Pseudo Orthogonal Designs as Space-Time Block Codes

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I. Introduction

The idea of applying the theory of *Orthogonal Designs* in *Space-Time Coding* was introduced in [1], [2]. The purpose was to create codes that provide full *spatial diversity* with maximal possible *transmission rate* R (preferably $R = 1$) while providing a linear decoding complexity. These designs have remarkably simple maximum-likelihood decoding algorithms via linear processing at the receiver. It was also shown that very few real orthogonal designs exist that provide full diversity and full rate. Assuming n to be the number of the transmit antennas, the 3 classical orthogonal designs for $n = 2, 4$ and 8 are the only existing square real orthogonal designs. The authors of [1] also introduce the concept of *Complex Orthogonal Designs* and prove that the case for complex orthogonal designs is even more stringent. The *Alamouti* design for $n = 2$ would turn out to be the only complex square orthogonal design [1]. Relaxing some of the conditions such as the rate would allow the authors to construct some new codes. The concept of generalized orthogonal designs was introduced that would provide codes for any possible value of n under the trade-off of having $R \leq 1/2$. The estimation of $R \leq 1/2$ would turn out not to be sharp and the authors were able to construct codes for sporadic values of n having higher rates. For instance, for $n = 3$ and $n = 4$ codes of rate $R = 3/4$ were constructed. Since the introduction of space-time block codes, there has been a lot of effort in designing higher rate codes, e.g. [3], [4]. The main conclusions are as follows. The maximum symbol transmission rate of a space-time block code from complex orthogonal designs is only 3/4 for three and four transmit antennas [3]. It means that it is impossible to improve the original orthogonal designs provided in [1] for three and four transmit antennas. Also, there are only few examples of codes with rate higher than 1/2 for more than four transmit antennas [4]. An orthogonal design is defined for any arbitrary indeterminate variables $x_1, x_2, \cdots, x_n$. The theory of orthogonal designs [5] and its generalization [1] provide designs that are orthogonal for any real or complex values of $x_1, x_2, \cdots, x_n$. However, in practice, in a communication system, $x_1, x_2, \cdots, x_n$ are members of a constellation that has a finite number of signals instead of the real (or complex) numbers with infinite number of possibilities. For example, when the transmitter uses QPSK constellation $x_1, x_2, \cdots, x_n$ can take one of the four possible QPSK symbols. From the results provided in [1] and the following studies in the literature, it is not clear if there exists designs that are orthogonal only for a finite set of constellation symbols. In this work, we study the possibility of such designs by defining *Pseudo-Orthogonal Designs* that are defined on a finite subset of real numbers. The case of complex pseudo-orthogonal designs will be handled in future.

In the following we will shortly revisit the orthogonal designs and review their connection with space-time coding and will introduce the concept of Pseudo-Orthogonal Designs.

II. Pseudo Orthogonal Designs

An orthogonal design is defined for any arbitrary indeterminate variables $x_1, x_2, \cdots, x_n$. In this section, we define pseudo-orthogonal designs that are defined for indeterminate variables $x_1, x_2, \cdots, x_n$ that are from a finite set of numbers. This finite set of numbers can be any constellation for example PSK or QAM. If such a pseudo-orthogonal design exists for any constellation , one can use that particular constellation to transmit with a full-rate and achieve maximum diversity and simple maximum-likelihood decoding.

In this work, we only define the real pseudo-orthogonal designs as follows:

Definition 1: Let S be an arbitrary subset of $\mathbb{R}$ with at least 3 different, nonzero elements s_1, s_2, and s_3. A real Pseudo-Orthogonal Design of order n is an n-square matrix G with polynomial entries of degree 1 in n variables $x_1, x_2, \cdots, x_n$, satisfying the real Hurwitz-Radon (HR) equation:

$$G^T G = (x_1^2 + x_2^2 + \cdots + x_n^2) I_n$$

for all possible values of $x_1, x_2, \cdots, x_n \in S$, where G^T is the transpose of G and I_n is the $n \times n$ identity matrix.

