sees its own transmitter’s signal interference-free over even time periods, while all interfering signals align with each other at the odd time periods. Thus each speaker is able to talk half the time and be heard interference-free by its desired audience. In terms of the interference channel, each user is able to communicate at a rate equal to half of his single-user capacity in the absence of all interference.

We now return to the classic information theoretic communication model without propagation delays. While interference is exactly aligned in the toy example above, perfect alignment is not feasible for the $K$ user interference channel with random coefficients (and no propagation delays). We circumvent this problem by adopting a novel approach of partial interference alignment where we do not seek to exactly align all interference terms but allow partial overlaps between interfering signals at various receivers. It turns out that while this imperfect alignment cannot exactly achieve $K/2$ degrees of freedom, it is capable of achieving arbitrarily close to $K/2$ degrees of freedom. With the degrees of freedom defined as a limit superior we say that the network has $K/2$ degrees of freedom.

B. Fallacy of the Cake-cutting Interpretation of Orthogonal Medium Access

The capacity characterization in (1) reveals the fallacy of the cake-cutting view of orthogonal medium access. Conventional design of wireless systems protects users from interference by an orthogonal allocation of resources (time/frequency/codes) among all interfering users so that each user obtains a fraction of the resources, and the sum of these fractions across all users is equal to 1. Intuitively, this reflects the cake-cutting view of orthogonal medium access since it achieves only a total of 1 degree of freedom. Since the capacity of each user in the absence of interference is $\log(\text{SNR}) + o(\log(\text{SNR}))$, the capacity characterization in (1) implies that at high SNR, every user in the $K$-user interference channel can achieve half of his capacity in absence of interference. Thus, each of the $K$ users in the $K$-user interference channel can get half the cake (where a full cake represents the degrees of freedom available to a single user in the absence of all interference).

II. System Model

Consider the $K$ user interference channel, comprised of $K$ transmitters and $K$ receivers. We assume coding may occur over multiple orthogonal frequency and time dimensions and the rates as well as the degrees of freedom are normalized by the number of orthogonal time and frequency dimensions. Each node is equipped with only one antenna. The channel output at the $k^{th}$ receiver over the $f^{th}$ frequency slot and the $t^{th}$ time slot is described as follows:

$$y[k](f,t) = H[kj](f)X[j](f,t) + \cdots + H[kK](f)X[K](f,t) + Z[k](f,t)$$

where, $k \in \{1,2,\cdots,K\}$ is the user index, $f \in \mathbb{N}$ is the frequency slot index, $t \in \mathbb{N}$ is the time slot index, $Y[k](f,t)$ is the output signal of the $k^{th}$ receiver, $X[k](f,t)$ is the input signal of the $k^{th}$ transmitter, $H[kj](f)$ is the channel fade coefficient from transmitter $j$ to receiver $k$ over the $f^{th}$ frequency slot and $Z[k](f,t)$ is the additive white Gaussian noise (AWGN) term at the $k^{th}$ receiver. The channel coefficients vary across frequency slots but are assumed constant in time (the results extend easily to the time-varying case as well). We assume all noise terms are i.i.d. (independent identically distributed) zero mean complex Gaussian with unit variance. We assume global channel knowledge, i.e. all channel coefficients $H[kj](f)$ are known to all transmitters and receivers. To avoid degenerate channel conditions (e.g. all channel coefficients are equal or channel coefficients are equal to either zero or infinity) we assume that the channel coefficient values are drawn i.i.d. from a continuous distribution and the absolute value of all the channel coefficients is bounded between a non-zero minimum value and a finite maximum value. Since the channel values are assumed constant in time, the time index $t$ is sometimes suppressed for compact notation.

We assume that transmitters 1,2,\cdots,$K$ have independent messages $W_1,W_2,\cdots,W_K$ intended for receivers 1,2,\cdots,$K$, respectively. The total power across all transmitters is assumed to be equal to $\rho$ per orthogonal time and frequency dimension. We indicate the size of the message set by $|W_i(\rho)|$. For codewords spanning $f_0 \times t_0$ channel uses (i.e. using $f_0$ frequency slots and $t_0$ time slots), the rates $R_i(\rho) = \log |W_i(\rho)|$ are achievable if the probability of error for all messages can be simultaneously made arbitrarily small by choosing an appropriately large $f_0t_0$.

