Maximum Likelihood Detection of Quasi-Orthogonal Space-Time Block Codes: Analysis and Simplification

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***Abstract*— In this paper, we propose a low complexity Maximum Likelihood (ML) decoding algorithm for quasi-orthogonal space-time block codes (QOSTBCs) based on the real-valued lattice representation and QR decomposition. We show that for a system with rate $r = n_s/T$, where n_s is the number of transmitted symbols per T time slots; the proposed algorithm decomposes the original complex-valued system into a parallel system with n_s 2×2 real-valued components, thus allowing for a simple joint decoding of two real symbols. For a square QAM constellation with L points (L-QAM), this algorithm achieves full diversity by properly incorporating two-dimensional rotation using the optimal rotation angle and the same rotating matrix for any number of transmit antennas ($N \geq 4$). We further show that for $N = 8$ and 16-QAM modulation scheme, the new approach achieves $> 98\%$ reduction in the overall complexity compared to conventional ML detection, and $> 93\%$ reduction compared to the most competitive reported algorithms in the literature. This complexity gain becomes greater when N or L becomes larger. We also show that the complexity of the proposed algorithm is linear with L and n_s.**

I. Introduction

Space-time block codes (STBCs) from orthogonal designs provide simple ML decoding and full diversity [1]. However, full-rate orthogonal STBCs do not exist for systems with more than $N = 2$ transmit antennas [2]. Consequently, Quasi-Orthogonal STBCs (QOSTBCs) were proposed in which the constraint of orthogonality is relaxed to obtain higher symbol transmission rate [3], [4], [5]. In general, QOSTBCs do not achieve the full diversity provided by the channel. One way to provide full diversity using QOSTBCs is to choose half of the transmitted symbols from a signal constellation set $\mathcal{A}$ and to choose the other half from a rotated constellation set $e^{j\phi}\mathcal{A}$ where ϕ is the rotation angle [6], [7], [8]. Another way is based on the multidimensional rotated constellations discussed in [9], [10], [11].

The encoding for these codes is very similar to the encoding of orthogonal STBCs (OSTBCs) [1], however, the decoding is more complicated. ML decoding of QOSTBCs is performed by searching over a subset of the total number of transmitted symbols. More specifically, a joint detection of at least two complex symbols is required for a full-rate system with $N = 4$ antennas, or $3/4$-rate system with $N = 8$. For the full-rate system with $N = 8$, four complex symbols are jointly detected to obtain the ML solution [6]- [8]. Unlike the OSTBCs, the decoding complexity is no longer linear, but rather, increases exponentially with N, i.e., $\mathcal{O}(L^{N/2})$ where L is the size of the L-QAM constellation [6], [12]. A number of techniques which allow for low complexity decoding have been proposed in the literature [11], [13], [14]. In [11], a simpler ML decoding was proposed resulting in a decoding complexity reduction from $\mathcal{O}(L^{N/2})$ to $\mathcal{O}(L^{N/4})$. However, this complexity is still exponential in N and will increase rapidly as L becomes large. In [13], an algorithm is presented based on the QR decomposition of the complex channel, which deals with QOSTBCs with $N = 4$ only. It achieves ML performance with significant reduction in the computational load by performing pairwise detection of the complex transmitted symbols in a way similar to that of conventional ML and by using a sorting rule. Minimum decoding complexity (MDC) for the class of QOSTBCs whose ML decoding requires only the joint detection of two real symbols was proposed in [14]. It was shown that the maximum achievable rate for the proposed MDC-QOSTBC is 1 for $N = 3$ and 4, and $3/4$ for $N = 5 - 8$. An algebraic structure for the code was constructed to guarantee that every complex transmitted symbol is orthogonal to all other complex symbols, but the in-phase and quadrature components within the same complex symbols need not be orthogonal. Sphere decoding (SD) [15], [16], [17], [18], on the other hand, was proposed as a low complexity algorithm for the detection of QOSTBCs [12]. The ML metric of QOSTBCs is written into two independent Euclidean norms in [12], making it possible to apply SD to operate on each independently, thus reducing the decoding complexity. Complexity of SD has been widely studied in [17], [19], [20]. In the sequel, we will compare the complexity of our algorithm with those discussed above in detail.

Our new decoding algorithm is based on QR decomposition of the real-valued lattice representation for the class of QOSTBCs discussed in [14]. Using this representation, we show that ML detection can be performed jointly considering only two real symbols. We further show that the equivalent channel as seen by every two real symbols is 2×2, upper

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triangular, and real-valued; a property which substantially reduces the decoding complexity. Full diversity is achieved by properly incorporating two-dimensional rotation. This rotation is performed using the same rotating matrix for any N. This becomes possible by an appropriate grouping of the complex transmitted symbols. We discuss the performance and complexity results for our algorithm, and compare them with conventional ML detection [3], those in [11], [13], [14], and sphere decoding [12].

The remainder of this paper is organized as follows: In Section II, we specify the system model and briefly review the STBCs from quasi-orthogonal designs. In Section III, we introduce the new algorithm for different number of antennas N and rates r. Section IV explains the application of a two-dimensional constellation rotation in order to improve the performance and achieve full diversity. A complexity discussion is provided and comparisons to other algorithms presented in the literature are given in Section V. Simulation results are included in Section VI. Finally, we conclude the paper in Section VII.