If G is a real orthogonal design of order n we may write

$$G = A_1 x_1 + A_2 x_2 + \cdots + A_n x_n.$$

Then the HR-equation is equivalent to

$$\Sigma_{1 \leq i < j \leq n} (A_i^T A_j + A_j^T A_i) x_i x_j + \Sigma_{1 \leq i \leq n} (A_i^T A_i - I_n) x_i^2 = 0. \quad (1)$$

The main idea in studying the orthogonal designs relies on the fact that this latter equation can only hold if all coefficients are simultaneously equal to 0. Since x_i's are simply indeterminate variables assuming any possible real number, Equation (1) results in:

Theorem 1: Let $G = A_1 x_1 + A_2 x_2 + \cdots + A_n x_n$ be a real square orthogonal design. Let also define n-1 matrices $B_1, B_2, \cdots, B_{n-1}$ via the relation $B_i = A_{i+1} A_1^T$, for $i = 1, 2, \cdots, n-1$. Then such a G can only exist if the family $\{B_1, B_2, \cdots, B_{n-1}\}$ build a so called *HR-family* of orthogonal matrices, i.e. matrices that satisfy the following conditions:

1) B_i's are all orthogonal
2) $B_i^2 = -I_n$
3) $B_i B_j = -B_j B_i$

Proof: For the details of the proof we refer to [5] and [1].

In the following we will try to find suitable finite sets S over which we allow the x_i to range in the hope to satisfy Equation (1) without requiring all the coefficients to be 0. The following theorem will settle this problem for the real pseudo orthogonal designs. It will turn out that for the real case no choice of S will provide us with any new result.

Theorem 2: For any arbitrary subset S of the reals having at least 3 non-zero elements s_1, s_2 and s_3, the 3 standard square orthogonal designs for $n = 2, 4, 8$ are the only real pseudo orthogonal designs.

But before we prove the theorem, we need to remark the following about the set S:

1) We do not consider the case where S is a singleton. Since firstly it would be less attractive from a coding theoretical point of view and secondly the HR-equation remains invariant under rescaling (simultaneous multiplication of all variables with a constant), therefore the case of S being a singleton is equivalent to all variables x_i's being equal to 1. In this case our orthogonal design becomes a scalar matrix A satisfying the equation

$$A^T A = n I_n.$$

Such a matrix is referred to as a *Hadamard* matrix in the literature and is extensively studied in [5]. It is shown that such a matrix can only exist if $n = 1, 2$ or $n = 4k$ for some natural number k. Therefore we will only focus on cases where $n = 4k$.

2) As mentioned above since rescaling leaves the HR-equation invariant we may always assume without loss of generality that $1 \in S$.
3) We can always assume that $0 \notin S$. Since otherwise let all x_i's except for one, x_j for instance, to be equal to 0. Then from Equation (1) it follows that $A_j^T A_j - I = 0$ for any arbitrary j. Let also all x_i's be 0 except for two specific x_j and x_k. Then from the same equation it follows that $(A_k^T A_j + A_j^T A_k) = 0$ for any arbitrary values for j and k. Then according to Theorem 1 any pseudo orthogonal design for any S containing 0 will be an orthogonal design.
4) Unlike the case of orthogonal designs, maximum diversity is not guaranteed for pseudo orthogonal designs. The result shows that, under the conditions of the theorem, even non full-rank pseudo orthogonal designs do not exist.

Proof of Theorem 2: Let us focus on a fixed entry ((k,l), for instance), of the matrices $A_i^T A_j + A_j^T A_i$ and $A_i^T A_i - I_n$ in Equation (1). Let

$$2a_{ij}^{kl} := (k,l) - entry\ of (A_i^T A_j + A_j^T A_i)$$

and

$$a_{ii}^{kl} := (k,l) - entry\ of (A_i^T A_i - I).$$

Then from Equation (1) it follows that

$$\Sigma_{1 \le i < j \le n} a_{ij}^{kl} x_i x_j + \Sigma_{1 \le i \le n} a_{ii}^{kl} x_i^2 = 0.$$

Defining the symmetric matrix $A^{kl} := (a_{ij}^{kl})$ will result in:

$$X^T A^{kl} X = 0, \tag{2}$$

where $X := (x_1, x_2, \cdots, x_n)^T$. Then the HR-equation would be equivalent to satisfying Equation (2) for all possible values of $X \in S^n$ for all matrices A^{kl}. We will have n^2 equations of type (2) for n^2 different A^{kl}'s. The proof is complete if we prove that for any such A^{kl}, $A^{kl} = 0$. Since this will prove that our pseudo orthogonal design satisfies Equation (1) and will turn out to be an orthogonal design.

Claim: $A^{kl} = 0$.