The capacity region $\mathcal{C}(\rho)$ of the $K$ user interference channel is the set of all achievable rate tuples $\mathbf{R}(\rho) = (R_1(\rho), R_2(\rho), \cdots, R_K(\rho))$. 
A. Degrees of Freedom

Similar to the degrees of freedom region definition for the MIMO X channel in [9] we define the degrees of freedom region $D$ for the $K$ user interference channel as follows:

\[
D = \left\{ (d_1, \ldots, d_K) \in \mathbb{R}_+^K : \forall (w_1, \ldots, w_K) \in \mathbb{R}_+^K \right. \\
\left. w_1 d_1 + w_2 d_2 + \cdots + w_K d_K \leq \limsup_{\rho \to \infty} \sup_{\mathbf{R}(\rho) \in C(\rho)} \left[ \sum_{\rho(\mathbf{R}) \in C(\rho)} \frac{|w_1 R_1(\rho) + \cdots + w_K R_K(\rho)|}{\log(\rho)} \right] \right\}
\]

III. DEGREES OF FREEDOM FOR THE K USER INTERFERENCE CHANNEL

The following theorem presents our main result.

**Theorem 1:** The number of degrees of freedom for the $K$ user interference channel with single antennas at all nodes is $K/2$.

\[
\max_{d \in D} d_1 + d_2 + \cdots + d_K = K/2
\]

Equivalently, the sum capacity $C_S(\rho)$ of the $K$ user interference channel can be characterized as

\[
C_S(\rho) = K/2 \log(\rho) + o(\log(\rho))
\]

The converse argument for the theorem follows directly from the bound for the $K$ user interference channel presented in [4]. The achievability proof is presented next. For simplicity of exposition, we present here the constructive proof for $K = 3$. The proof for general $K \geq 3$ is provided in the full paper [12].

A. Achievability Proof for Theorem 1 with $K = 3$

We show that \((d_1, d_2, d_3) = (\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})\) lies in the degrees of freedom region $\forall n \in \mathbb{N}$. Since the degrees of freedom region is closed, this automatically implies that

\[
\max_{(d_1, d_2, d_3) \in D} d_1 + d_2 + d_3 \geq \sup_{n} \frac{3n + 1}{2n + 1} = \frac{3}{2}
\]

This result, in conjunction with the converse argument proves the theorem.

To show that \((\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})\) lies in $D$, we construct a $2n + 1$ extension of the channel i.e. we collectively denote the $2n + 1$ symbols transmitted over the $2n + 1$ frequency slots at each time instant as a supersymbol. Over the extended channel, the signal vector at the $k^{th}$ user’s receiver can be expressed as

\[
\mathbf{Y}^{[k]} = \mathbf{H}^{[k]} \mathbf{X}^{[1]} + \mathbf{H}^{[k]} \mathbf{X}^{[2]} + \mathbf{H}^{[k]} \mathbf{X}^{[3]} + \mathbf{Z}^{[k]}
\]

where for $k \in \{1, 2, 3\}$, $\mathbf{X}^{[k]}$ is a $(2n + 1)$ \times 1 column vector representing the $2n + 1$ symbol extension of the transmitted symbol $X^{[k]}$, i.e.

\[
\mathbf{X}^{[k]}(t) \triangleq \begin{bmatrix} X^{[k]}(1, t) \\ X^{[k]}(2, t) \\ \vdots \\ X^{[k]}(2n+1, t) \end{bmatrix}
\]

Similarly $\mathbf{Y}^{[k]}$ and $\mathbf{Z}^{[k]}$ are $2n + 1$ symbol extensions of the $Y^{[k]}$ and $Z^{[k]}$ respectively. $\mathbf{H}^{[k]}$ is a diagonal $(2n + 1) \times (2n + 1)$ matrix representing the $2n + 1$ symbol extension of the channel i.e.

\[
\mathbf{H}^{[k]} \triangleq \begin{bmatrix} H^{[k]}(1) & 0 & \cdots & 0 \\ 0 & H^{[k]}(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H^{[k]}(2n+1) \end{bmatrix}
\]

Since the channel gains for each frequency slot are chosen independently from a continuous distribution, all diagonal entries of $\mathbf{H}^{[k]}$ are distinct with probability 1.