II. QOSTBCs and System Model

Consider a MIMO system with N transmit and M receive antennas, and an interval of T symbols during which the channel is constant. The received signal is given by

$$Y = \sqrt{\frac{\rho}{N}} C_N H + V \qquad (1)$$

where $Y = [y_t^j]_{T\times M}$ is the received signal matrix of size $T\times M$ and whose entry y_t^j is the signal received at antenna j at time t, $t = 1, 2, \ldots, T$, $j = 1, 2, \ldots, M$; $V = [v_t^j]_{T\times M}$ is the noise matrix; and $C_N = [c_t^i]_{T\times N}$ is the transmitted signal matrix whose entry c_t^i is the signal transmitted at antenna i at time t, $t = 1, 2, \ldots, T$, $i = 1, 2, \ldots, N$. $H = [h_{i,j}]_{N\times M}$ is the channel coefficient matrix of size $N\times M$ whose entry $h_{i,j}$ is the channel coefficient from transmit antenna i to receive antenna j. The entries of the matrices H and V are mutually independent, zero-mean, and circularly symmetric complex Gaussian random variables of unit variance; and the parameter ρ is the signal-to-noise-ratio (SNR) per receiving antenna.

In this paper, we consider the class of QOSTBCs for which decoding pairs of symbols independently is possible (see Chapter 5 of [1]). The analysis in the sequel applies to any QOSTBC that belongs to this class [3], [14]. For $N = 8$, we consider the QOSTBC proposed in [3] whose rate r is $3/4$ and $T = 8$ given by

$$C_8 = \begin{bmatrix} s_1 & s_2 & s_3 & 0 & s_4 & s_5 & s_6 & 0 \\ -s_2^* & s_1^* & 0 & -s_3 & s_5^* & -s_4^* & 0 & s_6 \\ s_3^* & 0 & -s_1^* & -s_2 & -s_6^* & 0 & s_4^* & s_5 \\ 0 & -s_3^* & s_2^* & -s_1 & 0 & s_6^* & -s_5^* & s_4 \\ -s_4 & -s_5 & -s_6 & 0 & s_1 & s_2 & s_3 & 0 \\ -s_5^* & s_4^* & 0 & s_6 & -s_2^* & s_1^* & 0 & s_3 \\ s_6^* & 0 & -s_4^* & s_5 & s_3^* & 0 & -s_1^* & s_2 \\ 0 & s_6^* & -s_5^* & -s_4 & 0 & s_3^* & -s_2^* & -s_1 \end{bmatrix}. \qquad (2)$$

Obviously, the number of transmitted symbols per block n_s is equal to 6 (recall $n_s = Tr$).

For $N = 4$, choosing any of the full-rate codes presented in [3], [11], or [21] leads to the same analysis. We consider the one presented in [11] with $n_s = 4$ and is defined by

$$C_4 = \begin{bmatrix} s_1 & s_3 & s_4 & s_2 \\ s_3^* & -s_1^* & s_2^* & -s_4^* \\ s_4^* & s_2^* & -s_1^* & -s_3^* \\ s_2 & -s_4 & -s_3 & s_1 \end{bmatrix}. \qquad (3)$$

Note that removing one or more columns of a QOSTBC results in new STBCs for smaller numbers of transmit antennas [1].

Assuming that the channel H is known at the receiver and setting $M = 1$ for simplicity, the ML estimate is obtained at the decoder by performing $\min_{C_N} ||Y - \sqrt{\frac{\rho}{N}} C_N H||_F^2$, where $||.||_F$ is the Frobenius norm. For all QOSTBCs, the measure $||Y - \sqrt{\frac{\rho}{N}} C_N H||_F^2$ can be decoupled into two parts, where each part solves $N/2$ symbols concurrently [12]. However, the complexity can still be reduced substantially as will become clearer in the sequel.

III. Proposed Algorithm

We start by decomposing the T-dimensional complex problem defined by (1) to a $2T$-dimensional real-valued problem. Using either of the real-valued representations proposed in [11] or [18] gives the same results in terms of our proposed algorithm. Recalling that $M = 1$, and rewriting (1) in matrix form, we have

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix} = C_N \begin{bmatrix} h_{1,1} \\ h_{2,1} \\ \vdots \\ h_{N,1} \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_T \end{bmatrix}. \qquad (4)$$

We specify the complex transmitted symbols of C_N by their real and imaginary parts as $s_m = x_{2m-1} + jx_{2m}$ for $m = 1, 2, \ldots, 6$ when $N = 8$, and for $m = 1, 2, 3, 4$ when $N = 4$. Applying the real-valued lattice representation defined in [18] to (4), we obtain

$$\check{y} = \check{H}\check{x} + \check{v} \qquad (5)$$

or equivalently

$$\begin{bmatrix} \Re(y_1) \\ \Im(y_1) \\ \vdots \\ \Re(y_T) \\ \Im(y_T) \end{bmatrix} = \check{H} \begin{bmatrix} \Re(s_1) \\ \Im(s_1) \\ \vdots \\ \Re(s_m) \\ \Im(s_m) \end{bmatrix} + \begin{bmatrix} \Re(v_1) \\ \Im(v_1) \\ \vdots \\ \Re(v_T) \\ \Im(v_T) \end{bmatrix}. \qquad (6)$$

The real-valued fading coefficients of $\check{H}$ are defined using the complex fading coefficients $h_{i,j}$ from transmit antenna i to receive antenna j as $h_{2n-1} = \Re(h_{n,1})$, and $h_{2n} = \Im(h_{n,1})$ for $n = 1, 2, \ldots, N$ (recall that we restricted ourselves to $M = 1$, and therefore we only consider $j = 1$). Let's define the number of transmitted symbols as n_s which is $n_s = 6$ when $N = T = 8$ and $n_s = 4$ when $N = T = 4$. Then, the equivalent real-valued channel $\check{H}$ is a $2N \times 2n_s$ matrix and is defined as given below.