Proof of the Claim: Since A^{kl} is symmetric, there exists an orthogonal matrix P such that $P^T A^{kl} P = diag(\lambda_1, \lambda_2, \cdots, \lambda_n) := \Lambda$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues (all real) of A^{kl}. The matrix Λ defines a symmetric bilinear form on $\mathbb{R}^n$, which can be nonsingular. Let M be the set of all isotropic vectors (vectors of length 0) with respect to Λ, i.e. the set of all points in $\mathbb{R}^n$ satisfying Equation (2). Then under P, M gets transferred into a *hypercone*

$$PM := \{(x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n | \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2 = 0\}$$

The following lemma is the crucial point of the proof:

Lemma 1: If a straight line L intersects PM in more than 2 points, then $L \subseteq PM$. In addition if L has the equation $X := Vt + W$, for some W and $V \in \mathbb{R}^n$. Then

$$< V, V > = < W, V > = < W, W > = 0,$$

where we adopt the notation $< x, y >_\Lambda$ or $< x, y >$ for short, instead of $x^T \Lambda y$. Consequently the subspace generated by V and W will be an isotropic subspace, i.e. a subspace on which the bilinear form vanishes.

Proof of Lemma:

Let $X := t(v_1, v_2, \cdots, v_n)^T + (w_1, w_2, \cdots, w_n)^T$ be an arbitrary point of L. Then the intersection of L with PM will satisfy the equation:

$$\lambda_1 (v_1 t + w_1)^2 + \lambda_2 (v_2 t + w_2)^2 + \cdots + \lambda_n (v_n t + w_n)^2 = 0.$$

This quadratic equation (in variable t) has at most 2 solutions. Knowing that L intersects PM at least 3 times, the equation can only hold if all its coefficients equal 0. Therefore it follows:

$$< V, V > = < W, V > = < W, W > = 0.$$

Considering that once all the coefficients are 0, the equation will hold for any possible value of t, proves that $L \subseteq PM$. □

Let $e_i := (0, \cdots, 0, 1, 0, \cdots, 0)^T$ be the standard i^{th} unit vector and $W := (s_1, s_1, \cdots, s_1)^T \in S^n$. Furthermore let L_i be the straight line having the equation: $X := te_i + W$. Then each L_i contains at least 3 points of S^n

$$Y_{i,1} := (s_1, s_1, \cdots, s_1, \cdots, s_1)^T,$$
$$Y_{i,2} := (s_1, s_1, \cdots, s_2, \cdots, s_1)^T,$$
$$Y_{i,3} := (s_1, s_1, \cdots, s_3, \cdots, s_1)^T.$$

Let us also define $e_i' := Pe_i$ and $W' := PW$. Considering that P is an orthogonal transformation, the image of L_i under P, PL_i, that is another line will intersect PM in more than 2 points (the images of $Y_{i,1}, Y_{i,2}, Y_{i,3}$ under P). Then according to the lemma:

$$< e_i', e_i' > = < W', e_i' > = < W', W' > = 0. \tag{3}$$

For an arbitrary choice of i and j ($i \neq j$) let $L_{i,j}$ be the straight line passing through the point $W + (s_2 - s_1)e_i$ and parallel to e_j. Then $L_{i,j}$ will contain 3 points of S^n. The line $PL_{i,j}$ will contain the images of these 3 points (under P) and will satisfy the equation:

$$X := PW + (s_2 - s_1)Pe_i + tPe_j = W' + (s_2 - s_1)e_i' + te_j'.$$

According to the lemma it follows:

$$< W' + (s_2 - s_1)e_i', e_j' > = 0.$$

Using Equation (3) it follows that for all $i \neq j$:

$$< e_i', e_j' > = 0. \tag{4}$$

Combining (3) and (4) it follows that for all i, j:

$$< e_i', e_j' > = 0. \tag{5}$$

Since P is an orthogonal transformation, the vectors $e_1', \cdots, e_n'$ generate the whole space $\mathbb{R}^n$. From Equation (5) it follows that the whole space $\mathbb{R}^n$ is Λ-isotropic, therefore $\Lambda = 0$ or consequently $A^{kl} = 0$. This proves the claim and the theorem. □

III. Conclusions

We have introduced the concept of Pseudo Orthogonal Designs. A pseudo orthogonal design is defined for a finite set of elements instead of all real or complex numbers. We have proved that a real pseudo orthogonal design does not exist. The case of complex pseudo orthogonal designs will be handled later.

References

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