We show that $(d_1, d_2, d_3) = (\frac{n+1}{2n+1}, \frac{n}{2n+1}, \frac{n}{2n+1})$ lies in the degrees of freedom region of the original channel.

In the extended channel, message $W_1$ is encoded at transmitter 1 into $n + 1$ independent streams $x^{[1]}_m(t), m = 1, 2, \ldots, (n + 1)$ sent along vectors $v^{[1]}_m$ so that $\mathbf{X}^{[1]}(t)$ is

\[
\mathbf{X}^{[1]}(t) = \sum_{m=1}^{n+1} x^{[1]}_m(t) v^{[1]}_m = \mathbf{V}^{[1]} \mathbf{X}^{[1]}(t)
\]

where $\mathbf{X}^{[1]}(t)$ is a $(n+1) \times 1$ column vector and $\mathbf{V}^{[1]}$ is a $(2n + 1) \times (n + 1)$ dimensional matrix. Similarly $W_2$ and $W_3$ are each encoded into $n$ independent streams by transmitters 2 and 3 as $\mathbf{X}^{[2]}(t)$ and $\mathbf{X}^{[3]}(t)$ respectively.

\[
\mathbf{X}^{[j]}(t) = \sum_{m=1}^{n} x^{[j]}_m(t) v^{[j]}_m = \mathbf{V}^{[j]} \mathbf{X}^{[j]}(t), j = 1, 2
\]

The received signal at the $i^{th}$ receiver is

\[
\mathbf{Y}^{[i]} = \mathbf{H}^{[i]} \mathbf{X}^{[1]} + \mathbf{H}^{[i]} \mathbf{X}^{[2]} + \mathbf{H}^{[i]} \mathbf{X}^{[3]} + \mathbf{Z}^{[i]}
\]

where dependence on time $t$ is suppressed for compact notation.

In this achievable scheme, receiver $i$ eliminates interference by zero-forcing all $\mathbf{Y}^{[j]}, j \neq i$ to decode $W_j$. At receiver 1, $n + 1$ desired streams are decoded after zero-forcing the interference to achieve $n + 1$ degrees of freedom. To obtain $n + 1$ interference free dimensions from a $2n+1$ dimensional received signal vector $\mathbf{Y}^{[1]}(t)$, the dimension of the interference should be not more than $n$. This can be ensured by perfectly aligning the interference from transmitters 2 and 3 as follows.

\[
\mathbf{H}^{[1]} \mathbf{V}^{[2]} = \mathbf{H}^{[1]} \mathbf{V}^{[3]}
\]
At the same time, receiver 2 zero-forces the interference from $\bar{X}^{[1]}$ and $\bar{X}^{[3]}$. To extract $n$ interference-free dimensions from a $2n+1$ dimensional vector, the dimension of the interference has to be not more than $n+1$. i.e.

$$\text{rank} \left( \begin{bmatrix} \bar{H}^{[21]} \bar{V}^{[1]} & \bar{H}^{[23]} \bar{V}^{[3]} \end{bmatrix} \right) \leq n+1$$