A. $\underline{N = 8}$

For $N = 8$ and using (2), the effective channel matrix is

$$\check{H} = \begin{bmatrix} h_1 & -h_2 & h_3 & -h_4 & \cdots & h_{13} & -h_{14} \\ h_2 & h_1 & h_4 & h_3 & \cdots & h_{14} & h_{13} \\ h_3 & h_4 & -h_1 & -h_2 & \cdots & h_{15} & -h_{16} \\ h_4 & -h_3 & -h_2 & h_1 & \cdots & h_{16} & h_{15} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -h_{15} & h_{16} & -h_{13} & -h_{14} & \cdots & h_3 & h_4 \\ -h_{16} & -h_{15} & -h_{14} & h_{13} & \cdots & h_4 & -h_3 \end{bmatrix}.$$

Setting $\check{H} = \begin{bmatrix} \check{h}_1 & \check{h}_2 & \cdots & \check{h}_{12} \end{bmatrix}$, where $\check{h}_k$ is the k-th column of $\check{H}$, we have

$$\begin{aligned} \langle \check{h}_1, \check{h}_i \rangle &= 0, i \neq 8, & \langle \check{h}_2, \check{h}_i \rangle &= 0, i \neq 7 \\ \langle \check{h}_3, \check{h}_i \rangle &= 0, i \neq 10, & \langle \check{h}_4, \check{h}_i \rangle &= 0, i \neq 9 \\ \langle \check{h}_5, \check{h}_i \rangle &= 0, i \neq 12, & \langle \check{h}_6, \check{h}_i \rangle &= 0, i \neq 11 \end{aligned} \quad (7)$$

where $\langle \check{h}_i, \check{h}_j \rangle$ is the inner product of columns $\check{h}_i$ and $\check{h}_j$. Interchanging the columns of $\check{H}$ such that every two columns that are not orthogonal to each other become adjacent makes the subsequent analysis appear in a compact form. Thus, we rewrite (5) as

$$\tilde{y} = \tilde{H}x + \tilde{v} \quad (8)$$

where $\tilde{H} = \begin{bmatrix} \check{h}_1, \check{h}_8, \check{h}_2, \check{h}_7, \check{h}_3, \check{h}_{10}, \check{h}_4, \check{h}_9, \check{h}_5, \check{h}_{12}, \check{h}_6, \check{h}_{11} \end{bmatrix}$. Consequently, the vectors $\tilde{y}$ and $\tilde{v}$ are the same as $\check{y}$ and $\check{v}$ in (5), whereas $x = \check{x}$ but with different order of elements given by $1, 8, 2, 7, 3, 10, 4, 9, 5, 12, 6, 11$. For example, $x_2 = \check{x}_8$. We rewrite (7) using the columns of $\tilde{H}$ defined by $\tilde{h}_k$ for $k = 1, 2, \ldots, 12$ as

$$\begin{aligned} \langle \tilde{h}_1, \tilde{h}_i \rangle &= 0, i \neq 2, & \langle \tilde{h}_7, \tilde{h}_i \rangle &= 0, i \neq 8 \\ \langle \tilde{h}_3, \tilde{h}_i \rangle &= 0, i \neq 4, & \langle \tilde{h}_9, \tilde{h}_i \rangle &= 0, i \neq 10 \\ \langle \tilde{h}_5, \tilde{h}_i \rangle &= 0, i \neq 6, & \langle \tilde{h}_{11}, \tilde{h}_i \rangle &= 0, i \neq 12. \end{aligned} \quad (9)$$

Applying QR decomposition to (8), we have

$$\begin{aligned} \tilde{y} &= QRx + \tilde{v} \\ Q^H \tilde{y} &= Rx + Q^H \tilde{v} \\ \bar{y} &= Rx + \bar{v} \end{aligned} \quad (10)$$

where $\bar{v}$ and $\tilde{v}$ have the same statistical properties since Q is unitary and so is Q^H. Recall that $\tilde{H}$ is a 16×12 matrix. Then Q^H is a 12×16 matrix and $\bar{y}$ is a vector of size 12. The matrix R is 12×12 block diagonal of the form

$$R = \begin{bmatrix} R_{1,2} & 0 & \cdots & 0 \\ 0 & R_{3,4} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{11,12} \end{bmatrix} \quad (11)$$

where

$$R_{i,i+1} = \begin{bmatrix} r_{i,i} & r_{i,i+1} \\ 0 & r_{i+1,i+1} \end{bmatrix} \qquad i = 1, 3, 5, \ldots, 11. \quad (12)$$

Note that the elements of the upper triangular matrix R are the inner products of columns of $\tilde{H}$ [22], and due to the properties given in (9), it is straightforward to obtain (11). Note also that the diagonal elements of R are the norm values of nonzero vectors [22], and thus $r_{i,i}$ and $r_{i+1,i+1}$ for $i = 1, 3, 5, \ldots, 11$ would never be zeros. Using (11), the ML problem is now simpler and rather than minimizing $||Y - C_N H||^2$, the solution is obtained by minimizing the metric $||\bar{y} - Rx||^2$ in a layered fashion over all different combinations of the vector x. To make this clearer, let the square L-QAM alphabet be given as Ω^2, where $\Omega = \{-\sqrt{L}+1, -\sqrt{L}+3, \ldots, \sqrt{L}-1\}$. Then

$$\hat{x} = \arg \min_{x \in \Omega^{12}} ||\bar{y} - Rx||^2.$$

Let x_i and x_{i+1} be given as

$$\arg \min_{\substack{x_i \in \Omega \\ x_{i+1} \in \Omega}} \left[|\bar{y}_{i+1} - r_{i+1,i+1} x_{i+1}|^2 + |\bar{y}_i - r_{i,i} x_i - r_{i,i+1} x_{i+1}|^2 \right]$$