This can be achieved by choosing $\bar{V}^{[3]}$ and $\bar{V}^{[1]}$ so that

$$\bar{H}^{[23]} \bar{V}^{[3]} \sim \bar{H}^{[21]} \bar{V}^{[1]}$$

where $P \prec Q$, means that the set of column vectors of matrix $P$ is a subset of the set of column vectors of matrix $Q$. Similarly, to decode $W_3$ at receiver 3, we wish to choose $\bar{V}^{[2]}$ and $\bar{V}^{[1]}$ so that

$$\bar{H}^{[32]} \bar{V}^{[2]} \sim \bar{H}^{[31]} \bar{V}^{[1]}$$

Thus, we wish to pick vectors $\bar{V}^{[1]}$, $\bar{V}^{[2]}$ and $\bar{V}^{[3]}$ so that equations (4), (5), (6) are satisfied. Note that the channel matrices $\bar{H}^{[ij]}$ have a full rank of $2n+1$ almost surely. Since multiplying by a full rank matrix (or its inverse) does not affect the conditions represented by equations (4), (5) and (6), they can be equivalently expressed as

$$\begin{align*}
B &= TC \quad (7) \\
B & \prec A \quad (8) \\
C & \prec A \quad (9)
\end{align*}$$

where

\begin{align*}
A &= \bar{V}^{[1]} \\
B &= (\bar{H}^{[21]})^{-1} \bar{H}^{[23]} \bar{V}^{[3]} \\
C &= (\bar{H}^{[31]})^{-1} \bar{H}^{[32]} \bar{V}^{[2]} \\
T &= \bar{H}^{[32]} (\bar{H}^{[21]})^{-1} \bar{H}^{[23]} (\bar{H}^{[32]})^{-1} \bar{H}^{[31]} (\bar{H}^{[31]})^{-1} (13)
\end{align*}

Note that $A$ is a $(2n+1) \times (n+1)$ matrix, $B$ and $C$ are $(2n+1) \times n$ matrices. Since all channel matrices are invertible, we can choose $A$, $B$ and $C$ so that they satisfy equations (7)-(9) and then use equations (10)-(13) to find $\bar{V}^{[1]}$, $\bar{V}^{[2]}$ and $\bar{V}^{[3]}$. Let $w$ be the $(2n+1) \times 1$ column vector $w = [1 \ 1 \ldots \ 1]^T$. We choose $A$, $B$ and $C$ as:

\begin{align*}
A &= \begin{bmatrix}
w & Tw & T^2 w & \cdots & T^n w
\end{bmatrix} \\
B &= \begin{bmatrix}
Tw & T^2 w & \cdots & T^n w
\end{bmatrix} \\
C &= \begin{bmatrix}
w & Tw & T^2 w & \cdots & T^n w
\end{bmatrix}
\end{align*}

It can be easily verified that $A$, $B$ and $C$ satisfy the three equations (7)-(9). Therefore, $\bar{V}^{[1]}$, $\bar{V}^{[2]}$ and $\bar{V}^{[3]}$ satisfy the interference alignment equations in (4), (5) and (6).

Now, consider the received signal vectors at Receiver 1. The desired signal arrives along the $n+1$ vectors $\bar{H}^{[1]} \bar{V}^{[1]}$ while the interference arrives along the $n$ vectors $\bar{H}^{[1]} \bar{V}^{[2]}$ and the $n$ vectors $\bar{H}^{[3]} \bar{V}^{[3]}$. As enforced by equation (4) the interference vectors are perfectly aligned. Therefore, in order to prove that there are $n+1$ interference free dimensions it suffices to show that the columns of the square, $(2n+1) \times (2n+1)$ dimensional matrix

$$\begin{bmatrix}
\bar{H}^{[11]} \bar{V}^{[1]} & \bar{H}^{[12]} \bar{V}^{[2]} \\
\bar{H}^{[13]} \bar{V}^{[3]}
\end{bmatrix}$$

are linearly independent almost surely. Multiplying by the full rank matrix $(\bar{H}^{[11]})^{-1}$ and substituting the values of $\bar{V}^{[1]}$, $\bar{V}^{[2]}$, equivalently we need to show that almost surely

$$S \triangleq [w \ Tw \ \cdots \ T^n w \ Dw \ DTw \ \cdots \ DT^{n-1} w]$$

has linearly independent column vectors where $D = (\bar{H}^{[11]})^{-1} \bar{H}^{[12]}$ is a diagonal matrix. In other words, we need to show $\det(S) \neq 0$ with probability 1. The proof is obtained by contradiction. If possible, let $S$ be singular with non-zero probability. I.e. $\Pr(|S| = 0) > 0$. Further, let the diagonal entries of $T$ be $\lambda_1, \lambda_2, \ldots, \lambda_{2n+1}$ and the diagonal entries of $D$ be $\kappa_1, \kappa_2, \ldots, \kappa_{2n+1}$. Then the following equation is true with non-zero probability.