for $i = 1, 3, \ldots, 11$. Then, the decoded message is

$$\hat{x} = (x_1, x_2, \ldots, x_{12})^T.$$

In other words, the ML solution is obtained by jointly decoding two real symbols through a simpler 2×2 real-valued upper triangular equivalent channel matrix. Note that this simplification is obtained through the orthogonality properties in (9) and the QR decomposition in (10), resulting in (11) and (12). This is similar to but simpler than [11] and [14] in that all have joint detection of two real symbols ($N/2$ times in [11], [14] and n_s times in this work) but with minimizing a norm of size $2N$ in [11] and [14] while minimizing a norm of size 2 in this work. This means that the original complex ML problem is decomposed into $n_s = 6$ parallel real-valued upper triangular problems, each of dimension 2. Writing this in matrix form, the ML solution is obtained by carrying out

$$\arg \min_{x_i \in \Omega, x_{i+1} \in \Omega} \left|\left| \begin{bmatrix} \bar{y}_i \\ \bar{y}_{i+1} \end{bmatrix} - \begin{bmatrix} r_{i,i} & r_{i,i+1} \\ 0 & r_{i+1,i+1} \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix} \right|\right|^2$$

for $i = 1, 3, \ldots, 11$.

Obviously, this approach allows finding the ML solution faster, and requires a small number of computational operations compared to conventional ML, [11] and [14], as will be shown in Section V.

B. $\underline{N = 4}$

For $N = 4$ and using the full rate code defined in (3), the equivalent channel matrix $\check{H} = \begin{bmatrix} \check{h}_1 & \check{h}_2 & \ldots & \check{h}_8 \end{bmatrix}$ is given by

$$\check{H} = \begin{bmatrix} h_1 & -h_2 & h_7 & -h_8 & h_3 & -h_4 & h_5 & -h_6 \\ h_2 & h_1 & h_8 & h_7 & h_4 & h_3 & h_6 & h_5 \\ -h_3 & -h_4 & h_5 & h_6 & h_1 & h_2 & -h_7 & -h_8 \\ -h_4 & h_3 & h_6 & -h_5 & h_2 & -h_1 & -h_8 & h_7 \\ -h_5 & -h_6 & h_3 & h_4 & -h_7 & -h_8 & h_1 & h_2 \\ -h_6 & h_5 & h_4 & -h_3 & -h_8 & h_7 & h_2 & -h_1 \\ h_7 & -h_8 & h_1 & -h_2 & -h_5 & h_6 & -h_3 & h_4 \\ h_8 & h_7 & h_2 & h_1 & -h_6 & -h_5 & -h_4 & -h_3 \end{bmatrix}.$$

We observe that

$$\begin{aligned} \langle \check{h}_1, \check{h}_i \rangle &= 0, i \neq 3, & \langle \check{h}_2, \check{h}_i \rangle = 0, i \neq 4 \\ \langle \check{h}_5, \check{h}_i \rangle &= 0, i \neq 7, & \langle \check{h}_6, \check{h}_i \rangle = 0, i \neq 8. \end{aligned} \quad (13)$$

Similar to $N = 8$, we interchange the columns of $\check{H}$ so that the subsequent analysis is in a compact form. In other words, we rewrite (5) as

$$\tilde{y} = \tilde{H} x + \tilde{v} \quad (14)$$

where $\tilde{H} = \left[\begin{array}{cccccccc} \check{h}_1 & \check{h}_3 & \check{h}_2 & \check{h}_4 & \check{h}_5 & \check{h}_7 & \check{h}_6 & \check{h}_8 \end{array} \right]$. Consequently, the vectors $\tilde{y}$ and $\tilde{v}$ are the same as $\check{y}$ and $\check{v}$, whereas $x = \check{x}$ with different elements order given by $1, 3, 2, 4, 5, 7, 6, 8$. Therefore, (13) can be rewritten using the columns of $\tilde{H}$ as

$$\begin{aligned} \langle \tilde{h}_1, \tilde{h}_i \rangle &= 0, i \neq 2, & \langle \tilde{h}_5, \tilde{h}_i \rangle = 0, i \neq 6 \\ \langle \tilde{h}_3, \tilde{h}_i \rangle &= 0, i \neq 4, & \langle \tilde{h}_7, \tilde{h}_i \rangle = 0, i \neq 8. \end{aligned} \quad (15)$$

Applying QR decomposition to $\tilde{H}$ produces an 8×8 block diagonal matrix of the form

$$R = \left[\begin{array}{cccc} R_{1,2} & 0 & 0 & 0 \\ 0 & R_{3,4} & 0 & 0 \\ 0 & 0 & R_{5,6} & 0 \\ 0 & 0 & 0 & R_{7,8} \end{array} \right] \quad (16)$$

where $R_{i,i+1}$ is as defined in (12) and $i = 1, 3, 5, 7$.

Analogous to $N = 8$, the ML problem is decomposed into $n_s = 4$ parallel real-valued simple problems each of dimension 2, and the optimal solution is obtained by performing

$$\arg \min_{x_i \in \Omega, x_{i+1} \in \Omega} \left\| \left[\begin{array}{c} \bar{y}_i \\ \bar{y}_{i+1} \end{array} \right] - \left[\begin{array}{cc} r_{i,i} & r_{i,i+1} \\ 0 & r_{i+1,i+1} \end{array} \right] \left[\begin{array}{c} x_i \\ x_{i+1} \end{array} \right] \right\|^2 \quad (17)$$

for $i = 1, 3, 5, 7$.

Finally, we observe that the proposed algorithm produces n_s parallel 2×2 real-valued subsystems for any N, resulting in a simplified ML problem that can be solved in a parallel fashion to obtain the optimal solution while substantially reducing the overall decoding complexity.