$$|S| = 0.$$

But $|S|$ is a polynomial in $\lambda_1, \lambda_2, \ldots, \lambda_{2n+1}, \kappa_1, \kappa_2, \ldots, \kappa_{2n+1}$ where each variable has a continuous distribution conditioned on all the rest. For this polynomial to be identically equal to zero with a probability greater than zero, all the coefficients must be equal to zero. It is easily verified that this is not the case. The details of the proof are provided in [12].

Thus, the $n+1$ vectors carrying the desired signal at receiver 1 are linearly independent of the $n$ interference vectors which allows the receiver to zero force interference and obtain $n+1$ interference free dimensions, and therefore $n+1$ degrees of freedom for its message.

Similarly, at receiver 2 all the interference streams align along $\bar{H}^{[21]} \bar{V}^{[1]}$. In order to show that all the desired signal streams are linearly independent of the interference, it is, therefore, enough to show that the columns of $(2n+1) \times (2n+1)$ matrix

$$\begin{bmatrix}
\bar{H}^{[22]} \bar{V}^{[2]} & \bar{H}^{[21]} \bar{V}^{[1]}
\end{bmatrix}$$

are linearly independent almost surely. This proof is identical to the proof for receiver 1. Using the same arguments we can show that both receivers 2 and 3 are able to zero force the $n+1$ interference vectors and obtain $n$ interference free dimensions for their respective desired signals so that they each achieve $n$ degrees of freedom.

Thus we established the achievability of $d_1 + d_2 + d_3 = \frac{3n+1}{2n+1}$ for any $n$. This scheme, along with the converse automatically imply that

$$\sup_{(d_1,d_2,d_3) \in \mathcal{D}} d_1 + d_2 + d_3 = \frac{3}{2}$$
ties in (16). Consider any point $(0,0,1)$ in $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. This implies that achievable by time sharing between the corner points, and $K$, $L$, $J$ as a convex combination of the corner points $D = (0,0,0), (1,0,0), (0,1,0), (1,0,0)$.(See Fig. 1). Since convex combinations are valid, any point in $D$ lies in $D'$ through trivial achievable schemes. Also, Theorem 1 implies that $N = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ lies in $D'$ (Note that this is the only point which achieves a total of $\frac{3}{2}$ degrees of freedom and satisfies the inequalities in (16)). Consider any point $(d_1, d_2, d_3) \in D$. Then, it can be shown ( [12]) that $(d_1, d_2, d_3)$ can be expressed as a convex combination of the corner points $K, L, J$ and $(0,0,0)$ (See Fig. 1). Since convex combinations are achievable by time sharing between the corner points, this implies that $D \subset D'$ and the proof is complete. \hfill \blacksquare

\section*{IV. Conclusion}

We have shown that with perfect channel knowledge the $K$ user interference channel has $K/2$ degrees of freedom. This result reveals the fallacy of the conventional cake-cutting view of orthogonal medium access which achieves only 1 degree of freedom in the interference channel. The result therefore indicates that the capacity of wireless networks may be much higher than previously believed. In theory, we find that at high SNR the capacity of the $K$ user interference channel is $50\%, 90\%$, and $4900\%$ higher than prior belief for $K = 3, 20$ and 100 users, respectively. Thus, the present result could guide future research of wireless networks along an optimistic path in the same manner that MIMO technology has shaped our view of the capacity of a wireless channel.

Finally, there is increasing evidence that unlike the random coding based achievability schemes typically used in single user and many multiuser capacity theorems, structured codes (e.g. lattice codes) and random codes with a limited amount of structure may be necessary for network theorems in general [13]. The remarkable capacity benefits of interference alignment over supersymbols also prompt further exploration of structured coding schemes for wireless networks.

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