IV. QOSTBC WITH FULL DIVERSITY

In order to achieve full diversity and improve the performance at high SNR, a conventional approach in the literature suggests that half of the symbols in a quasi-orthogonal design are chosen from a signal constellation set $\mathcal{A}$ and the other half are chosen from a rotated constellation set $e^{j\phi}\mathcal{A}$ [6]-[8]. Another approach is to apply multi-dimensional rotated constellations which exhibit full diversity and maximum coding gain [9]-[11]. However, no proper expressions for the rotation matrix of sizes greater than four exist. Applying the first approach to our algorithm introduces interference among the real symbols. Analytically, this means that the orthogonality properties defined in (9) and (15) are no longer valid, which results in having more than one nonzero sub-matrix in some rows of (11) and (16). Instead, a two-dimensional rotation of the real symbols $(x_1, x_2, \ldots, x_{2n_s})$ that are not orthogonal to each other is always applied, thus maintaining the orthogonality properties among the channel columns and maximizing the diversity. By this, we overcome the problem of finding proper expressions for the rotation matrix and maintain the orthogonality properties defined in (9) and (15). Moreover, this technique is applicable for an arbitrary number of antennas and is compatible with our proposed decoding algorithm.

The two-dimensional rotation matrix is defined by [10], [11] as

$$G = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right] \quad (18)$$

where the optimal angle is obtained by $\theta = \frac{1}{2}\mathrm{atan}(\frac{1}{2})$ for square QAM constellations [14]. In the following, we discuss the application of this rotation to QOSTBCs for $N = 8$ and $N = 4$.

A. $N = 8$

Using the properties defined in (9), we combine every two real symbols that are jointly decoded into one group, then we apply a two-dimensional rotation using the matrix G. The produced real symbols are defined as

$$\begin{aligned} \left[\begin{array}{c} \tilde{x}_1 \\ \tilde{x}_2 \end{array} \right] &= G \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right], & \left[\begin{array}{c} \tilde{x}_3 \\ \tilde{x}_4 \end{array} \right] = G \left[\begin{array}{c} x_3 \\ x_4 \end{array} \right], \\ \left[\begin{array}{c} \tilde{x}_5 \\ \tilde{x}_6 \end{array} \right] &= G \left[\begin{array}{c} x_5 \\ x_6 \end{array} \right], & \left[\begin{array}{c} \tilde{x}_7 \\ \tilde{x}_8 \end{array} \right] = G \left[\begin{array}{c} x_7 \\ x_8 \end{array} \right], \\ \left[\begin{array}{c} \tilde{x}_9 \\ \tilde{x}_{10} \end{array} \right] &= G \left[\begin{array}{c} x_9 \\ x_{10} \end{array} \right], & \left[\begin{array}{c} \tilde{x}_{11} \\ \tilde{x}_{12} \end{array} \right] = G \left[\begin{array}{c} x_{11} \\ x_{12} \end{array} \right]. \end{aligned} \quad (19)$$

Full diversity is achieved by replacing the symbols $s_1 = \check{x}_1 + j\check{x}_2, s_2 = \check{x}_3 + j\check{x}_4, \ldots, s_6 = \check{x}_{11} + j\check{x}_{12}$ defined in (2) by the rotated symbols $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_6$ [7], [10], where $\tilde{s}_1 = \tilde{x}_1 + j\tilde{x}_3, \tilde{s}_2 = \tilde{x}_5 + j\tilde{x}_7, \tilde{s}_3 = \tilde{x}_9 + j\tilde{x}_{11}, \tilde{s}_4 = \tilde{x}_4 + j\tilde{x}_2, \tilde{s}_5 = \tilde{x}_8 + j\tilde{x}_6, \tilde{s}_6 = \tilde{x}_{12} + j\tilde{x}_{10}$ (recall that the transmitted vector $x = \left[\begin{array}{cccc} x_1 & x_2 & \ldots & x_{12} \end{array} \right]$ is previously defined as $x = \left[\begin{array}{cccccc} \check{x}_1 & \check{x}_8 & \check{x}_2 & \check{x}_7 & \ldots & \check{x}_6 & \check{x}_{11} \end{array} \right]$, and that the vector $\tilde{x}$ is the rotation of x). Incorporating the rotation matrix G into the system will change the equivalent channel matrix $\tilde{H}$, but will keep the orthogonality properties defined in (9) unchanged. For $N = 8$, the equivalent channel matrix $\tilde{H}$ after the rotation is given by

$$\left[\begin{array}{ccc} h_1\cos\theta - h_{10}\sin\theta & \cdots & h_{13}\cos\theta + h_6\sin\theta \\ h_2\cos\theta + h_9\sin\theta & \cdots & h_{14}\cos\theta - h_5\sin\theta \\ \vdots & \ddots & \vdots \\ -h_{16}\cos\theta - h_7\sin\theta & \cdots & h_4\cos\theta + h_{11}\sin\theta \end{array} \right].$$

Applying QR decomposition to $\tilde{H}$ produces a block diagonal matrix R of the form previously discussed in (11). Thus, ML detection of $n_s = 6$ parallel 2×2 real-valued sub-systems becomes possible and achieves full diversity by properly choosing the optimal rotation angle θ. To make this clearer, we define the codeword difference matrix between any pair of distinct transmitted signals, $\mathbf{C}^1$ and $\mathbf{C}^2$, as $\Delta C = \mathbf{C}^1 - \mathbf{C}^2$ where $\mathbf{C}^1 = G(\check{s}_1, \check{s}_2, \ldots, \check{s}_6)$, $\mathbf{C}^2 = G(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_6)$, and $G(.)$ is the generator matrix that produces the codeword matrices $\mathbf{C}^1$ and $\mathbf{C}^2$. In order to achieve full diversity, the minimum value of the determinant of the product $(\Delta C)^H(\Delta C)$ has to be nonzero, and should be maximized in order to obtain the optimum coding gain [1], [6], [23]. With some algebra, we obtain

$$(\Delta C)^H(\Delta C) = \begin{bmatrix} (\Delta A)I_4 & (\Delta B)I_4 \\ (\Delta B)I_4 & (\Delta A)I_4 \end{bmatrix} \qquad (20)$$

where I_4 is the identity matrix of size $N/2 = 4$, and ΔA and ΔB are given by

$$\Delta A = \sum_{k=1}^{6} |\check{s}_k - \tilde{s}_k|^2$$

and

$$\Delta B = \sum_{k=1}^{3} -(\check{s}_k - \tilde{s}_k)(\check{s}_{k+3} - \tilde{s}_{k+3})^* + \sum_{k=1}^{3} (\check{s}_k - \tilde{s}_k)^*(\check{s}_{k+3} - \tilde{s}_{k+3}).$$

Now, we define $\tilde{\Delta}_i = \Delta_i \cos\theta - \Delta_j \sin\theta$ and $\tilde{\Delta}_j = \Delta_j \cos\theta + \Delta_i \sin\theta$ for $(i,j) \in \{(1,2),(3,4),(5,6),(7,8),(9,10),(11,12)\}$, where Δ_k and $\tilde{\Delta}_k$ are the differences between the real symbol $\check{x}_k$ in codeword $\mathbf{C}^1$ and $\tilde{x}_k$ in codeword $\mathbf{C}^2$ for $k = 1, 2, \ldots, 12$, before and after the two-dimensional rotation, respectively (see (19)). Thus, ΔA and ΔB can be rewritten as

$$\Delta A = \sum_{\substack{k=1 \\ k:odd}}^{12} (\tilde{\Delta}_k)^2 + (\tilde{\Delta}_{k+1})^2 \qquad (21)$$

and

$$\Delta B = 2 \sum_{\substack{k=1 \\ k:odd}}^{6} -(\tilde{\Delta}_k + j\tilde{\Delta}_{k+1})(\tilde{\Delta}_{k+6} - j\tilde{\Delta}_{k+7}). \qquad (22)$$

The determinant expression of $(\Delta C)^H(\Delta C)$ defined in (20) is derived in [23] as

$$\det(\Delta C)^H(\Delta C) = \left[(\Delta A + \Delta B)(\Delta A - \Delta B)\right]^4. \qquad (23)$$

Substituting (21) and (22) in (23), and denoting the set of pairs (i,j) $S = \{(1,2),(3,4),(5,6),(7,8),(9,10),(11,12)\}$, and the determinant value by d, we get

$$d = \left(\left(\sum_{(i,j)\in S} (\tilde{\Delta}_i + \tilde{\Delta}_j)^2 \right) \times \left(\sum_{(i,j)\in S} (\tilde{\Delta}_i - \tilde{\Delta}_j)^2 \right) \right)^4.$$

Following [6]-[8], [21], [23], we can assume that the worst-case (minimum) value of d is obtained when only one group of the real symbols (x_i, x_j) for $(i,j) \in S$, makes an error and the rest of the symbols are error free, hence d is simplified to [23]

$$d = \left[\Delta_i^2 \cos(2\theta) - 2\Delta_i\Delta_j \sin(2\theta) - \Delta_j^2 \cos(2\theta)\right]^8.$$

In order to achieve full diversity, we can properly choose the rotation angle θ so that d is nonzero. It was shown in [11], [14], that $\theta = \frac{1}{2}\text{atan}(\frac{1}{2})$ is the optimal rotation angle for square QAM constellations which not only provides a nonzero value of d and therefore achieves full diversity, but also maximizes the minimum determinant value d, and consequently, provides the optimum coding gain.

B. $N = 4$

Two-dimensional rotation is applied again using the same rotating matrix G and the optimal rotation angle θ defined above. Full diversity is achieved by replacing s_1, s_2, s_3, s_4 defined in (3) by $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4$ where $\tilde{s}_1 = \tilde{x}_1 + j\tilde{x}_3, \tilde{s}_2 = \tilde{x}_2 + j\tilde{x}_4, \tilde{s}_3 = \tilde{x}_5 + j\tilde{x}_7, \tilde{s}_4 = \tilde{x}_6 + j\tilde{x}_8$, and the real symbols $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_8$ are obtained by

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = G \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} = G \begin{bmatrix} x_3 \\ x_4 \end{bmatrix},$$
$$\begin{bmatrix} \tilde{x}_5 \\ \tilde{x}_6 \end{bmatrix} = G \begin{bmatrix} x_5 \\ x_6 \end{bmatrix}, \quad \begin{bmatrix} \tilde{x}_7 \\ \tilde{x}_8 \end{bmatrix} = G \begin{bmatrix} x_7 \\ x_8 \end{bmatrix}.$$

Then the equivalent channel matrix $\tilde{H}$ is given by

$$\begin{bmatrix} h_1\cos\theta + h_7\sin\theta & \cdots & -h_6\cos\theta + h_4\sin\theta \\ h_2\cos\theta + h_8\sin\theta & \cdots & h_5\cos\theta - h_3\sin\theta \\ \vdots & \ddots & \vdots \\ h_8\cos\theta + h_2\sin\theta & \cdots & -h_3\cos\theta + h_5\sin\theta \end{bmatrix}.$$

It is important to note that the orthogonality properties among the columns of $\tilde{H}$ presented in (15) still hold, and thus the matrix R produced by applying QR has the same structure of (16). The ML solution is obtained by (17) and achieves full diversity with the same optimal rotation angle $\theta = \frac{1}{2}\text{atan}(\frac{1}{2})$ as shown above for $N = 8$.

Finally, we emphasize the fact that the proposed decoding algorithm is generic, works and has the same computational complexity for rotated and non-rotated QOSTBCs.

V. Computational Complexity

In this section, we compare the computational complexity of our proposed algorithm with that of conventional ML detection and the complexity of other reported algorithms in the literature. Analogous to [11], the overall complexity is measured in terms of the number of operations required to decode the transmitted signals for each block period T. Using the same notation in [11], a complex multiplication is equivalent to 4 real multiplications C_M and 2 real additions C_A, while a complex addition is equivalent to 2 real additions. We split the complexity formula into two parts in order to represent C_M and C_A independently. We denote the complexity of our proposed algorithm by $\mathcal{C}_{PR}$, and show it as a two dimensional vector where the first dimension is the number of real multiplications and the second, the number of real additions, then

$$\mathcal{C}_{PR} = n_s L(6C_M, 8C_A). \qquad (24)$$

Besides (24), the proposed algorithm requires $144C_M$ and $88C_A$ for performing QR decomposition of the channel matrix for $N = 4$, and $360C_M$ and $204C_A$ for $N = 8$. Additionally, the computation of $\bar{y} = Q^H\tilde{y}$ requires $64C_M$ and $56C_A$ for $N = 4$, and $192C_M$ and $180C_A$ for $N = 8$. It is important to emphasize the fact that these numbers and (24) hold for rotated and non-rotated constellations, and that all comparisons given consider the complexity of QR and the computation of $\bar{y} = Q^H \tilde{y}$.

Conventional ML detection [3], on the other hand, performs pairwise complex symbol detection and the complexity $\mathcal{C}_{ML}$ is presented in [11] as (in the same notation as (24))

$$\mathcal{C}_{ML} = 2L^{\frac{N}{2}}\{(2N^2 + 4N)C_M, (2N^2 + 3N - 1)C_A\}.$$

Obviously, $\mathcal{C}_{ML}$ increases exponentially with N and polynomially with L, whereas $\mathcal{C}_{PR}$ is linear with n_s (or equivalently N) and L. Recall that the complexity gain obtained by $\mathcal{C}_{PR}$ is for free, since this algorithm gives the optimal ML solution. The decoding algorithm presented in [11] was able to achieve ML performance while reducing the complexity from $\mathcal{O}(L^{N/2})$ to $\mathcal{O}(L^{N/4})$ compared to conventional ML detection. The complexity is

$$\mathcal{C}_{[11]} = 4L^{\frac{N}{4}}\{(N^2 + 4N)C_M, (N^2 + 3N - 1)C_A\}. \quad (25)$$

This complexity is exponential in N and polynomial in L, thus $\mathcal{C}_{PR}$ is more desirable. For example, for $N = 4$ and 16-QAM modulation scheme, $\mathcal{C}_{PR}$ achieves $> 71\%$ reduction in the complexity compared to $\mathcal{C}_{[11]}$, and $> 97\%$ compared to $\mathcal{C}_{ML}$. However, for $N = 8$, $\mathcal{C}_{[11]}$ is slightly modified when considering the QOSTBC code defined in (2) (note that for $N = 8$, decoding pairs of complex symbols independently for the code defined in [11] is not possible). Thus, for $N = 8$ and 16-QAM, $\mathcal{C}_{PR}$ achieves $> 93\%$ reduction compared to $\mathcal{C}_{[11]}$, and $> 98\%$ compared to $\mathcal{C}_{ML}$. This complexity gain becomes greater as L or N is larger.

SD was proposed in [12] to decode QOSTBCs. The computational complexity is reduced from $\mathcal{O}((n_s)^6)$ for ML to $\mathcal{O}(2(n_s/2)^6)$ using SD, where n_s is the number of transmitted symbols. This complexity is exponential with n_s. It is also sensitive to the choice of the initial radius which is used to start the algorithm [16], [17]. The choice of the radius in [12] is based on the Gram matrix eigenvalues, which adds a computational complexity of $\mathcal{O}((2M)^3)$. Moreover, when no point is found inside the sphere, the radius is increased and the algorithm restarts, which additionally increases the computational complexity.

In [13], the decoding algorithm is based on QR decomposition of the complex channel matrix. It is only applicable to QOSTBCs with $N = 4$. It requires a number of complex computations for finding the partial metrics of the complex transmitted symbols [13]. Recall that a complex multiplication is equivalent to 4 real multiplications and 2 real additions. Furthermore, the complexity is not only a function of the signal points within the transmission constellation, but also a function of the sorting adopted there. Unfortunately, no complexity formula was presented. However, a complexity reduction of 82% compared to the conventional ML for 16-QAM was reported. Thus, taking the conventional ML as a baseline, we obtain a reduction of $> 97\%$ for the same L and N.

Finally, the decoding algorithm presented in [14] has a complexity which is comparable to the complexity of the algorithm proposed in [11] which is a function of $\mathcal{O}(L^{N/4})$, since both algorithms perform joint detection of two real symbols. Therefore, the same complexity gain of $\mathcal{C}_{PR}$ over $\mathcal{C}_{[11]}$ is applicable also over $\mathcal{C}_{[14]}$. However, the algorithm of [14] suffers a performance loss of 0.4 dB compared to ML.

VI. Simulation Results

We provide simulation results for the proposed algorithm for $N = 8$ and $N = 4$. We denote the algorithm by PR and compare the performance to the performance of optimal conventional ML detection. In all simulations, we consider one receive antenna. Moreover, we use the same rotating matrix G defined in (18), and the same rotation angle θ given by $\theta = \frac{1}{2}\text{atan}(\frac{1}{2})$ in order to achieve full transmit diversity.

Figure 1 provides simulation results for the transmission of 1.5 bits/s/Hz with $N = 8$ and 4-QAM using the quasi-orthogonal code defined in (2). We compare PR with conventional ML detection with and without rotation. PR achieves full diversity through the rotation matrix G, while ML detection achieves it by replacing the transmitted symbols s_4, s_5, s_6 with $s_4e^{j\phi}, s_5e^{j\phi}, s_6e^{j\phi}$ in (2) with $\phi = \pi/4$ (i.e., choosing half of the transmitted symbols from a rotated constellation set $e^{j\phi}\mathcal{A}$ as discussed earlier) [1]. PR achieves the same performance as conventional ML, with a substantial reduction in the decoding complexity. Using (24) and (25), we find that this complexity reduction reaches 99%.

Figure 2 shows similar results for $N = 4$ and 16-QAM, at 4 bits/s/Hz. ML detection achieves full diversity by replacing s_3, s_4 with $s_3e^{j\pi/4}, s_4e^{j\pi/4}$. Again, PR has the same performance as conventional ML, with a complexity gain of $> 97\%$.

The complexity is measured in terms of the number of real multiplications required to decode one block of transmitted symbols as a function of the constellation size L. The complexity of [14] is in the order of $\mathcal{O}(L^{N/4})$ which is equal to $\mathcal{C}_{[11]}$ and will be denoted $\mathcal{C}_{[11],[14]}$ in the sequel. In Table I, we give a comparison between $\mathcal{C}_{ML}$, $\mathcal{C}_{[11],[14]}$, and $\mathcal{C}_{PR}$ in

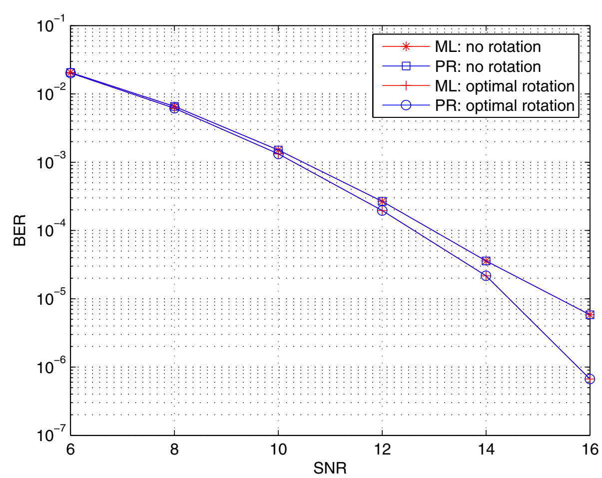

Fig. 1. BER vs SNR for PR and conventional ML for space-time block codes at 1.5 bits/s/Hz; 8 transmit and 1 receive antennas, 4-QAM.

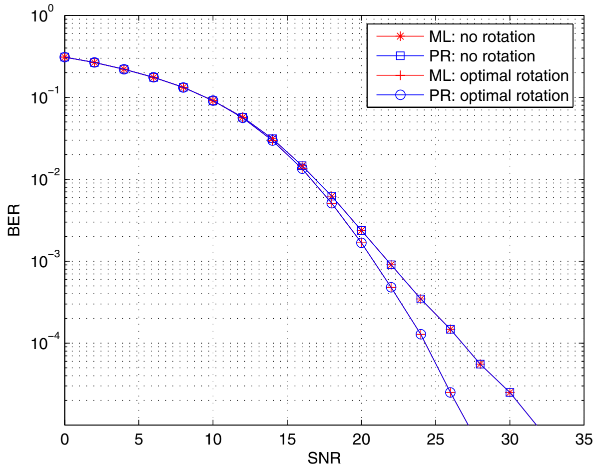

Fig. 2. BER vs SNR for PR and conventional ML for space-time block codes at 4 bits/s/Hz; 4 transmit and 1 receive antennas, 16-QAM.

terms of the number of real multiplication and real additions considering $N = 4$ for different constellation sizes. In Table II, we show the same comparison for $N = 8$. The number of multiplications and additions shown include the computation of QR and $\bar{y} = Q^H \tilde{y}$.

TABLE I

OF REAL MULTIPLICATIONS AND REAL ADDITIONS VS L FOR $N = 4$

	L	4	16	64	256
C_M	ML	1536	24576	393216	6291456
	[11], [14]	512	2048	8192	32768
	PR	304	592	1744	6352
C_A	ML	1376	22016	352256	5636096
	[11], [14]	432	1728	6912	27648
	PR	272	656	2192	8336

TABLE II

OF REAL MULTIPLICATIONS AND REAL ADDITIONS VS L FOR $N = 8$

	L	4	16	64	256
C_M	ML	5120	81920	1.3×10^6	2.1×10^7
	[11], [14]	4608	18432	73728	294912
	PR	696	1128	2856	9768
C_A	ML	4832	77312	1.2×10^6	1.9×10^7
	[11], [14]	4200	16800	67200	268800
	PR	576	1152	3456	12672

Apparently, the complexity gain obtained by PR is substantial. Finally, it is important to emphasize the fact that the complexity reduction, as shown in both tables becomes greater as N or L is larger.

VII. CONCLUSIONS

An efficient ML decoding algorithm based on QR decomposition is proposed for quasi-orthogonal space-time codes. The proposed algorithm is shown to achieve full diversity by applying a two-dimensional constellation rotation. The performance is shown to be optimal while reducing the decoding complexity significantly compared to conventional ML and best algorithms in the literature. Furthermore, we show that the complexity of this algorithm is linear with N and L.

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